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MULTIPLICITY RESULTS FOR SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS IN BESOV AND TRIEBEL–LIZORKIN SPACES

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The paper deals with superlinear elliptic boundary value problems depending on a parameter. Given appropriate hypotheses concerning the asymptotic behaviour of the nonlinearity, we prove lower bounds on the number of solutions. The results generalize a theorem due to Lazer and McKenna within the framework of quasi-Banach spaces of Besov and Triebel–Lizorkin spaces.

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1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ (the Euclidean n-space) with boundary $\partial\Omega$. Then we consider semilinear elliptic boundary value problems of the type

$$\begin{aligned} Lu &= f(u)+h(x) \qquad && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where L is a second-order uniformly elliptic, formally self-adjoint linear operator and h belongs to a function space of Triebel–Lizorkin and Besov type, respectively. Let $0<\lambda_1<\lambda_2\leqq\cdots\leqq\lambda_k\leqq\cdots$ denote the eigenvalues of L with Dirichlet boundary value conditions. Here f is a sufficiently smooth real-valued function with linear growth at infinity, and more precisely:

$$a\leqq f'(t)\leqq b \quad \text{for all} \quad t\in[-\infty,\infty], \text{ where } f'(\pm\infty)=\lim_{t\to\pm\infty} f'(t).$$

It is known that our problem (1.1) admits multiple solutions depending on the interaction between the values of f' and the spectrum of L if h belongs to the Hölder spaces $C^\alpha(\bar{\Omega})$, $0<\alpha<1$. First, note that if $[a,b]$ contains no eigenvalue λ_k, then (1.1) is uniquely solvable in $C^{2+\alpha}(\bar{\Omega})$, see Dolph [6].

The first result, where the nonlinearity f meets the first eigenvalue, was proved by Ambrosetti and Prodi [2]. They considered the case in which the range of f' contains only the first (simple) eigenvalue λ_1. They showed that the conditions $0<f'(-\infty)<\lambda_1$, $\lambda_1<f'(\infty)<\lambda_2$ and $f''>0$ on $(-\infty,\infty)$ imply the existence of a closed connected

C^1-manifold M_1 of codimension 1 in the Banach space $C^\alpha(\bar{\Omega})$ such that $C^\alpha(\bar{\Omega})\backslash M_1$ has exactly two components M_0 and M_2 with the property that

$$\begin{aligned} -\Delta u &= f(u) + h(x) \qquad \text{in } \Omega \\ u &= 0 \qquad\qquad\qquad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

has no solution if $h \in M_0$, exactly one solution if $h \in M_1$, and exactly two solutions if $h \in M_2$. A corresponding result within the framework of Besov and Triebel–Lizorkin spaces was proved in Geisler and Runst [12]. Manes and Micheletti [19] replaced the condition $0 < f'(-\infty) < \lambda_1$ by $-\infty < f'(-\infty) < \lambda_1$.

The next important result was obtained by Kazdan and Warner [16] for more general functions f. For example, they showed if one decomposes $h = h_1 + t\varphi_1$ (φ_1: normalized eigenfunction to λ_1, h_1: smooth function which satisfies $\int_\Omega h_1(x)\varphi_1(x)\,dx = 0$), then there exists $t_0 = t_0(h_1)$ such that (1.1) has no solution for $t > t_0$ and at least one solution for $t < t_0$. Simultaneously, Dancer [4] and Amann and Hess [1] showed that if f satisfies $f'(-\infty) < \lambda_1 < f'(\infty)$ and f' is bounded on $[0, \infty)$, then (1.1) has at least two solutions if $h = h_1 + t\varphi_1$ and $t < t_0(h_1)$ and at least one solution if $t = t_0$, see also Berger and Podolak [3]. In [17], Lazer and McKenna showed that if $f'(-\infty) < \lambda_1$ and $\lambda_{2k} < f'(\infty) < \lambda_{2k+1}$ ($k \geqq 1$), then there exists $t_1 \leqq t_0$ such that (1.1) has at least three solutions if $h = h_1 + t\varphi_1$ and $t < t_1$ (perform the change of variable $u \to -u$ in order to bring the problem considered in [17] to the present setting). Further results in this direction can be found in Hess and Ruf [13], Ruf [21] and Solimini [23]. Furthermore, Lazer and McKenna obtained in [18] that if $f'(-\infty) < \lambda_1$ and $\lambda_2 < f'(\infty) \leqq \lambda_3$, then (1.1) has at least four solutions if $h = h_1 + t\varphi_1$ and t is sufficiently small. There was also shown that if λ_3 has odd multiplicity, there exists $\beta > \lambda_3$ such that if $f'(-\infty) < \lambda_1$, $\lambda_3 < f'(\infty) \leqq \beta$ and $h = h + t\varphi_1$, then (1.1) has at least five solutions for t sufficiently small.

In this paper, we consider equations of type (1.1) within the framework of Triebel–Lizorkin spaces, $F^s_{p,q}$, and Besov spaces, $B^s_{p,q}$, with methods going back to [17]. For $0 < q < 1$ and/or $0 < p < 1$, $B^s_{p,q}$ and $F^s_{p,q}$ become quasi-Banach spaces (see 2.4). For instance, quasi-Banach spaces are not locally convex, in general. Hence Schauder's fixed point theorem is not applicable in these cases. Lazer and McKenna obtained their results in [17] using the Leray–Schauder degree. Klee [14] proved that it is possible to develop the Leray–Schauder theory in so-called admissible topological spaces. Up to now, it is an open problem whether every quasi-Banach space is admissible in the sense of Klee. However, in Franke and Runst [9], we obtained that the function spaces of Triebel–Lizorkin and Besov type are admissible. Hence we can carry over the theory of [17].

The paper is organized as follows. In Section 2, we describe the preliminaries (function spaces on smooth domains, mapping properties of linear differential operators and of nonlinear operators generated by smooth functions, results of the Leray–Schauder theory). Section 3 deals with the number of solutions of (1.1). The first result is an improvement of one obtained in Drabek and Runst [7]; the second is a generalization of the main result for Hölder spaces in Lazer and McKenna [17] formulated now within the framework of Besov and Triebel–Lizorkin spaces.

2. Preliminaries

2.1. Spaces

Let $\mathbb{R}^n$ be the real Euclidean n-space. In the following, we list some properties of the spaces $B^s_{p,q}$ and $F^s_{p,q}$, see Triebel [24] for details.

Let S be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ and let S' be its topological dual. The Fourier transform and its inverse on S' is denoted by F and F^{-1}, respectively. Now let $\varphi \in S$ be a real-valued even function with respect to the origin such that $\varphi(x)=\varphi(-x)$ if $x \in \operatorname{supp}\varphi \subset \{y \in \mathbb{R}^n, |y| \leqq 2\}$ and $\varphi(x)=1$ if $|x| \leqq 1$. Then we define a sequence $\{\varphi_j\}_{j=0}^{\infty}$ of functions by

$$\varphi_0(x)=\varphi(x), \quad \varphi_j(x)=\varphi(2^{-j}x)-\varphi(2^{-j+1}x), \qquad j=1,2,\ldots$$

for each $x \in \mathbb{R}^n$. We have $\sum_{j=0}^{\infty} \varphi_j(x)=1$ for all $x \in \mathbb{R}^n$.

If $-\infty < s < \infty$, $0 < p$, $q \leqq \infty$, then by definition

$$B^s_{p,q}(\mathbb{R}^n)=\left\{f \in S', \|f \,|\, B^s_{p,q}\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \|F^{-1}\varphi_j Ff \,|\, L_p(\mathbb{R}^n)\|^q\right)^{1/q} < \infty\right\}$$

and if $p < \infty$,

$$F^s_{p,q}(\mathbb{R}^n)\left\{f \in S', \|f \,|\, F^s_{p,q}\| = \left\|\left(\sum_{j=0}^{\infty} 2^{jsq} |F^{-1}\varphi_j Ff(\cdot)|^q\right)^{1/q} \Big| L_p(\mathbb{R}^n)\right\| < \infty\right\}$$

(usual modification if $p=\infty$ and/or $q=\infty$).

It can be shown (see Triebel [24]) that $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces if $\min(p,q) \geqq 1$).

Remark 2.1. By means of the fact that φ is a real-valued even function we can introduce the real part of the spaces $B^s_{p,q}(\mathbb{R}^n)$, etc., denoted by $\tilde{B}^s_{p,q}(\mathbb{R}^n), \ldots$ (for exact definitions see Franke and Runst [9, Subsection 3.2]).

Remark 2.2. These two scales of function spaces include many well-known classical spaces. We give some examples, for details see Triebel [24].

Let $s>0$, then $B^s_{\infty,\infty}(\mathbb{R}^n)=\mathscr{C}^s(\mathbb{R}^n)$ (Zygmund space). If $s>0$ is not an integer, then $B^s_{\infty,\infty}(\mathbb{R}^n)=C^s(\mathbb{R}^n)$ (Hölder space). Let $1<p<\infty$, $-\infty<s<\infty$, then $F^s_{p,2}(\mathbb{R}^n)=H^s_p(\mathbb{R}^n)$ (Bessel-potential space). If s is a natural number, then $F^s_{p,2}(\mathbb{R}^n)=W^s_p(\mathbb{R}^n)$ (Sobolev space).

For dealing with boundary value problems it is useful to define Besov and Triebel–Lizorkin spaces on domains. Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. Then one can introduce the spaces $B^s_{p,q}(\partial\Omega)$ and $F^s_{p,q}(\partial\Omega)$ by standard procedures via local charts (see Triebel [**24**, Subsection 3.2.2]). The spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ are defined usually by the restriction method (see Triebel [**24**, Subsection 3.3.1] and [**25**]).

(i) Let $0<p_0, p_1, q_0, q_1 \leqq \infty$ and $-\infty<s_1<s_0<\infty$. Then

$$D(\Omega)\subset B^{s_0}_{p_0,q_0}(\Omega)\subset B^{s_1}_{p_1,q_1}(\Omega)\subset D'(\Omega) \tag{2.1}$$

if $s_0-(n/p_0)>s_1-(n/p_1)$.

(ii) Let $0<p_0, p_1<\infty$, $0<q_0$, $q_1\leqq\infty$ and $-\infty<s_1<s_0<\infty$. Then

$$D(\Omega)\subset F^{s_0}_{p_0,q_0}(\Omega)\subset F^{s_1}_{p_1,q_1}(\Omega)\subset D'(\Omega) \tag{2.2}$$

if $s-(n/p_0)>s_1-(n/p_1)$.

Here $D(\Omega)$ denotes, as usual, the collection of all complex-valued infinitely differentiable functions f in $\mathbb{R}^n$ with $\operatorname{supp} f\subset\Omega$, and $D'(\Omega)$ is the dual space.

2.2. Traces and linear elliptic differential operators

Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$ and let f be a function defined in Ω belonging to some function spaces of the above type. In the following, R denotes the restriction operator given by $Rf=f\,|\,\partial\Omega$. The following results are known (see Triebel [**24**, Subsection 3.3.3] and Franke [**8**]).

If $0<p,q\leqq\infty$ and $s>s^*:=(n-1)(1/\min(p,1)-1)+1/p$, then R is a linear and continuous mapping from $B^s_{p,q}(\Omega)$ onto $B^{s-1/p}_{p,q}(\partial\Omega)$ and if $p>\infty$, then R is a linear and continuous mapping from $F^s_{p,q}(\Omega)$ onto $B^{s-1/p}_{p,q}(\partial\Omega)$.

Let

$$L=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j},\ a_{ij}(x)\in\tilde{C}^\infty(\bar{\Omega}),$$

satisfying $a_{ij}(x)=a_{ij}(x)$, be a second-order uniformly elliptic operator. In this paper, we only consider the corresponding homogeneous Dirichlet problem. We introduce (for admissible couples (s,p))

$$B^s_{p,q,0}(\Omega)=\{f\in B^s_{p,q}(\Omega), f\,|\,\partial\Omega=0\}$$

and

$$F^s_{p,q,0}(\Omega)=\{f\in F^s_{p,q}(\Omega), f\,|\,\partial\Omega=0\}.$$

Then the following result can be found in Franke [**8**] (see also Triebel [**24**, Subsection

3.3.3]), for $0<q\leqq\infty$ and $s>s^*$. If $0<p\leqq\infty$ then L yields an isomorphic mapping from $B^s_{p,q,0}(\Omega)$ onto $B^{s-2}_{p,q}(\Omega)$. If $0<p<\infty$ then L yields an isomorphic mapping from $F^s_{p,q,0}(\Omega)$ onto $F^{s-2}_{p,q}(\Omega)$.

By Fucik [**11**, Theorem 34.10], we obtain the following result. If λ_1 denotes the smallest eigenvalue of $L|_{\mathring{B}^2_{2,2}(\Omega)}$ with Dirichlet condition, then it holds $\lambda_1>0$ and λ_1 is a simple eigenvalue ($\mathring{B}^s_{p,q}(\Omega)$ denotes the completion of $D(\Omega)$ in $B^s_{p,q}(\Omega)$). Furthermore, there exists a unique normed positive eigenfunction $\varphi_1\in\tilde{C}^\infty(\bar{\Omega})$ to λ_1 with $\varphi_1(x)>0$ in Ω, $L\varphi_1=\lambda_1\varphi_1, (\partial\varphi_1/\partial\nu)<c<0$ on $\partial\Omega$ where ν is the normal, and $\int_\Omega\varphi_1(x)^2\,dx=1$.

In order to prove our main result we need some facts about sub- and supersolutions.

Definition 2.1. A distribution $\psi\in\tilde{D}'(\Omega)$ is said to be *non-negative* if $\psi(\varphi)\geqq0$ for any $\varphi\in\tilde{D}(\Omega)$ with $\varphi\geqq0$.

Remark 2.3. The set of non-negative distributions is $\sigma(D'(\Omega),D(\Omega))$-closed.

Definition 2.2. A function $u\in\tilde{C}(\bar{\Omega})$ is said to be a *supersolution* (*subsolution*) of (1.1) if $Lu\geqq f(u)+h(x)$ in Ω ($Lu\leqq f(u)+h(x)$ in Ω) in the above sense and $u|\partial\Omega=0$.

In Section 3 we use the following maximum principle.

Lemma 2.1. *Let* $v\in\bigcup_{\varepsilon>0}B^\varepsilon_{\infty,\infty}(\Omega)$ *and let* $\mu>-\lambda_1$. *If* $v|\partial\Omega=0$ *and* $(L+\mu)v\geqq0$ *(in the above sense of distributions) then* $v\geqq0$ *holds.*

Proof. *Step 1.* Let $w\in\tilde{B}^{\varepsilon-2}_{\infty,\infty}(\Omega)$, $0<\varepsilon<1$, be non-negative. If $\psi\in\tilde{C}^\infty(\bar{\Omega})$, $\psi|\partial\Omega=0$ then $\psi\in\tilde{B}^{2-\varepsilon}_{\infty,\infty}(\Omega)$. In Franke and Runst [**10**] (see also Triebel [**24**, Subsection 3.4.3]), it is proved: If ψ is non-negative, then ψ can be approximated in $\tilde{B}^{2-\varepsilon}_{1,1}(\Omega)$ by non-negative $\tilde{C}^\infty_0(\Omega)$-functions. Hence $\psi(w)$ is well defined (for the dual space of $(\mathring{\tilde{B}}^{2-\varepsilon}_{1,1}(\,):(\mathring{\tilde{B}}^{2-\varepsilon}_{1,1}(\Omega))'=\tilde{B}^{\varepsilon-2}_{\infty,\infty}(\Omega))$ and non-negative. If $f\in\tilde{C}^\infty_0(\Omega)$ is non-negative, then there exists a non-negative $g\in\tilde{C}^\infty(\bar{\Omega})$ with $g|\partial\Omega=0$, $(L+\mu)g=f$ (see Fucik [**11**, Chapter 34]). According to $\int_\Omega\psi(x)f(x)\,dx=\int_\Omega\varphi(x)g(x)\,dx$ we obtain the following: If $\varphi\in\tilde{C}^\infty_0(\Omega)$, $\psi|\partial\Omega=0$, ψ is non-negative if φ is. Here we used the fact that L is formally self-adjoint, i.e.

$$\int_\Omega Lu_1(x)u_2(x)\,dx=\int_\Omega u_1(x)Lu_2(x)\,dx$$

if $u_1,u_2\in\tilde{C}^\infty(\bar{\Omega})$, $u_1|\partial\Omega=u_2|\partial\Omega=0$.

Step 2. Let v be the same as in the formulation of Lemma 2.1. Let $\varphi\in\tilde{C}^\infty_0(\Omega)$ be non-negative, $\varphi=(L+\mu)\psi$ with $\psi\in\tilde{C}^\infty(\Omega)$, non-negative and $\psi|\partial\Omega=0$. Then an easy limiting argument proves

$$\int_\Omega\varphi(x)v(x)\,dx=\int_\Omega(L+\mu)\psi(x)v(x)\,dx$$

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$$=\int_{\Omega}\psi(x)(L+\mu)v(x)\,dx\geqq 0$$

which completes our proof.

Remark 2.4. The following version of the strong maximum principle holds also: Suppose that $v\in\bigcup_{\varepsilon>0}\tilde{B}^{\varepsilon}_{\infty,\infty}(\Omega)$, $c,\mu\in\mathbb{R}$ with $\mu>-\lambda_1$. If $v|\partial\Omega=0$ and $(L+\mu)v\geqq c>0$ (in the above sense of distributions) then $v>0$ in Ω holds.

2.3. Mapping properties

In this subsection, we list some results which can be found in Runst [**22**, Subsection 5.4]. C^{ρ} denotes as usual the classical Hölder space if $\rho>0$ is not an integer and the well-known Banach space of differentiable functions if $\rho>0$ is an integer. As mentioned above, $B^{\rho}_{\infty,\infty}=\mathscr{C}^{\rho}$ if $\rho>0$.

Lemma 2.2. *Let $f\in\tilde{C}^{\rho+1}$, $\rho>\max(1,s)$. Then $u\to f(u)$ is a completely continuous mapping*

$$\textit{from } \tilde{F}^{s+\varepsilon}_{p,q}(\Omega)\cap L_{\infty}(\Omega)\quad\textit{into}\quad \tilde{F}^{s}_{p,q}(\Omega)\cap L_{\infty}(\Omega)$$

$$(\textit{from } \tilde{B}^{s+\varepsilon}_{p,q}(\Omega)\cap L_{\infty}(\Omega)\quad\textit{into}\quad \tilde{B}^{s}_{p,q}(\Omega)\cap L_{\infty}(\Omega))$$

if $0<p<\infty$ $(0<p\leqq\infty)$, $0<q\leqq\infty$, $s>n((1/\min(p,1))-1)$ *and* $\varepsilon>0$.

Furthermore, there exists a function $g_f, g_f:[0,\infty)\to[0,\infty)$, which is independent of u such that

$$\|f(u)\,|\,F^{s}_{p,q}(\Omega)\|\leqq g_f(\|u\,|\,L_{\infty}\|)\|u\,|\,F^{s}_{p,q}\|$$

$$(\|f(u)\,|\,B^{s}_{p,q}(\Omega)\|\leqq g_f(\|u\,|\,L_{\infty}\|)\|u\,|\,B^{s}_{p,q}\|).$$

Remark 2.5. This result is a consequence of Runst [**22**, Subsection 5.4] and (2.1/2).

2.4. The Leray–Schauder degree

Let A be a (real or complex) linear vector space. $\|\cdot|A\|$ is said to be a quasi-norm if $\|\cdot|A\|$ satisfies the usual conditions of a norm with the exception of the triangle inequality, which is replaced by

$$\|a_1+a_2\,|\,A\|\leqq c(\|a_1\,|\,A\|+\|a_2\,|\,A\|), \tag{2.3}$$

i.e. there exists a positive number c such that (2.3) holds for all $a_1\in A$ and all $a_2\in A$. Of course $c\geqq 1$. (If $c=1$ is admissible, then A is a normed space.) A quasi-normed space is said to be a quasi-Banach space if it is complete. By a theorem due to Rolewicz (see Köthe [**15**, Subsection 18.10]) we may assume without loss of generality that

$$\|a_1+a_2|A\|^\lambda \leqq \|a_1|A\|^\lambda + \|a_2|A\|^\lambda$$

holds for suitable λ, $0<\lambda\leqq 1$. This makes A into a linear metric space with translation invariant metric

$$d_A(a_1,a_2):=\|a_1-a_2|A\|^\lambda.$$

Definition 2.3. Suppose that A is a quasi-normed space of type λ. Such a space is said to be *admissible* if for every compact subset $K\subset A$ and for every $\varepsilon>0$ there exists a continuous mapping $T:K\to A$ such that $T(K)$ is contained in a finite-dimensional subset of A and $x\in K$ implies $\|Tx-x|A\|\leqq\varepsilon$.

Remark 2.6. We introduced the notation "admissible" in the sense of Klee [**14**], see also Riedrich [**20**, Subsection 4.1].

In the following, we use essentially the fact that the spaces considered here are admissible. The next lemma has turned out to be very helpful.

Lemma 2.3 (Franke and Runst [**9**, Subsection 3.1]). *Let A and B be (real or complex) quasi-normed spaces. Furthermore, let $T_0:A\to B$ and $T_1:B\to A$ be continuous mappings. Suppose that T_1 is uniformly continuous on every bounded set and let $T_1T_0=I_A$ (identity of A). Then if B is admissible then A is also admissible.*

By Riedrich [**20**, Subsection 4.2] every normed space is admissible. Furthermore, the quasi-Banach spaces $B^s_{p,q}$ and $F^s_{p,q}$ are of type $\lambda=\min(p,q,1)$. (If $\lambda=1$, then they become Banach spaces.) Applying Lemma 2.3 one can prove the following result.

Lemma 2.4 (Franke and Runst [**9**, Subsection 3.2]). *Let $0<p,q\leqq\infty$ and $-\infty<s<\infty$.*

(i) *The spaces $B^s_{p,q}(\mathbb{R}^n)$, $\tilde{B}^s_{p,q}(\mathbb{R}^n)$, $B^s_{p,q}(\Omega)$, $\tilde{B}^s_{p,q}(\Omega)$, $B^s_{p,q}(\partial\Omega)$ and $\tilde{B}^s_{p,q}(\partial\Omega)$ are admissible.*

(ii) *Let $p<\infty$, then the spaces $F^s_{p,q}(\mathbb{R}^n)$, $\tilde{F}^s_{p,q}(\mathbb{R}^n)$, $F^s_{p,q}(\Omega)$, $\tilde{F}^s_{p,q}(\Omega)$, $F^s_{p,q}(\partial\Omega)$ and $\tilde{F}^s_{p,q}(\partial\Omega)$ are admissible.*

Suppose that X is an admissible quasi-Banach space, B is an open and bounded subset of X, $f:\bar{B}\to X$ is a completely continuous mapping and $y\notin(I-f)\,(\partial B)$. On these admissible triplets $(I-f,B,y)$ one can now introduce the Leray–Schauder degree denoted by $d_{\mathrm{LS}}(I-f,B,y)$. Then the following properties hold (also in admissible topological spaces).

Lemma 2.5 (Riedrich [20, Subsection 4.3]). *Let X be an admissible quasi-Banach space. Then we have*

(a) $d_{\mathrm{LS}}(I,B,y)=\begin{cases}0 \text{ if } y\in X\backslash\bar{B}\\ 1 \text{ if } y\in B\end{cases}.$

(b) *If $d_{\mathrm{LS}}(I-f,B,y)\neq 0$, then there exists a solution $x\in B$ such that $(I-f)(x)=y$ holds.*

(c) *$d_{\mathrm{LS}}(I-f,B,y)=d_{\mathrm{LS}}(I-f,B_1,y)+d_{\mathrm{LS}}(I-f,B_2,y)$ whenever B_1 and B_2 are disjoint open subsets of B such that $y\notin(I-f)(\bar{B}\backslash(B_1\cup B_2))$.*

(d) *$d_{\mathrm{LS}}(I-H(t,\cdot),B,y)$ is independent of $t\in[0,1]$ whenever $H:[0,1]\times\bar{B}\to X$ is a completely continuous mapping and $y\notin(I-H(t,\cdot))(\partial B)$ on $[0,1]$ (invariance under homotopy).*

In what follows, let X be an admissible quasi-Banach space. If x_0 is an isolated fixed point of f (i.e. $(I-f)(x_0)=0$ and $0\neq f(x)$ in $B_r(x_0\backslash\{x_0\}$, where $B_r(x_0)=\{y\in X,\|y-x_0|X\|<r\}$ and r is small enough), then we know that $d_{\mathrm{LS}}(I-f,B_\rho(x_0),0)$ is constant for all $\rho\in(0,r)$. This number is called the index of x_0 and is denoted by $i(I-f,x_0)$. Furthermore, we need some results about the Leray–Schauder degree of completely continuous linear operators acting in admissible quasi-Banach spaces.

Theorem 2.1. *Let X be a real admissible quasi-Banach space, L be a completely continuous linear operator acting in X, $0\neq\lambda\in R$ and λ^{-1} is not an eigenvalue of L. Then $d_{\mathrm{LS}}(I-\lambda L,B_R(0),0)=(-1)^{a(\lambda)}$, where $a(\lambda)$ is the sum of the algebraic multiplicities of the eigenvalues μ satisfying $\mu\lambda>1$, and $a(\lambda)=0$ if L has no eigenvalues of this kind.*

Proof. *Step 1.* By Williamson [26, p. 155] we know that there exists a smallest natural number $k=k(\lambda)$ such that $\ker[(I-\lambda L)^k]=\ker[(I-\lambda L)^{k+1}]$, $\dim\ker[(I-\lambda L)^k]<\infty$, $R[(I-\lambda L)^k]$ is closed and $X=\ker[(I-\lambda L)^k]+R[(I-\lambda L)^k]=:N_k^\lambda+R_k^\lambda$. Furthermore, we have $N_k^\lambda\cap R_k^\lambda=\{0\}$. Hence we obtain $X=N_k^\lambda\oplus R_k^\lambda$. Since $(I-\lambda L)^kL=(I-\lambda L)^k$ we see that N_k^λ and R_k^λ are invariant under L, $L|_{R_k^\lambda}$ is one-to-one and $LR_k^\lambda=R_k^\lambda$. Hence $L|_{R_k^\lambda}$ is a homeomorphism onto R_k. Furthermore, every eigenvalue $\lambda_0\neq 0$ is an isolated one. To prove it let $\tilde{L}_\lambda=L-\lambda I|_{R_k^{\mu_0}}$, $\mu_0=(1/\lambda_0)$. By the properties of L there exists a $c>0$ such that $\|L_{\lambda_0}|x|X\|\geqq c\|x|X\|$. Hence we get $\|\tilde{L}_\lambda x|X\|\geqq(\tilde{c}-|\lambda-\lambda_0|)\|x|X\|$, i.e. $\tilde{L}_\lambda$ is one-to-one for $|\lambda-\lambda_0|<\tilde{c}$. On the other hand, λ_0 is the only eigenvalue of L on $N_k^{\mu_0}$. Indeed, $L_\lambda x=0$ for some $x\in N_k^{\mu_0}$ implies $(L-\lambda I)x=(\lambda-\lambda_0)x$ and therefore $0=-\lambda_0^k(I-\mu_0L)^kx=(\lambda-\lambda_0)^k$ for $k=k(\lambda)$, i.e. $x=0$ if $\lambda\neq\lambda_0$.

Step 2. Now we can prove the above result. We may assume $\lambda=1$ for simplicity. By Step 1 there are at most finitely many eigenvalues $\mu_1,\dots,\mu_m$ of L such that $\mu_i\geqq 1$. Let $Y=N(\mu_1)\oplus\cdots\oplus N(\mu_m)$ and $Z=\bigcap_{i=1}^m R(\mu_i)$, where $N(\mu_i)=N_{k_i(\mu_i)}^{\mu_i}$ and $R(\mu_i)=R_{k_i(\mu_i)}^{\mu_i}$. It is straightforward to see that $X=Y\oplus Z$ holds. Now, any $x\in X$ can be written as $x=y+z$ with $y\in Y$ and $z\in Z$. It holds that $H(t,x):=(1-t)L(y+z)+2ty\neq x$ for all $x\in\partial B_r(0)$ and

$0 \leqq t \leqq 1$ because $H(t,x)=x$ implies $(1-t)Ly+2ty=y$ and $(1-t)Lz=z$. Hence we would obtain $0<t<1$, $Ly=((1-2t)/(1-t))y$ and $Lz=(1/(1-t))z$. Notice that $(1-2t)/(1-t)<1$ for all $t\in(0,1)$. By the properties of Y and Z we get $y=z=0$, i.e. a contradiction to $x\in\partial B_r(0)$. Applying Lemma 2.5 it follows that

$$\begin{aligned} d_{\mathrm{LS}}(I-L, B_r(0), 0) &= d_{\mathrm{LS}}(I-H(0,\cdot), B_r(0), 0) \\ &= d_{\mathrm{LS}}(I-H(1,\cdot), B_r(0), 0) = d_{\mathrm{LS}}(-y, B_r(0), 0) \\ &= \operatorname{sgn}\det(-I|_Y) = (-1)^{a(1)}, \end{aligned}$$

where $a(1)=\dim Y$. Here we used the properties of the Leray–Schauder degree in finite dimensions (see Zeidler [**27**, Subsection 12.5]). Finally, if $a(1)=0$, we may consider the homotopy $H(t,x):=(1-t)Lx$.

Remark 2.7. Theorem 2.1 is a generalization of the so-called index theorem to admissible quasi-Banach spaces. We used an idea similar to one due to Zeidler [**27**, Subsection 14.2], see also Deimling [**5**, Subsection 8.6].

Corollary 2.1. *Let X be a real admissible quasi-Banach space, let $f:B_r(x_0)\subset X\to X$ be a completely continuous mapping with $f(x_0)=x_0$. Furthermore, f is (Frechet-) differentiable at x_0 and $\lambda=1$ is not an eigenvalue of $L:=f'(x_0)$. Then x_0 is an isolated fixed point of f and $i(I-f,x_0)=d_{\mathrm{LS}}(I-L,B_r(0),0)$.*

Proof. The proof is essentially the same as for Banach spaces (see Zeidler [**27**, Korollar 14.1]).

3. Boundary value problems

3.1. On the existence of solutions

Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$ and let

$$\begin{aligned} Lu &= f(u)+h_1(x)+t\varphi_1(x) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{P_t}$$

be a semilinear elliptic boundary value problem, where L and φ_1 are the same as in 2.2. The function h_1 belongs to a real Besov and Triebel–Lizorkin space, respectively, and satisfies $\int_\Omega h_1(x)\varphi_1(x)\,dx=0$.

Theorem 3.1. *Let $0<p,q\leqq\infty$, $s>(n/p)$, $t\in\mathbb{R}$ and $f\in\tilde{C}^{\rho+1}(\mathbb{R})$, $\rho>\max(1,s)$, satisfying the conditions:*

(f1) *there exists a constant c such that $f(x)-\lambda_1 x>c$,*

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(f2) $\lim_{r\to-\infty} \frac{f(r)}{r} < \lambda_1$.

(i) *Let* $h_1 \in \tilde{B}^{s-2}_{p,q}(\Omega) \cap L_\infty(\Omega)$. *Then there is a* $t_0(h_1) \in \mathbb{R}$ *such that* (P_t) *has at least one solution* $u \in \tilde{B}^s_{p,q}(\Omega)$ *if* $t < t_0$, *but no solution if* $t > t_0$.

(ii) *Let* $p < \infty$ *and* $h_1 \in \tilde{F}^{s-2}_{p,q}(\Omega) \cap L_\infty(\Omega)$. *Then there is a* $t_0(h_1) \in \mathbb{R}$ *such that* (P_t) *has at least one solution* $u \in \tilde{F}^s_{p,q}(\Omega)$ *if* $t < t_0$, *but no solution if* $t > t_0$.

Proof. We consider (i). The proof of (ii) is the same.

Step 1. Assume one can solve (P_{t_1}), and let u be a solution. Then u is a strict supersolution of (P_t) for all $t < t_1$. By means of hypothesis (f2) and Franke and Runst [**10**, Theorem 3.4/1], we can find a strict subsolution $u_- \in \tilde{B}^s_{p,q}(\Omega)$ of (P_t), $u_- < u_+$. The following conditions ensure the existence of a subsolution $u_- \in \tilde{B}^s_{p,q}(\Omega)$ of (P_t) (see Franke and Runst [**10**, Subsection 3.4]):

There exists a real number s_- and a bounded $\tilde{C}^\infty$-function $h_-:\bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ with

$$h_1(x) + t\varphi_1(x) + f(v(x)) - \lambda_1 v(x) \geqq h_-(x, v(x)),$$

$$\int_\Omega h_-(x, v(x))\varphi_1(x)\,dx \geqq 0 \quad \text{if} \quad v \in \tilde{C}^\infty(\bar{\Omega}),$$

$$v > s_-\varphi_1 \text{ in } \Omega$$

By analogy with Drabek and Runst [7, Section 3] one can show: If u_+ is a supersolution of (P_t) and u_- is a subsolution of (P_t), $u_- < u_+$ in Ω, then there exists a function $u \in \tilde{B}^s_{p,q}(\Omega)$ such that $u_- \leqq u \leqq u_+$ in Ω and u is a solution of (P_t). Hence one can solve (P_t) for all $t \leqq t_1$.

Step 2. We show that $t_0 > -\infty$. It is enough to find for some $t \in \mathbb{R}$ a supersolution of (P_t), since as in Step 1 we can then find a subsolution $u_- < u_+$. By our assumption we have $h_1 \in \tilde{B}^{s-2}_{p,q}(\Omega) \cap L_\infty(\Omega)$ and $f \in \tilde{C}^{\rho+1}$. Hence we can choose $t(h_1) < 0$ so small such that $f(0) + h_1(x) + t\varphi_1(x) < 0$ holds for $x \in \Omega$. Then $u_+ \equiv 0$ is a supersolution of (P_t).

Step 3. If we put $t_0 = \sup\{t \in \mathbb{R}, (P_t) \text{ is solvable}\}$, then we get $t_0 > -\infty$. To see that $t_0 < \infty$, we apply hypothesis (f1). Let u be a solution of (P_t). Then we get

$$0 = \int_\Omega (Lu - \lambda_1 u)(x)\varphi_1(x)\,dx$$

$$= \int_\Omega (f(u) - \lambda_1 u)(x)\varphi_1(x)\,dx + t$$

which together with (f1) implies that $t_0 < \infty$ holds. Our proof is finished.

Remark 3.1. Notice that the above proof yields the folowing result: For each $h_1 \in \tilde{B}^s_{p,q}(\Omega) \cap L_\infty(\Omega)$ there exists a $t_1(h_1) \leqq t_0(h_1)$ such that (P_t) has at least one negative solution for all $t \leqq t_1(h_1)$.

Remark 3.2. Theorem 3.1 is a generalization of the assertion in Kazdan and Warner [**16**, Corollary 3.11].

Now we consider the solvability of (P_t) for $t = t_0(h_1)$. For it we need an additional condition on f. Then we can apply an idea similar to one used by Hess (see Lazer and McKenna [**17**, Section 3]).

Theorem 3.2. *Let the assumption on s, p, q and ρ of Theorem 3.1 be satisfied and let* $f \in \tilde{C}^{\rho+1}(\mathbb{R})$ *satisfies the following conditions:*

(f3) $f(x) - \lambda_1 x \geqq c_1|x| - b$ *for all* $x \in \mathbb{R}$, *where* $c_1 > 0$, $b \geqq 0$ *and* λ_1 *is, as usual, the first eigenvalue of L,*

(f4) $f(x)$ *is bounded on* $[0, \infty)$.

Then there exists a solution $u_0 \in \tilde{B}^s_{p,q}(\Omega)(u_0 \in \tilde{F}^s_{p,q}(\Omega))$ *of* (P_t) *when* $t = t_0$.

Proof. We consider the case when $u_0 \in \tilde{B}^s_{p,q}(\Omega)$ holds. The other part is almost the same.

Step 1. Let $\{t_n\}_{n=1}^\infty \subset \mathbb{R}$, $t_n \uparrow t_0$, and let u_n be the corresponding solutions. Then $g_n(x) = h_1(x) + t_n\varphi_1(x)$ is bounded in $\tilde{B}^{s-2}_{p,q}(\Omega) \cap L_\infty(\Omega)$. We prove that u_n is bounded in $L_\infty(\Omega)$. Assume the contrary. Then there exists a sequence $\{g_n\}_{n=1}^\infty$ in $\tilde{B}^{s-2}_{p,q}(\Omega) \cap L_\infty(\Omega)$ with $\|g_n|L_\infty\| \leqq M$ and a corresponding sequence $\{u_n\}_{n=1}^\infty$ in $\tilde{B}^s_{p,q}(\Omega)$ satisfying $Lu_n = f(u_n) + g_n$ in Ω, $u_n = 0$ on $\partial\Omega$ and $\|u_n|L_\infty\| \to \infty$ as $n \to \infty$. By (f3), we obtain, if $0 < \lambda_1 - \gamma < c_1$, the existence of a real number c^* such that for all $n \geqq 1$, $Lu_n - \gamma u_n = g_n + (\lambda_1 - \gamma)u_n + (f(u_n) - \lambda_1 u_n) \geqq c^*$ on Ω. By the properties of L (see 2.2) there exists a function $v \in \tilde{B}^s_{p,q,0}(\Omega)$ satisfying $(L-\gamma)v = c^*$. Since $\gamma < \lambda_1$, Lemma 2.1 implies $u_n(x) \geqq \min_{x\in\Omega} v(x)$ for $x \in \Omega$ and all n.

If we define $w_n = u_n/\|u_n|L_\infty\|$, from (f4) it follows for some $K > 0$, $\|Lw_n|L_\infty\| < K$ for all $n \geqq 1$. By the mapping properties of L and compactness results, see 2.1, it follows that we may assume $w_n \to w$ in $L_\infty(\Omega)$. Then $\|w|L_\infty\| = 1$ and since

$$w_n(x) \geqq \frac{\min_{x\in\Omega} v(x)}{\|u_n|L_\infty\|},$$

we obtain $w(x) \geqq 0$ in Ω.

On the other hand, by (f3) we have

$$0 = \int_\Omega (Lw_n - \lambda_1 w_n)(x)\varphi_1(x)\,dx$$

$$=\frac{1}{\|u_n|L_\infty\|}\left(\int_\Omega [f(u_n)-\lambda_1 u_n](x)\varphi_1(x)\,dx+\int_\Omega g_n(x)\varphi_1(x)\,dx\right)$$

$$\geqq\frac{1}{\|u_n|L_\infty\|}\left(\int_\Omega c_1|u_n(x)|\varphi_1(x)\,dx+\int_\Omega (g_n(x)-b)\varphi_1(x)\,sx\right).$$

Hence

$$\int_\Omega w_n(x)\varphi_1(x)\,dx=\frac{1}{\|u_n|L_\infty\|}\int_\Omega u_n(x)\varphi_1(x)\,dx$$

$$\leqq\frac{1}{\|u_n|L_\infty\|}\int_\Omega |u_n(x)|\varphi_1(x)\,dx$$

$$\leqq\frac{1}{c_1\|u_n|L_\infty\|}\int_\Omega (b-g_n(x))\varphi_1(x)\,dx.$$

Now $n\to\infty$, implies a contradiction to $w(x)\geqq 0$ in Ω, $\|w|L_\infty\|=1$.

Step 2. We show that $\{u_n\}_{n=1}^\infty$ is also bounded in $\tilde{B}^s_{p,q}(\Omega)$. Let $0<\varepsilon<2$ be small enough such that $s-\varepsilon>(n/p)$. Step 1 and Lemma 2.2 yield $\|f(u_n)|B^{s-2}_{p,q}\|\leqq c_2(1+\|u_n|B^{s-\varepsilon}_{p,q}\|)$. Because of the imbedding $L_\infty(\Omega)\subset B^0_{p,2}(\Omega)$, see 2.1, and the inequality $\|g|B^{\theta s_0+(1-\theta)s_1}_{p,q}\|\leqq c_3\|g|B^{s_0}_{p,2}\|^\theta\|g|B^{s_1}_{p,q}\|^{1-\theta}$, $0<\theta<1$, we get from $Lu_n=f(u_n)+g_n$

$$\|u_n|B^s_{p,q}\|\leqq c_4(\|g_n|B^{s-2}_{p,q}\|+\|f(u_n)|B^{s-2}_{p,q}\|)$$

$$\leqq c_5(1+\|u_n|B^s_{p,q}\|^\theta).$$

Now we conclude $\|u_n|B^s_{p,q}\|\leqq M_1$. This proves the boundedness of $\{u_n\}_{n=1}^\infty$ in $\tilde{B}^s_{p,q}(\Omega)$.

Step 3. We have proved that the solutions u_n of $u_n=L^{-1}[f(u_n)+h_1(x)+t_n\varphi_1(x)]$, $t_n\uparrow t_0$, are bounded in $\tilde{B}^s_{p,q}(\Omega)$. Applying compactness arguments, we see $u_n\to u$ in $\tilde{B}^s_{p,q,0}(\Omega)$ and u is a solution of (P_t) when $t=t_0$. Our proof is finished.

Remark 3.3. Theorem 3.2 generalizes a result obtained by Amann and Hess [1] (see also Dancer [4] and Lazer and McKenna [17]).

3.2. Multiplicity results

In this subsection, we prove results concerning the number of solutions of the problem (P_t) within the framework of Besov and Triebel–Lizorkin spaces.

Lemma 3.1. *Let* $0<p,q\leqq\infty$, $2>s>(n/p)$, $\rho>\max(1,s)$ *and* $f\in\tilde{C}^{\rho+1}(\mathbb{R})$. *If* $t_1<t<t_2$

and if for $i=1,2, u_i(x)$ is a solution of (P_t) when $t=t_i$ such that $u_1(x)\leqq u_2(x)$ then there exists a number r such that

$$d_{LS}(u-L^{-1}(f(u)+h_1+t\varphi_1), \operatorname{Int} K, 0)=1 \tag{3.1}$$

where

$$K=\{u\in \tilde{B}^s_{p,q,0}(\Omega), u_1\leqq u\leqq u_2 \text{ in } \Omega, \|u|B^s_{p,q}\|\leqq r\}.$$

Remark 3.4. A corresponding result holds also in Triebel–Lizorkin spaces.

Proof of Lemma 3.1. We use an argument of Fucik [11, Theorem 34.7] (see also Lazer and McKenna [17]). By our assumption we have

$$Lu_1<f(u_1)+h_1+t\varphi_1 \text{ in } \Omega,\ u_1|\partial\Omega=0,$$

and

$$Lu_2>f(u_2)+h_1+t\varphi_1 \text{ in } \Omega,\ u_2|\partial\Omega=0.$$

By Theorem 3.1, Step 1 we obtain that there is a solution $u\in\tilde{B}^s_{p,q,0}(\Omega)$ satisfying (P_t) and $u_1\leqq u\leqq u_2$. By analogy with Drabek and Runst [7, Section 3], we choose a number $\omega>0$ such that $\omega+f'(\xi)>0$ for $\xi\in[\min_{x\in\bar\Omega}u_1(x), \max_{x\in\bar\Omega}u_2(x)]$. Notice that $\tilde{B}^s_{p,q}(\Omega)\subset\tilde{C}(\bar\Omega)$ if $s>(n/p)$. Then we get

$$(L+\omega)(u_2-u)=f(u_2)-f(u)+\omega(u_2-u)+(t_2-t_1)\varphi_1$$

and

$$(L+\omega)\left(u_2-u-\left(\frac{t_2-t}{\omega+\lambda_1}\right)\varphi_1\right)=f(u_2-f(u)+\omega(u_2-u_1)>0.$$

Since $u|\partial\Omega=u_2|\partial\Omega=\varphi_1|\partial\Omega=0$, Lemma 2.1 implies $u_2(x)>u(x)$ for $x\in\Omega$. Analogously, we can prove $u_1(x)<u(x)$ for $x\in\Omega$. Now we define $\tilde{f}:\Omega\times\mathbb{R}\to\mathbb{R}$ by

$$\tilde{f}(x,u)=\begin{cases} f(u_2(x))+h_1(x)+t\varphi_1(x) & \text{if } u\geqq u_2(x)\\ f(u)+h_1(x)+t\varphi_1(x) & \text{if } u_1(x)\leqq u\leqq u_2(x)\\ f(u_1(x))+h_1(x)+t\varphi_1(x) & \text{if } u\leqq u_1(x).\end{cases}$$

Notice that by our assumptions $L_{p_1}(\Omega)\subset B^{s-2+\varepsilon}_{p,q}(\Omega)$ for some $p_1<1$ and $\varepsilon>0$ sufficiently small. By definition of $\tilde{f}$ it holds that $L^{-1}\tilde{f}$ is completely continuous in $\tilde{B}^s_{p,q,0}(\Omega)$ and $\|L^{-1}\tilde{f}(x,u)|B^s_{p,q}\|<R$ for $u\in\tilde{B}^s_{p,q,0}(\Omega)$ if R is chosen sufficiently large. For it we apply

the same arguments as in Step 2 of Theorem 3.2. Furthermore, Lemma 2.1 shows that $u=L^{-1}\tilde{f}(x,u)$ implies $u\in \operatorname{Int} K$. Now standard arguments prove

$$1=d_{\mathrm{LS}}(u, B_R(0), 0)=d_{\mathrm{LS}}(u-L^{-1}\tilde{f}(x,u), B_R(0), 0)$$

$$=d_{\mathrm{LS}}(u-L^{-1}(f(u)+h_1+t\varphi_1), \operatorname{Int} K, 0), \tag{3.2}$$

where $\operatorname{Int} K\subset B_R(0)=\{u\in \tilde{B}^s_{p,q,0}(\Omega), \|u\,|\,B^s_{p,q}\|<R\}$.

Lemma 3.2. *Let all the hypotheses of Theorem 3.2 be satisfied and t be a real number.*

(i) *Let $B_R(0)=\{u\in B^s_{p,q,0}(\Omega), \|u\,|\,B^s_{p,q}\|<R\}$. Then there exists $R_0=R_0(t)>0$ such that*

$$d_{\mathrm{LS}}(u-L^{-1}(f(u)+h_1+t\varphi_1), B_{R_1}(0), 0)=0 \tag{3.3}$$

for $R_1>R_0$.

(ii) *Let $p<\infty$ then a corresponding result is also true for $\tilde{F}^s_{p,q}$.*

Proof. We use an argument of Dancer [4] and Theorem 3.2. Let $t_1>t_0(h_1)$ and let R_0 be sufficiently large such that

$$Lu=f(u)+h_1(x)+[t+s(t_1-t)]\varphi_1(x) \qquad \text{in } \Omega$$

$$u=0 \qquad \text{on } \partial\Omega,\ 0\leqq s\leqq 1,$$

imply $u\in B_{R_0}(0)$. By Theorem 3.1 we know that there is no solution for $s=1$. Hence, we obtain by the *a priori* bound and the homotopy invariance

$$0=d_{\mathrm{LS}}(u-L^{-1}(f(u)+h_1+t_1\varphi_1), B_{R_1}(0), 0)$$

$$=d_{\mathrm{LS}}(u-L^{-1}(f(u)+h_1+t\varphi_1), B_{R_1}(0), 0),$$

for $R_1>R_0(t)$.

In the following, we describe the existence of at least three solutions of (P_t).

Theorem 3.3. *Let $0<p,q\leqq\infty$, $2>s>(n/p)$ and $f\in\tilde{C}^{\rho+1}(\mathbb{R})$, $\rho>\max(1,s)$, satisfying (f3) and $\lim_{x\to+\infty} f'(x)=\alpha$, where $\lambda_2<\alpha<\lambda_3$, and λ_2 has multiplicity one.*

(i) *Let $h_1\in\tilde{B}^{s-2}_{p,q}(\Omega)\cap L_\infty(\Omega)$. Then there exists a $t_1(h_1)\in\mathbb{R}$ such that if $t>t_1$, (P_t) has at least three solutions.*

(ii) *Let $p<\infty$ and $h_1\in\tilde{F}^{s-2}_{p,q}(\Omega)\cap L_\infty(\Omega)$. Then there exists a $t_1(h_1)\in\mathbb{R}$ such that (P_t) has at least three solutions if $t<t_1$.*

Proof. We give an outline of the proof of (i). The other part is almost the same. We apply a method used in Lazer and McKenna [**17**, Lemma 3.7].

Step 1. Let $z(x)=\varphi_1(x)/(\lambda_1-\alpha)$. Since $z<0$ in Ω, $\partial z/\partial v>c>0$ on $\partial\Omega$, there exists $\delta_1>0$ such that if $v\in\tilde{B}^s_{p,q,0}(\Omega)$ then $\|v-z|B^s_{p,q}\|<\delta_1$ implies $v<0$ in Ω. Here we used $\tilde{B}^s_{p,q}(\Omega)\subset\tilde{C}(\bar{\Omega})$ if $s>(n/p)$. Let $\delta>0$ be given. Then we choose some $x_0\in\Omega$ and put $r=\min(\delta,\delta_1,(-z(x_0))/2$. The mapping $u\to L^{-1}(\varphi_1+\alpha u)$ is completely continuous on $\tilde{B}^s_{p,q,0}(\Omega)$. Since $z=L^{-1}(\varphi_1+\alpha z)$ holds, we obtain that there exists $\eta>0$ such that $\|u-z|B^s_{p,q}\|=r$ implies $\|u-L^{-1}(\varphi_1+\alpha u)|B^s_{p,q}\|\geqq\eta$. Applying Corollary 2.1 and Theorem 2.1, we get

$$d_{LS}(u-L^{-1}(\varphi_1+\alpha u),V_r,0)=1, \tag{3.3}$$

where $V_r=\{u\in\tilde{B}^s_{p,q,0}(\Omega),\|u-z|B^s_{p,q}\|<r\}$. (Notice that αL^{-1} has precisely two eigenvalues larger than 1.)

Step 2. Now we consider the boundary value problem

$$\begin{aligned} Lu&=\frac{f(tu)-\alpha tu}{t}+\varphi_1+\frac{h_1}{t}+\alpha u && \text{in }\Omega\\ u&=0 && \text{on }\partial\Omega \end{aligned} \tag{3.4}$$

Notice that u is a solution of (P_t) if and only if tu is a solution of (3.4). We get for $t<0$

$$\begin{aligned} &\left\|L^{-1}\left[\varphi_1+\alpha u+\frac{1}{t}(h_1+(f(tu)-\alpha tu)\right]-L^{-1}[\varphi_1+\alpha u]\middle|B^s_{p,q}\right\|\\ &=\left\|L^{-1}\left[\frac{1}{t}(h_1+(f(tu)-\alpha tu))\right]\middle|B^s_{p,q}\right\|\\ &\leqq c\left\|\frac{1}{t}(h_1+(f(tu)-\alpha tu))\middle|L_{p_1}\right\| \end{aligned} \tag{3.5}$$

for some $p_1>1$. Here we used the mapping properties of L and imbedding results (see 2.1/2.2).

From $\lim_{s\to\infty}f'(s)=\alpha$ we infer that for $t\leqq T<0\|u-z|B^s_{p,q}\|<r$ (which implies $u<0$ in Ω) the following estimate holds:

$$c\left\|\frac{1}{t}(h_1+(f(tu)-\alpha tu))\middle|L_{p_1}\right\|<\eta. \tag{3.6}$$

Hence by Step 1, from (3.3), (3.6) it follows that

$$1=d_{\mathrm{LS}}(u-L^{-1}[\varphi_1+\alpha u],V_r,0)$$
$$=d_{\mathrm{LS}}\left(u-L^{-1}\left[\varphi_1+\alpha u+\frac{1}{t}(h_1+f(tu)-\alpha tu)\right],V_r,0\right) \tag{3.7}$$

(homotopy invariance). This implies the existence of a solution $v\in\tilde{B}^s_{p,q,0}(\Omega)$ of (3.10) satisfying $\|v-z|B^s_{p,q}\|<r$. Then tv is a solution of (P_t). Hence we have shown that for given $\delta>0$ there exists $T(\delta)$ so that if $t<T(\delta)$, (P_t) has a solution w such that $\|z-(w/t)|B^s_{p,q}\|<\delta$. Similarly, one can prove

$$1=d_{\mathrm{LS}}(u-L^{-1}[t\varphi_1-\alpha u],B(t),0)$$
$$=d_{\mathrm{LS}}(u-L^{-1}[t\varphi_1+h_1+f(u)],B(t),0), \tag{3.8}$$

where $B(t)=\{u\in\tilde{B}^s_{p,q,0}(\Omega),\|u-tz|B^s_{p,q}\|<|t|r\}$. Notice that $u\in B(t)$ and $r\geqq-(z(x_0)/2)$, where $x_0\in\Omega$, implies $u(x_0)>c(tz(x_o)/2)$. Since $z(x_0)<0$ and since $u\in K(t)$ implies $u(x_0)\leqq u_2(x_0)$, where K and u_2 as in Lemma 3.1 we get

$$\overline{B(t)}\cap K(t)=\varnothing \tag{3.9}$$

for $t<T(\delta)$ and some fixed $r\in(0,\delta)$.

Step 3. For given $\delta>0$ we choose $t<T(\delta)$ such that (3.9) holds. Now we can find by Lemma 3.2 $B_R(0)\supset(\overline{B(t)}\cup K(t))$ such that (3.3) is satisfied. By the properties of the Leray–Schauder degree (Lemma 2.5(c)) we deduce that

$$d_{\mathrm{LS}}(u-L^{-1}[t\varphi_1+h_1+f(u)],B_R(0)\backslash(\overline{B(t)}\cup K(t)),0)\neq 0.$$

Hence in each of the three disjoint sets $B(t)$, $\operatorname{Int}K(t)$ and $B_R(0)\backslash(\overline{B(t)}\cup K(t))$ there exists at least one solution of (P_t). This completes the proof.

Remark 3.5. The conclusion of Theorem 3.3 holds also if $\lambda_{2m}<\alpha<\lambda_{2m+1}$. In the proof of (3.7), (3.8) we use the homotopy invariance of the Leray–Schauder degree. Therefore it was essential that (3.5) and (3.6) hold. In general,

$$\left\|\frac{1}{t}(h_1+(f(tu)-\alpha tu))\middle|B^{s-2}_{p,q}\right\|<\eta$$

does not hold for $t<T$, $\|u-z|B^s_{p,q}\|\leqq r$ and $u\leqq 0$ in $\bar{\Omega}$ if $s-2>(n/p)(\tilde{B}^{s-2}_{p,q}(\Omega))\subset\tilde{C}(\bar{\Omega})$. A simple counter-example is $f(x)\sim c|x|^m$ near the origin and $\lim_{s\to\infty}f'(s)=\alpha$ (see Runst [**22**, Subsection 5.4]).

Remark 3.6. If $\lambda_1<f'(+\infty)<\infty$ holds then we get by Lemma 3.1 and Lemma 3.2

that (P_t) admits at least two solutions if $t < t_0(h_1)$. It is a generalization of a result obtained by Amann and Hess [1] (see also Dancer [4]).

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