J. Aust. Math. Soc. **106** (2019), 31–55
doi:10.1017/S1446788718000046

THE LATTICE OF VARIETIES OF STRICT LEFT RESTRICTION SEMIGROUPS

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(Received 22 February 2017; accepted 7 February 2018; first published online 30 May 2018)

Communicated by J. East

Abstract

Left restriction semigroups are the unary semigroups that abstractly characterize semigroups of partial maps on a set, where the unary operation associates to a map the identity element on its domain. This paper is the sequel to two recent papers by the author, melding the results of the first, on membership in the variety $\mathbf{B}$ of left restriction semigroups generated by Brandt semigroups and monoids, with the connection established in the second between subvarieties of the variety $\mathbf{B}_R$ of two-sided restriction semigroups similarly generated and varieties of categories, in the sense of Tilson. We show that the respective lattices $\mathcal{L}(\mathbf{B})$ and $\mathcal{L}(\mathbf{B}_R)$ of subvarieties are almost isomorphic, in a very specific sense. With the exception of the members of the interval $[\mathbf{D}, \mathbf{D} \vee \mathbf{M}]$, every subvariety of $\mathbf{B}$ is induced from a member of $\mathbf{B}_R$ and vice versa. Here $\mathbf{D}$ is generated by the three-element left restriction semigroup D and $\mathbf{M}$ is the variety of monoids. The analogues hold for pseudovarieties.

2010 *Mathematics subject classification*: primary 20M07; secondary 08A15.

Keywords and phrases: left restriction semigroup, weakly left E-ample semigroup, variety, lattice of varieties, pseudovariety.

1. Introduction

This paper is the sequel to the author's papers [10, 9], melding the results of the former, on membership in the variety $\mathbf{B}$ of left restriction semigroups generated by Brandt semigroups and monoids, with the connection established in the latter between subvarieties of the variety $\mathbf{B}_R$ of two-sided restriction semigroups similarly generated and varieties of categories, in the sense of Tilson [13]. While there is a stark contrast between $\mathbf{B}$ and $\mathbf{B}_R$ – the former is definable by a single unary identity while the latter is nonfinitely based as a biunary variety [7] – we show here that the respective lattices of subvarieties are almost isomorphic, as demonstrated by a comparison of Figure 1 with the corresponding diagram in [10]. There $\mathbf{B_2}$, $\mathbf{B_0}$ and $\mathbf{D}$ denote the varieties respectively generated by the five-element Brandt semigroup B_2 and its subsemigroups B_0 and D; while $\mathbf{M}$ denotes the variety of monoids, $\mathbf{S}$ denotes the variety of semilattices and $\mathbf{T}$ the variety of trivial semigroups.

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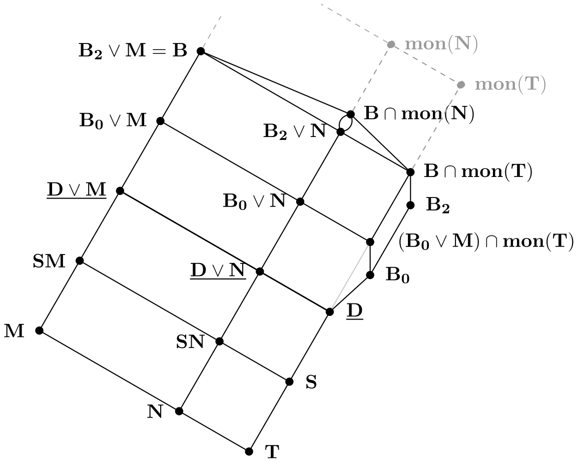

FIGURE 1. The lattice $\mathcal{L}(\mathbf{B})$.

In addition, again in parallel to [9], we consider the pseudovariety generated by finite Brandt semigroups and finite monoids, and show that its lattice of subpseudovarieties may also be represented by Figure 1, with suitable modification of notation.

This paper follows a well-trodden path in studying algebras by the identities that they satisfy. The study of varieties of both left- and two-sided restriction semigroups was initiated by the author in [6]. As noted above, the matter was pursued in [7, 9] for the two-sided case, and in [10] for the one-sided case.

direct continuation of [10], we shall keep the preliminaries on left restriction semigroups to a minimum and refer the reader there for more depth. As noted in [10], left restriction semigroups are of particular interest in that they abstractly characterize the semigroups of partial mappings of a set, under the unary operation $\alpha \mapsto \alpha^+$ that associates with such a map the identity map on its domain. Regarded as unary semigroups, they form the variety **LR**, a standard set of defining identities being

$$x^+x = x, \quad (x^+y)^+ = x^+y^+, \quad x^+y^+ = y^+x^+, \quad xy^+ = (xy)^+x.$$

Recall that the Brandt semigroups are the completely 0-simple inverse semigroups. Given that every inverse semigroup may be regarded as a restriction semigroup and, further, as a left restriction semigroup, they respectively generate the varieties $\mathbf{B}_R$ and **B** defined above. It is shown in the cited papers that, as for inverse semigroups, these varieties are in a sense the 'lowest' in the respective lattices of varieties that do not consist of semilattices of monoids.

As always, the Brandt semigroup of primary interest is $B_2 = \{a, b, e = ab, f = ba, 0\}$. It may also be regarded both as a restriction semigroup and as a left restriction semigroup. Its subsemigroup $B_0 = \{e, a, b, 0\}$ is naturally a restriction semigroup and therefore also a left restriction semigroup. The further subsemigroup $D = \{e, a, 0\}$ is a left restriction semigroup only.

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The lattice of varieties of *inverse* semigroups generated by Brandt semigroups has long been known to be simply described [12]. First, the only proper subvarieties of the variety generated by B_2 are the semilattices and the trivial inverse semigroups; then the larger lattice is the product of this three-element lattice with the lattice of varieties of groups. (Regarded as 'plain' semigroups, Lee [11] has determined the entire, countably infinite, lattice of subvarieties of the variety generated by B_2.)

The focus in [7, 10] was on describing membership in $\mathbf{B}_R$ and $\mathbf{B}$, respectively, in key subvarieties such as those generated by B_0 and B_2, and in the joins of these with the variety of monoids (which plays the role here analogous to that for groups in the case of inverse semigroups). This was achieved in terms of identities, structural characterizations and subdirect decompositions into primitive members. The one-sided case is summarized in Section 4. The necessary material in the two-sided case may be found in Section 6. While there were strong structural parallels between the two-sided and one-sided cases, there was a contrast in varietal terms, illustrated by the fact that every finite restriction semigroup in which the left and right unary operations are distinct is inherently nonfinitely based [7] while, in the case of left restriction semigroups, all the relevant varieties are finitely based (modulo identities for the monoids they contain).

Here we focus on the lattice $\mathcal{L}(\mathbf{B})$ of subvarieties of $\mathbf{B}$. As witnessed in Figure 1, in [10] it was shown that this sublattice is the disjoint union of the ideal $\mathcal{L}(\mathbf{D} \vee \mathbf{M})$ with the filter $[\mathbf{B_0}, \mathbf{B}]$ and that the first of these is isomorphic to the product of the lattice $\mathcal{L}(\mathbf{D}) = \{\mathbf{T}, \mathbf{S}, \mathbf{D}\}$ with $\mathcal{L}(\mathbf{M})$ and so known, modulo varieties of monoids. The second is the disjoint union of the intervals $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$ and $[\mathbf{B_2}, \mathbf{B_2} \vee \mathbf{M} = \mathbf{B}]$.

The main result of this paper is Theorem 7.1, asserting that the interval $[\mathbf{B_0}, \mathbf{B}]$ is isomorphic to the corresponding interval $[(\mathbf{B_0})_R, \mathbf{B}_R]$ in the lattice of varieties of restriction semigroups. (The subscript is used to distinguish the two-sided varieties from the one-sided ones.) In fact, the subintervals $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$ and $[\mathbf{B_2}, \mathbf{B}]$ are individually isomorphic to the corresponding subintervals in $[(\mathbf{B_0})_R, \mathbf{B}_R]$. It was shown in [9] that $[(\mathbf{B_0})_R, (\mathbf{B_0})_R \vee \mathbf{M}]$ is isomorphic to the lattice of varieties of monoids, with an adjoined zero, and that $[(\mathbf{B_2})_R, \mathbf{B}_R]$ is isomorphic to the lattice of varieties of categories, in the sense of [13].

One of the key tools in [10] was an embedding of certain special primitive left restriction semigroups S into restriction semigroups. It is proved here that, in all but two of the cases of interest, the left restriction reduct of the extension belongs to the variety generated by S. This relates varieties of left restriction semigroups to varieties of restriction semigroups; the relation in the opposite direction proceeds by a different mechanism. In each case, the proof is greatly facilitated by Tilson's bonded component theorem [13] (using the connection espoused in [9]).

The only difference between the lattice in Figure 1 and the corresponding one for restriction semigroups is the insertion of the interval sublattice $[\mathbf{D}, \mathbf{D} \vee \mathbf{M}]$, signified by the interval containing underlined varieties. In [9] the join $\mathbf{V} \vee \mathbf{M}$ was condensed to $\mathbf{VM}$, but we shall not do so here, except in the case of semilattices of monoids.

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In Section 4, we show that the basic results on varieties of left restriction semigroups from [6] and [10] carry over to pseudovarieties. In Section 8 we then show that the same is true for the lattice of subpseudovarieties.

A final remark is warranted in this introduction. In [9] a very natural correspondence was established between categories and primitive restriction semigroups, leading to an isomorphism between the lattice of varieties of categories and the interval $[(\mathbf{B}_2)_R, \mathbf{B}_R]$ in the lattice of varieties of restriction semigroups. It was also shown there that an analogous isomorphism exists for pseudovarieties. The results of this paper demonstrate that the lattice of varieties of categories is also isomorphic to the corresponding interval $[\mathbf{B}_2, \mathbf{B}]$ in the lattice of varieties of left restriction semigroups, and analogously for pseudovarieties, a connection that seems rather less natural to the author's mind.

2. Preliminaries

We briefly review the background on left restriction semigroups. For a general background on semigroups, see [5]. Useful introductions to both left and two-sided restriction semigroups, including different ways in which they have arisen as topics of interest, are [4] and [3]. We shall need only elementary universal algebra, which may be found in [2], and elementary general properties of pseudovarieties, as found in [1], for instance.

From the first two defining identities for left restriction semigroups stated above it follows that if S is such a semigroup, then for all $x \in S$, x^+ is idempotent and, in conjunction with the second identity, $(x^+)^+ = x^+$. These idempotents (there may be others) are the *projections* of S. The set P_S of projections forms a semilattice, ordered in the usual fashion, by virtue of the third identity. The last identity (or a variation of it) is often termed the 'left ample' identity.

A *restriction* semigroup is a biunary semigroup $(S, \cdot, ^+, ^*)$ that is a left restriction semigroup with respect to $^+$, satisfies the 'dual' identities obtained by replacing $^+$ by * and reversing the order of each expression, and further satisfies $(x^+)^* = x^+$ and $(x^*)^+ = x^*$. Thus $P_S = \{x^+ : x \in S\} = \{x^* : x \in S\}$. Every restriction semigroup may be regarded as a left restriction semigroup, by 'forgetting' the second unary operation.

For the purposes of this paper, the relevant generalized Green relations may be defined as follows. In a left restriction semigroup, $\mathbb{R} = \{(a, b) : a^+ = b^+\}$. In a restriction semigroup, the relation $\mathbb{L} = \{(a, b) : a^* = b^*\}$ is defined dually; and there $\mathbb{H} = \mathbb{R} \cap \mathbb{L}$ and $\mathbb{D} = \mathbb{R} \vee \mathbb{L}$ (not in general equal to $\mathbb{R} \circ \mathbb{L}$). We have reverted to the notation of [6, 7] after having misguidedly changed notation in [9]. The natural partial order on a left restriction semigroup S is defined by $a \leq b$ if $a = eb$ for some $e \in P_S$ and, by application of the left ample identity, is easily seen to be compatible with the operations. On a restriction semigroup, this partial order is self-dual (and so compatible with both unary operations).

In general, the terms 'homomorphism', 'congruence' and 'divides' will be used appropriate to context; that is, they should respect the unary operation for left

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restriction semigroups and both unary operations for restriction semigroups. In the case of subsemigroups, we shall generally use the modifier 'left restriction' (or 'unary') or 'restriction' to clarify. If S is a left restriction semigroup and T is a restriction semigroup, when we say that T is a left restriction subsemigroup of S, the second operation on T is forgotten. Note that in this situation the $\mathbb{R}$-relations coincide and P_T is a subsemilattice of P_S (because the projections are determined by either unary operation). One congruence of note on left restriction semigroups is the greatest projection-separating congruence, denoted μ. Equivalently, μ is the greatest congruence contained in $\mathbb{R}$. Elements a and b are μ-related if and only if $(ae)^+ = (be)^+$ for all $e \in P_S$.

In the standard terminology, restriction semigroups S (either one- or two-sided) with $|P_S| = 1$ are termed *reduced*. Since, in essence, they are just monoids, regarded as restriction semigroups by setting $a^+(= a^*) = 1$ for all a, we will generally omit the qualifier 'reduced', except in cases of possible ambiguity.

The second congruence of note is the least monoid congruence, denoted σ. A left restriction semigroup is *proper* if $\sigma \cap \mathbb{R}$ is the identical relation. Since $\mu \subseteq \mathbb{R}$, it is clear from the definition that on any such semigroup, $\sigma \cap \mu$ is the identical relation. A *proper cover* for a left restriction semigroup S is a proper left restriction semigroup T for which there is a projection-separating homomorphism upon S. An elementary proof of the following was given by the author.

Result 2.1 [8, Theorem 9.1]. *Every [finite] left restriction semigroup has a [finite] proper cover.*

A (unary) subsemigroup T of a left restriction semigroup S is a *submonoid* if it contains a unique projection e. By [6, Lemma 4.6], the maximal submonoids of S have the form $M_e = \{a \in \mathbb{R}_e : a = ae\}$. By [10, Lemma 2.2], the term is independent of the one-sided or two-sided context.

A $^+$*-ideal* I of a left restriction semigroup S is an ideal of S that is also a left restriction subsemigroup. It is easily seen that the Rees quotient semigroup S/I is again a left restriction semigroup. As usual, for technical reasons it is convenient to allow the empty set to be an ideal and, in that case, to put $S/I = S$.

3. Primitive and strict left restriction semigroups

The term *primitive* refers to any (left or two-sided) restriction semigroup with zero in which each nonzero projection is 0-minimal.

A left restriction semigroup is *strict* if it is a subdirect product of primitive left restriction semigroups and monoids. Strict restriction semigroups are defined analogously. These definitions are the natural extensions of strictness for inverse semigroups [12, II.4]. In each case, by [10, 7, 12], strictness characterizes membership in the respective variety generated by Brandt semigroups, together with monoids or groups, as appropriate. Thus in the varietal context, primitivity is the essential concept. We now review the background in the left restriction case (for the two-sided case, see Section 5).

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Let S be any left restriction semigroup. Let $S^{RI} = \{s \in S : se = s \text{ for some } e \in P_S\}$. By a *right identity* for an element of S we will always mean a *projection* with that property. Clearly S^{RI} is a $^+$-subsemigroup and left ideal of S. For each $e \in P_S$, put $\mathrm{RF}(e) = \{s \in S : as^+ = a \text{ for some } a \in \mathbb{R}_e\}$, the union of the $\mathbb{R}$-classes of those projections of S that are right identities for some element of $\mathbb{R}_e$. By [10, Lemma 2.4], $\mathrm{RF}(e) = \{s \in S : as \in \mathbb{R}_e \text{ for some } a \in \mathbb{R}_e\}$. Observe that for any $a \in \mathbb{R}_e$ and $f \in P_S$, f is a right identity for a if and only if $af \in \mathbb{R}_e$, since $af = (af)^+a$, the left 'ample' identity. Note that the submonoid M_e consists of those members of $\mathbb{R}_e$ having e as right identity.

If $x, y \in S$ and $xy \in \mathrm{RF}(e)$ then $x, y \in \mathrm{RF}(e)$. Thus $I_e = S \setminus \mathrm{RF}(e)$ is a $^+$-ideal of S unless $S = \mathrm{RF}(e)$. As usual, we may identify the Rees quotient S/I_e with the union of $\mathrm{RF}(e)$ and 0 when convenient. If $S = \mathrm{RF}(e)$, then e must be the least projection of S and $\mathbb{R}_e = M_e$. By convention, $S/I_e = S$ in that case.

Returning now to the specific topic of this section, let S be a primitive left restriction semigroup (with zero) and suppose x has a right identity (that is, $x \in S^{RI}$). Then x has a *unique* right identity, since the set of right identities for x is a subsemilattice of P_S. Denote this right identity by x^* (shown in [10] to be consistent with its use in two-sided restriction semigroups). Clearly $e^* = e$ for every projection e of S. We collect some basic tools in the following.

Result 3.1 [10, Lemma 5.1, Proposition 5.2]. *Let S be a primitive left restriction semigroup and let $x, y \in S$, $x, y \neq 0$. Then $xy \neq 0$ if and only if (x^* exists and) $x^* = y^+$, in which case $xy \,\mathbb{R}\, x$, and if xy has a right identity, then so does y, and $(xy)^* = y^*$.*

It follows that in any primitive left restriction semigroup S, the left restriction subsemigroup S^{RI} is a primitive restriction semigroup, under the additional operation * defined above. Equivalently, if every element of a primitive left restriction semigroup S has a right identity, then S is also a primitive restriction semigroup in this way.

Call a primitive left restriction semigroup S *primitive with base e* if e is a nonzero projection of S with the property that $S\setminus\{0\} = \mathrm{RF}(e)$, in other words if every nonzero projection of S is a right identity for some (not necessarily unique) element of $\mathbb{R}_e$. If g is a right identity for $a \in \mathbb{R}_e$, then $g = a^*$. Note that $M_e = \{a \in \mathbb{R}_e : a^* = e\}$.

Relating primitive left restriction semigroups to their two-sided counterparts played a major role in proving the main results in [10] and is the crux of the current paper. One connection is established through the following embedding.

Let S be a primitive left restriction semigroup with base e. If $S = S^{RI}$, put $S^* = S$. Otherwise, let $S^* = S \cup \{h\}$, where h is an element distinct from S. Extend the binary operation on S to S^* by putting $h^2 = h$, $hs = 0$ for all $s \in S\setminus\{h\}$; $sh = s$ for all $s \notin S^{RI}$; and $sh = 0$ for all $s \in S^{RI}$. Put $h^+ = h$. The following result will be greatly enhanced in Theorem 7.6.

Result 3.2 [10, Proposition 5.4]. *Let S be a primitive left restriction semigroup with base e. Then S^* is a primitive restriction semigroup in which S is $^+$-embedded. Each submonoid of S^* is either a submonoid of S or is trivial. If S is finite, so is S^*.*

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4. Varieties and pseudovarieties of left restriction semigroups

The elementary material on varieties that follows is extracted from [6] and the material on strict varieties is from [10]. With modest modifications, we adapt it all to pseudovarieties, along the lines followed in [9, Section 6] for the two-sided case.

A pseudovariety of left restriction semigroups is a collection of finite such semigroups that is closed under finite products and division. The finite members of any variety $\mathbf{V}$ of universal algebras form a pseudovariety, which we shall denote $F\mathbf{V}$. Given a collection of finite members of $\mathbf{V}$, the pseudovariety that they generate will not in general comprise the finite members of the variety they generate. However, in the case of a single finite algebra (or finitely many such), the pseudovariety so generated does indeed consist of the finite members of the variety so generated (e.g. see [1, page 60]).

Recall the notation and terminology for specific semigroups and varieties from Section 1. Also recall that monoids in this context are left restriction semigroups with one projection and so may be defined by the identity $x^+ = y^+$. Subvarieties of $\mathbf{M}$ are essentially varieties of monoids, and we shall treat them as such. It is clear that a variety $\mathbf{V}$ of left restriction semigroups consists of monoids if and only if $\mathbf{V} \cap \mathbf{S} = \mathbf{T}$. Analogous statements hold for pseudovarieties.

For any variety $\mathbf{V}$ of left restriction semigroups, denote by $\mathcal{L}(\mathbf{V})$ its lattice of subvarieties. If $\mathbf{U}, \mathbf{V} \in \mathcal{L}(\mathbf{V})$ and $\mathbf{U} \subseteq \mathbf{V}$, $[\mathbf{U}, \mathbf{V}]$ denotes the interval sublattice $\{\mathbf{W} : \mathbf{U} \subseteq \mathbf{W} \subseteq \mathbf{V}\}$. The notation $\mathbf{V} \succ \mathbf{U}$, $\mathbf{V}$ *covers* $\mathbf{U}$, means that the interval consists only of the two given varieties. If X is a set of left restriction semigroups, $\langle X \rangle_{LR}$ will denote the variety of left restriction semigroups it generates. Again, analogous notation applies to pseudovarieties.

Proposition 4.1. *If* $\mathbf{V} \in \mathcal{L}(\mathbf{LR})$, *then* $\mathbf{V} \vee \mathbf{M} = \{S \in \mathbf{LR} : S/\mu \in \mathbf{V}\}$. *Hence on the lattice* $\mathcal{L}(\mathbf{LR})$ *the map* $\mathbf{V} \longrightarrow \mathbf{V} \vee \mathbf{M}$ *is a complete lattice homomorphism. The map* $\mathbf{V} \longrightarrow \mathbf{V} \cap \mathbf{M}$ *is a lattice homomorphism.*

From the first statement, it follows that $F(\mathbf{V} \vee \mathbf{M}) = F\mathbf{V} \vee F\mathbf{M}$.

The analogues of the statements in the first paragraph hold for pseudovarieties.

Proof. The first paragraph is [6, Theorems 4.1, 4.2]. We need to outline the proof of the first statement to see how finiteness is respected. Then we immediately obtain its finitary analogue, from which the analogue of the second statement will follow, along with the statement in the second paragraph.

The original statement, that if $S \in \mathbf{V} \vee \mathbf{M}$ then $S/\mu \in \mathbf{V}$, clearly respects finiteness. To prove the converse, we apply Result 2.1: any left restriction semigroup S has a proper cover T, which can be chosen to be finite if S is finite. As noted in the paragraph prior to that result $\sigma \cap \mu$ is the identical relation on T, so it embeds in $T/\sigma \times T/\mu$, where, because the covering map separates projections, $T/\mu \cong S/\mu$. Thus if $S/\mu \in \mathbf{V}$, then T, and so also S, belongs to $\mathbf{V} \vee \mathbf{M}$. If S, and so T, is finite, the same is true of the quotients of the latter, and so $S \in F\mathbf{V} \vee F\mathbf{M}$. If $\mathbf{V}$ is a pseudovariety to begin with, $S \in \mathbf{V} \vee F\mathbf{M}$.

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The proof in [6] of the last statement in the first paragraph made use of properties of the free left restriction semigroup and thus a new proof is needed for pseudovarieties, as follows.

Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of left restriction semigroups and let $M \in \mathbf{V} \vee \mathbf{W}$. Thus there exist $S \in \mathbf{V}$, $T \in \mathbf{W}$, a unary subsemigroup U of $S \times T$ and a unary homomorphism of U upon M. By finiteness, there exists a least $e \in P_U$ that is mapped to the identity of M, whence eUe is mapped onto M itself. By minimality, e is the only projection in eUe, that is, $eUe = \mathbb{H}_e$. Routinely, if $e = (e_1, e_2) \in P_S \times P_T$, then $\mathbb{H}_e$ embeds into $\mathbb{H}_{e_1} \times \mathbb{H}_{e_2} \in (\mathbf{V} \cap \mathbf{M}) \vee (\mathbf{W} \cap \mathbf{M})$. Therefore $(\mathbf{V} \vee \mathbf{W}) \cap \mathbf{M} = (\mathbf{V} \cap \mathbf{M}) \vee (\mathbf{W} \cap \mathbf{M})$. □

The usual argument shows that $\mathbf{S}$ is the only atom of $\mathcal{L}(\mathbf{LR})$ that does not consist of monoids, and likewise for pseudovarieties. Denote by $\mathbf{SM}$ the variety of semilattices of monoids. The proof of the varietal portion of the following result is straightforwardly modified to the case of pseudovarieties, using the result just proved.

Result 4.2 [6, Theorems 4.4, 4.5]. *The following are equivalent for a left restriction semigroup S: (a) $S \in \mathbf{SM}$; (b) S satisfies $(xy)^+ = x^+y^+$; (c) S becomes a restriction semigroup under the assignment $a^* = a^+$; (d) S is a (strong) semilattice of monoids.*

The sublattice $\mathcal{L}(\mathbf{SM})$ of $\mathcal{L}(\mathbf{LR})$ is isomorphic to the direct product of the lattice $\mathcal{L}(\mathbf{S}) = \{\mathbf{T}, \mathbf{S}\}$ and the lattice $\mathcal{L}(\mathbf{M})$, under the map $\mathbf{V} \mapsto (\mathbf{V} \cap \mathbf{S}) \vee (\mathbf{V} \cap \mathbf{M})$. If $\mathbf{V}$ is not simply a variety of monoids, then it consists of all (strong) semilattices of monoids from $\mathbf{V} \cap \mathbf{M}$.

The analogues hold for pseudovarieties.

Next we review further coverings and show that the analogues hold for pseudovarieties. According to [6, Theorem 4.20], there are precisely two subvarieties of $\mathbf{LR}$ that are minimal with respect to not being contained in $\mathbf{SM}$. One is $\mathbf{D}$, introduced in Section 1. The other is $\mathbf{L_2^1}$, which is generated by the left restriction semigroup L_2^1 that is obtained from the two-element left zero semigroup $L_2 = \{g, h\}$ by adjoining an identity 1, and setting $g^+ = g$ and $h^+ = 1^+ = 1$. Note that, when regarded as a 'plain' semigroup, L_2^1 is a monoid, but it is not when regarded as a left restriction semigroup. It is the union of the submonoids $M_1 = \{1, h\}$ and M_g.

The pseudovarietal analogue of [6, Theorem 4.20] (as in Corollary 4.4 below) can be deduced from the varietal result. However both follow immediately from the next, significantly sharper, lemma, which is potentially useful in its own right. A left restriction semigroup that is a union of its submonoids is a *union of monoids*. According to [6, Theorem 4.8], the unions of monoids form a variety, defined by $xx^+ = x$.

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LEMMA 4.3.

(1) *The semigroup D divides any left restriction semigroup that is not a union of monoids.*
(2) *The semigroup L_2^1 divides any left restriction semigroup that is a union of monoids but not a semilattice of monoids.*

PROOF. To prove the first statement, observe from the theorem just cited that if S is not a union of monoids, then it contains an element a such that $aa^+ \neq a$. The proof of [6, Theorem 4.17] shows that the unary subsemigroup T generated by a contains as a $^+$-ideal the subset $K_2 = \{(a^n)^+a^k : n \geq 2, 0 \leq k \leq n\}$ (where a^0 is a nominal identity), and that $T \backslash K_2 = \{a^+, a\}$ and $T/K_2 \cong D$.

To prove the second statement, suppose S is a union of monoids that is not a semilattice of monoids. Then, by Result 4.2, S contains an element a and a projection e such that $a = se \neq es$, say. The proof of [6, Proposition 4.16] shows that the unary subsemigroup U generated by $\{a, se\}$ is a projection-separating quotient of the (infinite) left restriction semigroup B_{01}^+. The precise nature of the latter semigroup – a subsemigroup of the bicyclic semigroup – is not needed, merely that $B_{01}^+/\mu \cong L_2^1$ [6, Proposition 4.14]. It follows that L_2^1 is a quotient of U. □

COROLLARY 4.4 (See [6, Theorem 4.20]). *The varieties* $\mathbf{D}$ *and* $\mathbf{L_2^1}$ *are the subvarieties of* $\mathbf{LR}$ *that are minimal with respect to not being contained in* $\mathbf{SM}$. *The analogue holds for pseudovarieties.*

We now move to the results of [10] on the variety $\mathbf{B}$ of left restriction semigroups generated by the Brandt semigroups and the monoids, together with certain of its subvarieties, and also consider the analogues for pseudovarieties. First recall from the cited paper that if $\mathbf{V}$ is any variety of left restriction semigroups, $\mathbf{loc}(\mathbf{V})$ denotes the variety consisting of the left restriction semigroups S that are 'locally' in $\mathbf{V}$, meaning that $eSe \in \mathbf{V}$ for all $e \in P_S$. In a related vein, for any variety $\mathbf{N}$ of monoids, let $\mathbf{mon}(\mathbf{N})$ consist of the left restriction semigroups all of whose submonoids belong to $\mathbf{N}$. Note that $\mathbf{loc}(\mathbf{SN}) \subseteq \mathbf{mon}(\mathbf{N})$. In general the latter is not a variety.

RESULT 4.5 [10, Proposition 4.3]. *The inclusion* $\mathbf{B} \subset \mathbf{loc}(\mathbf{SM})$ *holds. For any variety* $\mathbf{N}$ *of monoids,* $\mathbf{B} \cap \mathbf{mon}(\mathbf{N}) = \mathbf{B} \cap \mathbf{loc}(\mathbf{SN})$ *and so forms a subvariety.*

In particular, the members of $\mathbf{B}$ having trivial submonoids form the subvariety $\mathbf{B} \cap \mathbf{mon}(\mathbf{T}) = \mathbf{B} \cap \mathbf{loc}(\mathbf{S})$.

It follows from the first statement of Result 4.5 that $L_2^1 \notin \mathbf{B}$, since $L_2^1 = 1L_2^1 1 \notin \mathbf{SM}$. Thus varieties consisting of unions of monoids play no role in this paper and in our context $\mathbf{D}$ is the unique relevant cover of $\mathbf{S}$ (there being other covers in the interval $[\mathbf{S}, \mathbf{SM}]$).

The analogues of all the above statements regarding $\mathbf{B}$ then carry over the pseudovariety $F\mathbf{B}$.

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Recall that $\mathbf{B_2}$, $\mathbf{B_0}$ and $\mathbf{D}$ are the subvarieties generated respectively by B_2, B_0 and D. By the general argument noted earlier in this section, the *pseudovarieties* generated by these last three finite left restriction semigroups are respectively $F\mathbf{B_2}$, $F\mathbf{B_0}$ and $F\mathbf{D}$.

It will be seen in Result 4.11 below that $\mathbf{B} = \mathbf{B_2} \vee \mathbf{M}$. By the second paragraph of Proposition 4.1, $F\mathbf{B} = F\mathbf{B_2} \vee F\mathbf{M}$ and therefore this pseudovariety is indeed that generated by the finite Brandt semigroups and finite monoids, to which the 'analogous results' will refer.

Result 4.6 characterizes membership in $\mathbf{B}$ in several further ways. For our purposes, rather than the identity stated in (i), it is the paraphrase (ii) in terms of right identities and the $\mathbb{R}$-relation that is more useful in practice, along with the subdirect decomposition (iii) that plays a central role in all applications.

Result 4.6 [10, Theorem 6.6]. *For a left restriction semigroup S, the following are all equivalent to membership in $\mathbf{B}$, the variety generated by Brandt semigroups and monoids:*

(i) *S satisfies the identity $(xz)^+(yz^+w)^+ = (yz)^+(xz^+w)^+$;*

(ii) *if two $\mathbb{R}$-related elements of S share any right identity, they share all right identities;*

(iii) *S is a subdirect product of monoids and primitive left restriction semigroups with a specified base;*

(iv) *S is a subdirect product of monoids and primitive left restriction semigroups, that is, S is strict.*

In view of the observations before the last result and the obvious fact that the factors in the subdirect products given by (iii) and (iv) preserve finiteness, the following is clear.

Corollary 4.7. *The analogue of Result 4.6 holds for the pseudovariety $F\mathbf{B}$.*

Next we consider $\mathbf{B_2}$ and its subvarieties $\mathbf{B_0}$ and $\mathbf{D}$ in the context of Result 4.6. As well as the characterizations stated here, in each case identities were also provided in [10]; naturally, there is an additional characterization in terms of subdirect products whose factors satisfy the respective identities.

Result 4.8 [10, Theorem 8.11, Corollaries 8.12, 7.7]. *Let S be a left restriction semigroup. Then*

(1) *$S \in \mathbf{B_2}$ if and only if for any projection e, distinct elements of $\mathbb{R}_e$ do not share a common right identity;*

(2) *$S \in \mathbf{B_0}$ if and only if $S \in \mathbf{B_2}$ and every regular element of S is a projection;*

(3) *$S \in \mathbf{D}$ if and only if the only elements with right identities are the projections.*

Again, in view of the remarks preceding Result 4.6, the following is clear.

Corollary 4.9. *The analogue of Result 4.8 holds for finite left restriction semigroups, with respect to the pseudovarieties $F\mathbf{B_2}$, $F\mathbf{B_0}$ and $F\mathbf{D}$.*

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The sublattice $\mathcal{L}(\mathbf{B_2})$ was shown to be as follows, from which we deduce the pseudovarietal analogue.

Proposition 4.10 [10, Theorem 8.16]. *The lattice of subvarieties of* $\mathbf{B_2}$ *consists of the chain of coverings* $\mathbf{T} \prec \mathbf{S} \prec \mathbf{D} \prec \mathbf{B_0} \prec \mathbf{B_2}$.

The analogue holds for the lattice of subpseudovarieties of $F\mathbf{B_2}$.

Proof. To prove the second statement, let $S \in F\mathbf{B_2}$. Then the *variety* $\mathbf{V}$ generated by S is one of those listed in the first statement and so the pseudovariety $F\mathbf{V}$ generated by S is one of the corresponding pseudovarieties. Since any pseudovariety is the join of its finitely generated subvarieties, every subpseudovariety of $F\mathbf{B_2}$ has one of the stated forms. □

Finally, we consider the joins of each of these varieties with $\mathbf{M}$. It was noted prior to Result 4.6 that $\mathbf{B} = \mathbf{B_2} \vee \mathbf{M}$. The proof of this fact in [10, Section 8] used the corresponding fact for restriction semigroups (see the comments following Result 6.4). We include a direct proof here, since the corollary that follows will play a key role in the last two sections.

Result 4.11. $\mathbf{B} = \mathbf{B_2} \vee \mathbf{M}$ *and* $F\mathbf{B} = F\mathbf{B_2} \vee F\mathbf{M}$.

Proof. By Proposition 4.1, it suffices to show that if $S \in \mathbf{B}$, then $S/\mu \in \mathbf{B_2}$. We apply Results 4.6 and 4.8. Let $a, b \in S$ and suppose both that $(a\mu)^+ = (b\mu)^+$ and that $a\mu$ and $b\mu$ share a common right identity $f\mu$, where $f \in P_S$. Since μ separates projections, $(af)^+ = a^+ = b^+ = (bf)^+$, so $af = a$ and $bf = b$ (see Section 3). Now given any $e \in P_S$, $ae = (ae)^+a \,\mathbb{R}\, (ea)^+b$, and these two elements share the right identity f. By Result 4.6, from $(ae)e = ae$ it follows that $(ea)^+b = (ea)^+be$. Thus $(ea)^+ \leq (be)^+$ and, by symmetry, equality holds. Thus $a\mu = b\mu$ and so $S/\mu \in \mathbf{B_2}$.

The statement on pseudovarieties is immediate from Proposition 4.1. □

Corollary 4.12. *Any finitely generated strict left restriction semigroup has only finitely many projections.*

Proof. By Result 4.11 and Proposition 4.1, for any such semigroup there is a projection-separating homomorphism onto a member of the locally finite variety generated by B_2. □

Result 4.13 [10, Corollary 8.13, Theorem 7.5]. *Let* S *be a left restriction semigroup. Then*

(1) $S \in \mathbf{B_0} \vee \mathbf{M}$ *if and only if* $S \in \mathbf{B}$ *and (a) for all* $x, y \in S$, xyx *belongs to a submonoid of* S *or, equivalently, (b) if* $x, y \in S$, $yx^+ = y$ *and* $xy^+ = x$, *then* $x^+ = y^+$;

(2) $S \in \mathbf{D} \vee \mathbf{M}$ *if and only if any element* a *with a right identity belongs to the monoid* M_{a^+}.

By Proposition 4.1, $F\mathbf{B_0} \vee F\mathbf{M} = F(\mathbf{B_0} \vee \mathbf{M})$ *and* $F\mathbf{D} \vee F\mathbf{M} = F(\mathbf{D} \vee \mathbf{M})$. *Thus the analogous statements hold for these pseudovarieties.*

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Note that since the element $a \in B_2$ satisfies $a = aba$ but is not in a submonoid, $B_2 \notin \mathbf{B_0} \vee \mathbf{M}$; regarded as an element of B_0, a has a right identity (namely f) and so, similarly, $B_0 \notin \mathbf{D} \vee \mathbf{M}$ (and likewise in the pseudovarietal case).

In conjunction with Proposition 4.10, it then follows that $\mathcal{L}(\mathbf{B})$ is the disjoint union of $\mathcal{L}(\mathbf{D} \vee \mathbf{M})$ and the interval $[\mathbf{B_0}, \mathbf{B}]$, which in turn is the union of the intervals $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$ and $[\mathbf{B_2}, \mathbf{B_2} \vee \mathbf{M} = \mathbf{B}]$. The analogue is true for pseudovarieties.

It was shown in [10, Theorem 7.8] that the varieties in the interval $[\mathbf{D}, \mathbf{D} \vee \mathbf{M}]$ are those of the form $\mathbf{D} \vee \mathbf{N}$, for some variety of monoids. The proof there (in particular, the use of Proposition 7.3) respects finiteness. In combination with Result 4.2, this yields the following straightforward description of the ideal $\mathcal{L}(\mathbf{D} \vee \mathbf{M})$ (refer to Figure 1).

Result 4.14 [10, Corollary 7.9]. *The sublattice $\mathcal{L}(\mathbf{D} \vee \mathbf{M})$ is isomorphic to the direct product of the lattice $\mathcal{L}(\mathbf{D}) = \{\mathbf{T} \prec \mathbf{S} \prec \mathbf{D}\}$ and the lattice $\mathcal{L}(\mathbf{M})$, under the map $\mathbf{V} \mapsto (\mathbf{V} \cap \mathbf{D}) \vee (\mathbf{V} \cap \mathbf{M})$. The analogue holds for the lattice of subpseudovarieties of $F(\mathbf{D} \vee \mathbf{M})$.*

5. Primitive and strict restriction semigroups

The material in this section is from [7, 9]. As noted in Section 3, a restriction semigroup is strict if it is a subdirect product of primitive restriction semigroups; equivalently, if it is a subdirect product of monoids and *completely 0-r-simple* restriction semigroups. Here we shall instead term the latter semigroups *connected*, based on the connection with categories that will soon be elucidated.

A *restriction ideal* (r-ideal in [7, 9]) of a restriction semigroup S is an ideal that is closed under both unary operations. In the terminology of [7, 9], a restriction semigroup S with zero is *0-r-simple* if $\{0\}$ and S are its only restriction ideals and *completely 0-r-simple* (here, *connected*) if, in addition, it is primitive. It is connected if and only if it is primitive and its nonzero elements form a single $\mathbb{D}$-class. Again motivated by the connection with categories, if it is, further, the case that for any nonzero projections e, f of S, $(e, f) \in \mathbb{R} \circ \mathbb{L}$ (and so likewise for any two nonzero elements a, b of S), we shall call S *strongly connected*.

The basic computational tool is as follows. Note that if such a semigroup has only one nonzero projection, then it is a monoid with adjoined zero, and so a semilattice of monoids.

Result 5.1 [9, Lemma 2.1]. *(See Result 3.1.) If a and b are nonzero elements of a connected restriction semigroup S, then $ab \neq 0$ if and only if $a^* = b^+$, in which case $(ab)^+ = a^+$ and $(ab)^* = b^*$.*

Various characterizations of strict restriction semigroups were given in [7, Theorem 8.1], including the varietal one stated in Result 6.2 below and another by identities. Here we shall need the following structural one (see the characterization of strict inverse semigroups [12] in terms of $\mathcal{D}$).

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RESULT 5.2. *A restriction semigroup is strict if and only if it satisfies* $\mathbb{D}$*-majorization: whenever* f, g, h *are projections,* $f > g, h$ *and* $g \mathbb{D} h$*, then* $g = h$.

We also need to refer to a decomposition of strict restriction semigroups, for which a little further background from [7, Section 5] is needed. If S is a restriction semigroup and $a \in S$, the principal restriction ideal generated by a is denoted $rI(a)$. Let $\mathbb{J} = \{(a, b) : rI(a) = rI(b)\}$. The set $rQ(a) = rI(a) \backslash \mathbb{J}_a$ is a restriction ideal of $rI(a)$ and the Rees factor semigroup $rI(a)/rQ(a)$ is the *r-principal factor* associated with a. As usual, the r-principal factor may be regarded as $\mathbb{J}_a \cup \{0\}$ (or just $\mathbb{J}_a$ if $rQ(a)$ is empty). Each such factor is 0-r-simple (or r-simple if $rQ(a)$ is empty).

RESULT 5.3 [7, Propositions 6.3, 6.1]. *A strict restriction semigroup is a subdirect product of its r-principal factors. On such a semigroup* $\mathbb{D} = \mathbb{J}$ *and each r-principal factor is either a connected restriction semigroup or a monoid.*

As in the one-sided case, in our context the primitive restriction semigroups are therefore the essential concept. In [9] a correspondence was established between primitive restriction semigroups (including monoids, in that treatment) and (small) categories. The application to varieties will be elucidated in the next section.

A *category* is a (directed) graph C, consisting of a set Obj C of objects (or vertices) and a set Arr C of arrows, where for each $e, f \in$ Obj C, Arr (e, f), or sometimes $C(e, f)$, denotes the *homset* of arrows from e to f. The product of consecutive arrows is defined, associative in the natural way, along with (partial) identity arrows 1_e, $e \in$ Obj C. From this point of view, a monoid may also be regarded as a one-object category. The sets $C(e) = C(e, e)$ are called the *local monoids* of C.

A category is *connected* if its underlying *undirected* graph is connected, and *strongly connected* (*bonded* in [13]) if its underlying *directed* graph is connected. In the terminology of [13], a category is *trivial* if each homset has at most one member. A category is *locally trivial* if its local monoids are trivial. In [9], we termed a category C *anticyclic* if for any distinct $e, f \in$ Obj C, either $C(e, f)$ or $C(f, e)$ is empty. Clearly, if such a category is strongly connected, it must be a monoid.

With each primitive restriction semigroup S (with zero) is associated the category $\mathrm{C}(S)$, defined as follows:

- Obj $\mathrm{C}(S) = P_S \backslash \{0\}$;
- Arr $\mathrm{C}(S) = S \backslash \{0\}$, where Arr (e, f) is the $\mathbb{H}$-class $\mathbb{R}_e \cap \mathbb{L}_f$, if nonempty, and otherwise empty, so that $a \colon a^+ \to a^*$, for any nonzero $a \in S$;
- if $a \in$ Arr (e, f) and $b \in$ Arr (f, g), then the product ab is that in S.

If S is a restriction semigroup that is a monoid, define $\mathrm{C}(S) = S$, now treating S as a category.

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Given a (small) category C, $\mathrm{R}(C)$ is its *consolidation*, as termed by Tilson [13] (but there denoted C_{cd} and expressed in somewhat different terms):

- if C is simply a monoid, $\mathrm{R}(C) = C$, regarded as a restriction semigroup;
- otherwise, $\mathrm{R}(C) = \mathrm{Arr}\, C \cup \{0\}$, using the previously defined products and setting all previously undefined products to zero.

For instance, if C is the category with two objects, two arrows from the first to the second and two identity arrows, then $R(C)$ is the restriction semigroup A_2, which will appear in the next section. Here we may write $A_2 = \{e, a, c, f, 0\}$, where $e, f, 0$ are the projections and $a \,\mathbb{H}\, c$.

It is immediate from the definitions that $\mathrm{R}(\mathrm{C}(S)) = S$ for all primitive restriction semigroups S and monoids, and $\mathrm{C}(\mathrm{R}(C)) = C$ for all categories C.

Result 5.4 [9, Proposition 4.1]. *The mappings* $\mathrm{R}\colon C \mapsto \mathrm{R}(C)$ *and* $\mathrm{C}\colon S \mapsto \mathrm{C}(S)$ *are mutually inverse bijections between the class of (small) categories and the union of the classes of primitive restriction semigroups and of monoids. Under this correspondence:*

(1) *monoids correspond to monoids;*
(2) *connected categories that are not monoids correspond to connected restriction semigroups;*
(3) *strongly connected categories that are not monoids correspond to strongly connected restriction semigroups;*
(4) *locally trivial categories that are not trivial correspond to primitive restriction semigroups with trivial submonoids;*
(5) *anticyclic categories correspond to square-free primitive restriction semigroups (see Result 6.2).*

In fact, the third item in this result was not actually stated in [9], but it follows immediately from the definitions.

6. Varieties of restriction semigroups

After reviewing material parallelling that in Section 4, we outline the application to varieties of the categorical connection made in Section 5.

Let **R** denote the variety of all restriction semigroups. As a general rule we shall use the same letters to denote 'familiar' varieties as we do varieties of left restriction semigroups. So **B** denotes the variety of restriction semigroups generated by Brandt semigroups, and $\mathbf{B_2}$ and $\mathbf{B_0}$ denote the subvarieties generated by B_2 and B_0, respectively. Recall that monoids may be regarded as both left restriction semigroups and restriction semigroups. In fact, the same is true for the variety **SM** of semilattices of monoids, and its subvarieties, since these are precisely the restriction semigroups on which $a^* = a^+$ [6].

On the occasions where confusion would otherwise arise, in particular in Section 7, we shall distinguish two-sided varieties by using the subscript R. So, for example,

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$\mathbf{B}_R$ and $(\mathbf{B_0})_R$ then denote the obvious varieties of restriction semigroups. We shall not make that distinction in the case of subvarieties of **SM**. Similarly, if X is a set of restriction semigroups $\langle X\rangle_R$ will denote the variety of restriction semigroups it generates. In addition, we may use terms such as *LR*-homomorphism (or $^+$-homomorphism), *LR*-variety, *R*-variety, and so on, as necessary.

To conclude this section, we make some elementary observations regarding relationships between varieties of left restriction semigroups and varieties of restriction semigroups, based on the association of every restriction semigroup $S = (S, \cdot, ^+, ^*)$ with its left restriction reduct $(S, \cdot, ^+)$.

If X is a set of restriction semigroups, the notation $\langle X\rangle_{LR}$, introduced in Section 4, will perforce denote the variety of left restriction semigroups generated by the reducts of the members of X. Let $\mathbf{V}$ be a variety of left restriction semigroups. Then $\mathbf{V}^R$ will denote the collection of two-sided restriction semigroups whose reducts belong to $\mathbf{V}$.

Result 6.1 [10, Proposition 4.8]. *Let* $\mathbf{V}$ *be a variety of left restriction semigroups and let* X *be a set of restriction semigroups. Then*

(1) $\mathbf{V}^R$ *is a variety of restriction semigroups and* $\langle \mathbf{V}^R\rangle_{LR} \subseteq \mathbf{V}$;
(2) $\langle X\rangle_R \subseteq (\langle X\rangle_{LR})^R$;
(3) $\langle X\rangle_{LR} = \langle\langle X\rangle_R\rangle_{LR}$;
(4) *as a result,* $\mathbf{B}_R \subset \mathbf{B}^R$ *and* $\mathbf{B} = \langle \mathbf{B}_R\rangle_{LR}$.

Returning now to varieties of restriction semigroups, first note that direct analogues of Proposition 4.1 and Result 4.2 hold in the two-sided case [6, Theorems 3.1, 3.3]. The analogue of Result 4.5 also holds in the two-sided case [7, Proposition 8.3].

For simplicity's sake, we do not use the subscript notation in the remainder of the section, since only varieties of restriction semigroups are considered.

Result 6.2 [7, Theorems 8.1, 10.6, 9.3 and 10.3]. *A restriction semigroup* S *belongs to the variety* $\mathbf{B}$ *if and only if it is a subdirect product of monoids and primitive restriction semigroups, that is, it is strict. In that event:*

(1) $S \in \mathbf{B_2}$ *if and only if* $\mathbb{H}$ *is the identical relation;*
(2) $S \in \mathbf{B_0}$ *if and only if* $S \in \mathbf{B_2}$ *and, further, the only regular elements of* S *are the projections;*
(3) $S \in \mathbf{B_0} \vee \mathbf{M}$ *if and only if for distinct projections* e *and* f *of* S, *both* $\mathbb{R}_e \cap \mathbb{L}_f$ *and* $\mathbb{L}_e \cap \mathbb{R}_f$ *cannot be nonempty (*S *is 'square-free').*

Corollary 6.3. *Any subvariety of* $\mathbf{B}$ *is generated by its connected members, together with its monoids.*

Result 6.4 [7, Corollaries 10.2, 9.4] (See Result 4.10). *In the lattice* $\mathcal{L}(\mathbf{R})$, *the sublattice* $\mathcal{L}(\mathbf{B_2})$ *comprises the chain* $\mathbf{T} \prec \mathbf{S} \prec \mathbf{B_0} \prec \mathbf{B_2}$.

Therefore $\mathcal{L}(\mathbf{B})$ *is the disjoint union of the interval sublattices* $[\mathbf{T}, \mathbf{M}]$, $[\mathbf{S}, \mathbf{S} \vee \mathbf{M}]$, $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$ *and* $[\mathbf{B_2}, \mathbf{B_2} \vee \mathbf{M}]$, *where* $\mathbf{S} \vee \mathbf{M} = \mathbf{SM}$ *and* $\mathbf{B_2} \vee \mathbf{M} = \mathbf{B}$.

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The proof that $\mathbf{B} = \mathbf{B_2} \vee \mathbf{M}$ is analogous to that for the one-sided case in Result 4.11. Before proving the next corollary, we need a general well-known result.

LEMMA 6.5. *Let S be a restriction semigroup and X a set that generates S as such. Then $S = P_S T^1$, where T is the plain subsemigroup generated by X, and S is therefore generated as a semigroup by $P_S \cup X$.*

PROOF. Since the subset $P_S T^1$ contains each $x = x^+ x$ in X, it suffices to show that it is a restriction subsemigroup of S. Since it contains P_S, it is closed under the unary operations. Let $e, f \in P_S$ and $u, v \in T^1$. If $u = 1$, there is nothing to prove, so assume otherwise. Applying the left ample identity, $(eu)(fv) = e(uf)v = e(uf)^+ uv \in P_S T^1$. □

COROLLARY 6.6. *Any finitely generated strict restriction semigroup has only finitely many projections and, as a result, is finitely generated as a left restriction semigroup.*

PROOF. The proof of the first statement proceeds analogously to that of Corollary 4.12. As a result, by the previous lemma the semigroup is then finitely generated as a (plain) semigroup. □

In the rest of this section, we review from [9] the application of Result 5.4 to varieties of strict restriction semigroups and use it to prove some results needed in the next section. The precise nature of the universal algebraic treatment of varieties is not needed, so we refer the reader to [9] (or to [13]). The variety of all categories is denoted by **Cat** and its lattice of subvarieties by $\mathcal{L}(\mathbf{Cat})$; the trivial categories form a variety **I** that is the minimum member of this lattice.

For any variety **W** of categories, let $\mathcal{R}(\mathbf{W})$ be the variety of strict restriction semigroups generated by $\{\mathrm{R}(C) : C \in \mathbf{W}\}$. In view of Results 6.2 and 5.4, $\mathcal{R}(\mathbf{Cat}) = \mathbf{B}$. Further, $\mathcal{R}(\mathbf{I}) = \mathbf{B_2}$, since the latter variety consists precisely of the strict restriction semigroups on which $\mathbb{H}$ is the identical relation, according to [7, Theorem 9.3], and is generated by B_2 itself.

Conversely, for any variety **V** of strict restriction semigroups that contains $\mathbf{B_2}$, let $\mathcal{C}(\mathbf{V}) = \{\mathrm{C}(S) : S \text{ is a primitive member of } \mathbf{V}\}$. Using Result 5.4, $\mathcal{C}(\mathbf{V}) = \{C : \mathrm{R}(C) \in \mathbf{V}\}$ and so, by [9, Proposition 5.1], it is a variety of categories.

RESULT 6.7 [9, Theorem 5.2]. *The mappings $\mathcal{R}$ and $\mathcal{C}$ are mutually inverse isomorphisms between the lattice $\mathcal{L}(\mathbf{Cat})$ and the interval $[\mathbf{B_2}, \mathbf{B}]$.*

This correspondence was first used to prove the following useful result regarding the lattice of varieties of strict restriction semigroups. The restriction semigroup A_2 was defined prior to Result 5.4. By definition, it belongs to **B**, in fact to $\mathbf{B_0} \vee \mathbf{M}$, by an application of Result 6.2(3), however, since $\mathbb{H}$ is nontrivial, it does not belong to $\mathbf{B_2}$.

RESULT 6.8 [9, Theorems 5.4, 5.6]. *In the lattice $\mathcal{L}(\mathbf{B})$, the unique cover of $\mathbf{B_2}$ is $\mathbf{B} \cap \mathbf{mon}(\mathbf{T})$, generated by A_2 and B_2. Likewise, in the interval $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$, the unique cover of $\mathbf{B_0}$ is $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})$, generated by A_2.*

A powerful theorem of Tilson is the 'bonded component theorem'.

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Result 6.9 [13, Theorem 11.3]. *Let C be a category that is not locally trivial. Then C divides a product of its strongly connected components (and divides a finite product if it is itself finite). Thus any such variety is generated by its strongly connected members.*

Proposition 6.10. *Let S be a connected restriction semigroup having more than one nonzero projection and at least one nontrivial submonoid. Let the strongly connected components of the category $C(S)$ be C_i, $i \in I$. For each $i \in I$, $R(C_i)$ is (isomorphic to) either a strongly connected restriction subsemigroup, or a submonoid, of S itself. Then S divides the product of $\prod_{i\in I} R(C_i)$ with a semigroup in $\mathbf{B_2}$. If $S \in \mathbf{B_0} \vee \mathbf{M}$, then the latter semigroup belongs to $\mathbf{B_0}$.*

Proof. The vertex set of the component C_i is a subset P_i of the nonzero projections of S. The semigroup $R(C_i)$ may be regarded as a subsemigroup of S, consisting of the union of the $\mathbb{H}$-classes $\mathbb{R}_e \cap \mathbb{L}_f$, $e, f \in P_i$, together with the zero of S. By assumption, each $\mathbb{H}$-class is nonempty. The fact that $R(C_i)$ is then a strongly connected restriction subsemigroup of S is immediate from the properties of the operations given by Result 5.1.

The assumption on S is equivalent to $C(S)$ not being locally trivial, by Result 5.4(4). Thus, by Result 6.9, $C(S)$ divides $\prod_{i\in I} C_i$. According to [9, Lemma 4.5], $S = R(C(S))$ divides the product of $R(\prod_{i\in I} C_i)$ with a restriction semigroup that belongs to $\mathbf{B_2}$ and, further, belongs to $\mathbf{B_0}$ if $C(S)$ is anticyclic, that is, $S \in \mathbf{B_0} \vee \mathbf{M}$ (by the combination of Results 5.4 and 6.2). But according to [9, Lemma 4.8], $R(\prod_{i\in I} C_i)$ divides $\prod_{i\in I} R(C_i)$. □

We now interpret this result in varietal terms. Observe that the first of these results only has a real consequence for those varieties that strictly contain $\mathbf{B} \cap \mathbf{mon}(\mathbf{T})$.

Corollary 6.11. *Any variety of restriction semigroups in the interval $[\mathbf{B_2}, \mathbf{B}]$, other than $\mathbf{B_2}$ itself, is generated by its strongly connected members and its monoids, together with the members of $\mathbf{B} \cap \mathbf{mon}(\mathbf{T})$.*

Proof. Let $\mathbf{V}$ be such a variety. By Corollary 6.3 it is generated by its connected restriction semigroups and monoids. If $\mathbf{V} = \mathbf{B} \cap \mathbf{mon}(\mathbf{T})$, then the result holds trivially. Otherwise, $\mathbf{V}$ contains this variety, by Result 6.8. If a connected member has a single nonzero projection, it is a semilattice of monoids and so, by the two-sided version of Result 4.2, it belongs to $\mathbf{S} \vee (\mathbf{V} \cap \mathbf{M})$, where $\mathbf{S} \subset \mathbf{B_2} \subset \mathbf{B} \cap \mathbf{mon}(\mathbf{T})$. For any connected member of $\mathbf{V}$ that does not belong to $\mathbf{V} = \mathbf{B} \cap \mathbf{mon}(\mathbf{T})$ and has more than one nonzero projection, Proposition 6.10 applies, so this term belongs to the variety generated by the strongly connected members of $\mathbf{V}$. □

Corollary 6.12 [9, Theorem 5.5]. *Any variety of restriction semigroups in the interval $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$, other than $\mathbf{B_0}$ itself, is generated by its monoids, together with the members of $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})$.*

Thus every such variety $\mathbf{V}$ in the interval $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$, other than $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})$, has the form $\mathbf{B_0} \vee (\mathbf{V} \cap \mathbf{M})$. As a result, for any nontrivial variety $\mathbf{N}$ of monoids, $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{N}) = \mathbf{B_0} \vee \mathbf{N}$.

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Proof. The proof of the first statement follows that of the previous corollary, with the critical distinction that, by Result 6.2, subvarieties of $\mathbf{B_0} \vee \mathbf{M}$ contain *no* strongly connected members other than monoids.

Turning to the first statement in the second paragraph, let $\mathbf{V} \in [\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$. Clearly $\mathbf{B_0} \vee (\mathbf{V} \cap \mathbf{M}) \subseteq \mathbf{V}$ always holds and the opposite inclusion certainly holds for $\mathbf{B_0}$, since $\mathbf{B_0} \cap \mathbf{M} = \mathbf{T}$. Otherwise, by the first part of this corollary, $\mathbf{V} = ((\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})) \vee (\mathbf{V} \cap \mathbf{M})$. But if $\mathbf{V} \neq (\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})$, then $\mathbf{V} \cap \mathbf{M} \neq \mathbf{T}$ and, by Result 6.8, $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T}) \subset \mathbf{B_0} \vee (\mathbf{V} \cap \mathbf{M})$, yielding the opposite inclusion in the desired equality.

The final statement now follows immediately, since $((\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{N})) \cap \mathbf{M} = \mathbf{N}$. □

The corresponding result does not hold in $[\mathbf{B_2}, \mathbf{B} = \mathbf{B_2} \vee \mathbf{M}]$. In fact, by [9, Theorem 5.3], if $\mathbf{N}$ is a nontrivial variety of monoids, the equation $\mathbf{B} \cap \mathbf{mon}(\mathbf{N}) = \mathbf{B_2} \vee \mathbf{N}$ holds if and only if $\mathbf{N}$ is *local* in the sense of Tilson [13].

7. Comparing the two lattices

In this section, we use the notation established in Section 6 to distinguish between the left- and two-sided cases. Thus when citing results from that section, named varieties must be subscripted with the letter R. Result 6.1 and the notation that precedes it will be used without further comment.

Before stating our main theorem, we should note that the (reducts of) restriction semigroups in $\mathbf{B}$ need not be members of $\mathbf{B}_R$, that is, the inclusion $\mathbf{B}_R \subset \mathbf{B}^R$ in the cited result is strict. For instance, for $k \geq 2$ the restriction semigroups Λ_k of [7] are not strict, when regarded as such (the key to showing there that $\mathbf{B}_R$ is nonfinitely based), but are strict when regarded as left restriction semigroups [10, Proposition 8.15]. In fact that example demonstrates that $(\mathbf{B}_0)^R \neq (\mathbf{B}_0)_R$.

Theorem 7.1. *The maps* $\mathbf{V} \to \mathbf{V}^R \cap \mathbf{B}_R$ *and* $\mathbf{W} \to \langle \mathbf{W} \rangle_{LR}$ *are mutually inverse order isomorphisms between the interval* $[\mathbf{B_0}, \mathbf{B}]$ *in the lattice of varieties of left restriction semigroups and the corresponding interval* $[(\mathbf{B_0})_R, \mathbf{B}_R]$ *in the lattice of varieties of restriction semigroups.*

Under the first isomorphism, for each monoid variety $\mathbf{N}$ *the interval* $[\mathbf{B_0} \vee \mathbf{N}, \mathbf{B} \cap \mathbf{mon}(\mathbf{N})]$ *in the lattice of LR-varieties maps to the corresponding interval in the lattice of R-varieties.*

Before considering the general case, we treat the extremities, and then the atoms, of the two subintervals $[\mathbf{B_2}, \mathbf{B}]$ and $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$ that make up $[\mathbf{B_0}, \mathbf{B}]$.

Lemma 7.2. *The following equalities hold:*

(a) $\mathbf{B_2}^R \cap \mathbf{B}_R = (\mathbf{B_2})_R$; $\mathbf{B_0}^R \cap \mathbf{B}_R = (\mathbf{B_0})_R$; $(\mathbf{B_0} \vee \mathbf{M})^R \cap \mathbf{B}_R = (\mathbf{B_0})_R \vee \mathbf{M}$; *and* $\mathbf{B}^R \cap \mathbf{B}_R = \mathbf{B}_R$;

(b) $\langle (\mathbf{B_2})_R \rangle_{LR} = \mathbf{B_2}$; $\langle (\mathbf{B_0})_R \rangle_{LR} = \mathbf{B_0}$; $\langle \mathbf{B}_R \rangle_{LR} = \mathbf{B}$; $\langle (\mathbf{B_0})_R \vee \mathbf{M} \rangle_{LR} = \mathbf{B_0} \vee \mathbf{M}$.

https://doi.org/10.1017/S1446788718000046 Published online by Cambridge University Press

Proof. In each case of (a), the right-hand side is clearly contained in the left. Now suppose $S \in \mathbf{B_2}^R \cap \mathbf{B}_R$. We apply Results 4.8(1) and 6.2(1). Suppose $a \mathbb{H} b$. Then $a \mathbb{R} b$ in both S and its reduct, and a and b have the common right identity $a^* = b^*$, so by the former cited result, $a = b$. Therefore, by the latter, $S \in (\mathbf{B_2})_R$. The second case follows similarly, using the second parts of the cited results.

Next let $S \in (\mathbf{B_0} \vee \mathbf{M})^R \cap \mathbf{B}_R$. We apply Results 4.13(1) and 6.2(3). Let $e, f \in P_S$ and suppose both $\mathbb{R}_e \cap \mathbb{L}_f$ and $\mathbb{R}_f \cap \mathbb{L}_e$ are nonempty, containing respectively x and y, say. Then $xy^+ = xx^* = x$ and $yx^+ = yy^* = y$. Applying the former cited result in the reduct, $e = f$. Therefore, by the latter, $S \in (\mathbf{B_0})_R \vee \mathbf{M}$. The final case of (a) is immediate.

The third equality in (b) was part of Result 6.1(4), which itself results from applying the earlier parts of that result, with X consisting of B_2 together with all monoids. The other equalities in (b) follow likewise, with X consisting respectively of B_2, B_0, and B_0 together with all monoids. □

The mappings in Theorem 7.1, being order-preserving, therefore respect the subintervals $[\mathbf{B_2}, \mathbf{B}]$ and $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$.

The varieties with trivial submonoids need some special treatment, leading to the analogue of Result 6.8, part of which is as follows. Recall that the restriction semigroup A_2 that appeared in that result belongs to $(\mathbf{B_0})_R \vee \mathbf{M}$ but not to $(\mathbf{B_2})_R$. Its left restriction reduct belongs to $\mathbf{B_0} \vee \mathbf{M}$ but not to $\mathbf{B_2}$ (by the very first equation in Lemma 7.2).

Lemma 7.3. *In the lattice $\mathcal{L}(\mathbf{B})$, the unique cover of $\mathbf{B_2}$ is $\langle A_2, B_2 \rangle_{LR}$. Likewise the unique cover of $\mathbf{B_0}$ in the interval $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$ is $\langle A_2 \rangle_{LR}$.*

Proof. First let $\mathbf{V}$ be any LR-variety in the interval $[\mathbf{B_0}, \mathbf{B}]$ that is not contained in $\mathbf{B_2}$ itself. By Result 4.6, $\mathbf{V}$ contains a primitive left restriction semigroup S, with base e, say, that does not belong to $\mathbf{B_2}$ and, applying Result 4.8(1), distinct members a, c of $\mathbb{R}_e$, with a common right identity f, say. Then $\{e, a, c, f, 0\}$ is a (left) restriction subsemigroup isomorphic to A_2. Thus $A_2 \in \mathbf{V}$ and the second statement of the lemma holds. If $\mathbf{V}$ contains B_2, then the first also holds. □

The next preliminary lemma is a special case of Theorem 7.6. Recall that the restriction semigroup S^* of Result 3.2 is obtained from S by the adjunction of a new projection h, and that $h = a^*$ for all $a \notin S^{RI}$. Since $hS^* = \{h, 0\}$, $\mathbb{R}_h = \{h\}$. Since S^* is primitive, it follows from Result 6.2 that it belongs to $\mathbf{B}_R$.

Lemma 7.4. *Let S be a primitive left restriction semigroup, with base e, that belongs to $\mathbf{B_0} \vee \mathbf{M}$. Then the restriction semigroup S^* of Result 3.2 belongs to $(\mathbf{B_0} \vee \mathbf{M})^R$. Hence for any variety $\mathbf{N}$ of monoids, the corresponding statement holds for $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{N})$ and for $\mathbf{B} \cap \mathbf{mon}(\mathbf{N})$.*

Proof. If $S^* = S$, there is nothing to prove. Otherwise, $S \neq S^{RI}$ and the only new element in S^* is the projection h. We apply the first characterization in Result 4.13(1): xyx belongs to a submonoid for all $x, y \in S$, and we must prove the same holds in S^*.

https://doi.org/10.1017/S1446788718000046 Published online by Cambridge University Press

Now, by construction, h is a left zero for all but itself, so the only new case of xyx that need be considered is where $x = y = h$, in which case xyx is the projection h itself. So $S^* \in (\mathbf{B_0} \vee \mathbf{M})^R$.

The final statements are immediate from the observation made in Result 3.2 that no new nontrivial monoids appear in S^* (and that, as noted in [10, Lemma 2.2], the submonoids of a restriction semigroup are the same as those in its reduct). □

Proposition 7.5 (See Result 6.8). *The variety* $\mathbf{B} \cap \mathbf{mon}(\mathbf{T}) = \langle A_2, B_2 \rangle_{LR}$. *It is therefore the unique cover of* $\mathbf{B_2}$ *in the interval* $[\mathbf{B_2}, \mathbf{B}]$. *Likewise* $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T}) = \langle A_2 \rangle_{LR}$. *It is therefore the unique cover of* $\mathbf{B_0}$ *in the interval* $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$.

Proof. In view of Lemma 7.3, it remains to prove that, in each case, the variety on the left-hand side is contained in that on the right. First consider $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})$. It is enough to take a primitive left restriction semigroup S, with base e, say. Again consider its embedding in the restriction semigroup S^*, as above. Then $S^* \in (\mathbf{B_0} \vee \mathbf{M})^R \cap \mathbf{B}_R$, by Lemma 7.4, and so $S^* \in (\mathbf{B_0})_R \vee \mathbf{M}$, by the second-to-last case of Lemma 7.2(a). By the second part of Result 6.8, $((\mathbf{B_0})_R \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T}) = \langle A_2 \rangle_R$. By Result 6.1(3), $S \in \langle A_2 \rangle_{LR}$ and so the second of the two stated equations holds.

A similar, but more straightforward, argument applies to the first equation. □

The next result is the first major key to proving Theorem 7.1. In [10, Section 8], it was shown that if $S \in \mathbf{B_2}$ or $S \in \mathbf{B_0}$, the conclusion need not hold.

Theorem 7.6. *Let S be a primitive left restriction semigroup, with base e, such that $B_0 \in \langle S \rangle_{LR}$ but $\langle S \rangle_{LR}$ is neither* $\mathbf{B_0}$ *nor* $\mathbf{B_2}$. *Then the restriction semigroup S^* belongs to $(\langle S \rangle_{LR})^R$.*

Proof. The assumption on S implies that $\langle S \rangle_{LR}$ belongs to the half-open interval $(\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$ if $S \in \mathbf{B_0} \vee \mathbf{M}$, or to $(\mathbf{B_2}, \mathbf{B}]$ if not.

Once again we may assume $S \neq S^*$. If the submonoids of S are trivial, then by the assumption the variety $\langle S \rangle_{LR}$ generated by S is either $\mathbf{B} \cap \mathbf{mon}(\mathbf{T})$ or $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})$, applying Proposition 7.5. The conclusion follows from Lemma 7.4.

Otherwise, S contains a nontrivial submonoid and so the same is true of S^*, whether regarded as a left- or a two-sided restriction semigroup. Referring to Proposition 6.10, consider the strongly connected components C_i, $i \in I$, of S^*. Regarded as a restriction semigroup, but also therefore as a left restriction semigroup, S^* then divides the product of the strongly connected restriction semigroups $R(C_i)$, $i \in I$, together with a semigroup T in $(\mathbf{B_2})_R$ that, further, belongs to $(\mathbf{B_0})_R$ if $S \in \mathbf{B_0} \vee \mathbf{M}$. Then by Result 6.1(2), $T \in \mathbf{B_2}^R$ or $T \in \mathbf{B_0}^R$, respectively. By hypothesis, T therefore belongs to $(\langle S \rangle_{LR})^R$ in either case.

But the strongly connected component containing h is simply $\{h\}$ itself and the remaining components are contained in $C(S^{RI})$. It follows that all of the terms in the above product belong to $(\langle S \rangle_{LR})^R$, so the same is true of S^*. □

Since the two mappings in Theorem 7.1 are clearly order-preserving, the next proposition completes half the proof of the first statement of the theorem.

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Corollary 7.7. *Let* $\mathbf{V}$ *be a variety of left restriction semigroups in the interval* $[\mathbf{B_0}, \mathbf{B}]$. *Then* $\mathbf{V} = \langle \mathbf{V}^R \cap \mathbf{B}_R \rangle_{LR}$.

Proof. The cases $\mathbf{V} = \mathbf{B_0}$ and $\mathbf{V} = \mathbf{B_2}$ were treated in Lemma 7.2.

In the general case, one inclusion is obvious. To prove the opposite inclusion, again let $S \in \mathbf{V}$ be primitive, with a specified base. If $S \in \mathbf{B_0}$ or if $S \in \mathbf{B_2} \backslash \mathbf{B_0}$, the previous paragraph applies (noting that in the latter case, $\mathbf{V}$ then contains $\mathbf{B_2}$). Otherwise, if $S \notin \mathbf{D} \vee \mathbf{M}$, then by the comments following Result 4.13, $\mathbf{B_0} \subset \langle S \rangle_{LR}$ and by Theorem 7.6, $S^* \in \mathbf{V}^R \cap \mathbf{B}_R$, so that $S \in \langle \mathbf{V}^R \cap \mathbf{B}_R \rangle_{LR}$. Finally, if $S \in \mathbf{D} \vee \mathbf{M}$, then by Theorem 4.14 and the comments that precede it, $S \in \mathbf{D} \vee \mathbf{N}$, where $\mathbf{N} = \mathbf{V} \cap \mathbf{M} = (\mathbf{V}^R \cap \mathbf{B}_R) \cap \mathbf{M}$. Then $\mathbf{N} \subset \langle \mathbf{V}^R \cap \mathbf{B}_R \rangle_{LR}$ and since $\mathbf{D} \subset \mathbf{B_0}$, the conclusion follows. □

The simple nature of the interval $[(\mathbf{B_0})_R, (\mathbf{B_0})_R \vee \mathbf{M}]$, as stated in Corollary 6.12, allows the interval $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$ to be determined, from which the other half of the proof of Theorem 7.1 will almost immediately follow for *these* subintervals.

Corollary 7.8. *Every variety* $\mathbf{V}$ *in the interval* $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$, *other than* $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})$, *has the form* $\mathbf{B_0} \vee (\mathbf{V} \cap \mathbf{M})$. *As a result, for any nontrivial variety* $\mathbf{N}$ *of monoids,* $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{N}) = \mathbf{B_0} \vee \mathbf{N}$.

Proof. By the last corollary, any variety in $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$ has the form $\langle \mathbf{W} \rangle_{LR}$, for some $\mathbf{W} \in [(\mathbf{B_0})_R, (\mathbf{B_0})_R \vee \mathbf{M}]$. According to Corollary 6.12, the proof breaks into three parts.

If $\mathbf{W} = (\mathbf{B_0})_R$, then $\langle \mathbf{W} \rangle_{LR} = \mathbf{B_0}$, by Lemma 7.2(b).

If $\mathbf{W} = ((\mathbf{B_0})_R \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})$, then $\mathbf{W} = \langle A_2 \rangle_R$, by the second statement of Result 6.8. Thus $\langle \mathbf{W} \rangle_{LR} = \langle A_2 \rangle_{LR} = (\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})$, by Proposition 7.5, the one-sided analogue of that result.

Finally, if $\mathbf{W} = ((\mathbf{B_0})_R \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{N}) = (\mathbf{B_0})_R \vee \mathbf{N}$ for some nontrivial variety $\mathbf{N}$ of monoids, then $\langle \mathbf{W} \rangle_{LR} = \langle (\mathbf{B_0})_R \vee \mathbf{N} \rangle_{LR} = \mathbf{B_0} \vee \mathbf{N}$. Note that $\mathbf{B_0} \vee \mathbf{N}$ is the unique member of $[\mathbf{B_0}, \mathbf{B_0} \vee \mathbf{M}]$ whose intersection with $\mathbf{M}$ is $\mathbf{N}$ (using Proposition 4.1). Thus $(\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{N}) = \mathbf{B_0} \vee \mathbf{N}$. □

As alluded to above, the other half of the proof of the first statement in Theorem 7.1 will be completed in two steps, corresponding to the two subintervals of $[\mathbf{B_0}, \mathbf{B}]$. The first step is now easily accomplished.

Proposition 7.9. *Let* $\mathbf{W}$ *be any variety of restriction semigroups in the interval* $[(\mathbf{B_0})_R, (\mathbf{B_0})_R \vee \mathbf{M}]$. *Then* $\mathbf{W} = (\langle \mathbf{W} \rangle_{LR})^R \cap \mathbf{B}_R$.

Proof. The proof breaks into the same three parts as that of the previous corollary. For each part we use the value of $\langle \mathbf{W} \rangle_{LR}$ computed in that proof.

If $\mathbf{W} = (\mathbf{B_0})_R$, then Lemma 7.2(a) gives the desired outcome. Before considering the remaining cases, observe first that it is immediate from the third case of part (b) of that same lemma that, for any variety $\mathbf{N}$ of monoids, $((\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{N}))^R \cap \mathbf{B}_R = ((\mathbf{B_0})_R \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{N})$. This equation will be applied in each of the following paragraphs.

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If $\mathbf{W} = ((\mathbf{B_0})_R \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T})$, then $(\langle \mathbf{W} \rangle_{LR})^R \cap \mathbf{B}_R = ((\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T}))^R \cap \mathbf{B}_R = ((\mathbf{B_0})_R \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{T}) = \mathbf{W}$.

Finally, if $\mathbf{W} = ((\mathbf{B_0})_R \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{N}) = (\mathbf{B_0})_R \vee \mathbf{N}$ for some nontrivial variety $\mathbf{N}$ of monoids, then $(\langle \mathbf{W} \rangle_{LR})^R \cap \mathbf{B}_R = ((\mathbf{B_0} \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{N}))^R \cap \mathbf{B}_R = ((\mathbf{B_0})_R \vee \mathbf{M}) \cap \mathbf{mon}(\mathbf{N}) = \mathbf{W}$. □

The key to understanding the interval $[\mathbf{B_2}, \mathbf{B}]$ is Proposition 6.10, based on the general form of Tilson's bonded component theorem (the argument above having at its heart the special case of this theorem for anticyclic categories).

Lemma 7.10. *Let S be a strongly connected restriction semigroup, finitely generated as such. If S LR-divides a strict restriction semigroup U, then it belongs to the R-variety generated by U.*

Proof. Let S and U be such semigroups. If S is simply a two-element semilattice, then there exists a two-element subsemilattice of P_U that maps onto it, so assume otherwise. By assumption, there exists a left restriction subsemigroup T of U and a surjective LR-homomorphism ϕ of T upon S. Since $U \in \mathbf{B}_R$, $T \in \langle \mathbf{B}_R \rangle_{LR} = \mathbf{B}$. Since T is a unary subsemigroup of U, the $^+$-operations coincide and therefore the $\mathbb{R}$-relation on T is the restriction of that on U. Moreover, $P_T \subseteq P_U$.

By Corollary 6.6, there is a finite subset X of S that generates it as a left restriction semigroup. For each $x \in X$, let x_0 be a preimage under ϕ in T and $X_0 = \{x_0 : x \in X\}$. Without loss of generality, it may now be assumed that T is LR-generated by X_0 and U is R-generated by X_0. Again by Corollary 6.6 P_U, and so P_T, is finite.

For each nonzero projection e of S, let e' be its least preimage in the semilattice P_T and put $P' = \{e' : e \in P_S, e \neq 0\}$, consisting of incomparable projections that map bijectively onto $P_S \setminus \{0\}$ under ϕ. Let $s \in S$, $s \neq 0$, $s^+ = e$, say, and suppose t is a preimage of s in T under ϕ. Since ϕ is an LR-homomorphism, $t^+\phi = e$ and, by minimality, $e' \leq t^+$, so that $e't \,\mathbb{R}\, e'$ (both in T and in U), where again $(e't)\phi = s$.

Let $D = T \cap \bigcup\{\mathbb{R}_{e'} : e \in P_S, e \neq 0\}$. Thus $D\phi = S \setminus \{0\}$. By replacing each x_0 by $(x^+)'x_0$, we may assume that $X_0 \subset D$. We shall show that D is an entire 'strongly connected' $\mathcal{D}$-class of U.

Let $a \in D$, with $(a\phi)^+ = e$ and $(a\phi)^* = f$. We have established already that $a^+ = e'$ and now we show that $a^* = f'$ in U. *A priori*, a^* need not belong to T. Note that since $(af')\phi = (a\phi)f = a\phi$, $(af')^+ = a^+$ and so $af' = a$. In U, therefore, $a^* \leq f'$. In the case that $e = f$, $e' = f'$ and the equality $ae' = a$ ensures that a belongs to the maximal submonoid with identity e', considered either in T or in U (see the comments following Result 2.1) which is $\mathbb{H}_{e'}$; that is, $a^* = e'$.

If $e \neq f$, then since S is strongly connected, there exists $s \in S$ such that $s^+ = f$ and $s^* = e$. As above, there exists $t \in T$, with $t^+ = f'$ and $te' = t$, such that $t\phi = s$. Thus $(ta)^+ = (ta^+)^+ = (te')^+ = t^+ = f'$. In U, therefore, $f' \,\mathbb{R}\, ta \,\mathbb{L}\, (ta)^* \leq a^* \leq f'$. By Result 5.2, U has $\mathbb{D}$-majorization, whereby $f' = (ta)^*$ and so $f' = a^*$.

Therefore for each $a \in D$, $a^* \in D$ and $a^*\phi = (a\phi)^*$. It follows that for any distinct nonzero projections $e, f \in S$, $(e', f') \in \mathbb{R} \circ \mathbb{L}$. Suppose a projection g of U is $\mathbb{D}$-related in U to a projection in D. Then there is a sequence of elements of U from some $e' \in P'$

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to g that begins either $e' \mathbb{R}\, u \,\mathbb{L}\, h$ or $e' \mathbb{L}\, u \,\mathbb{R}\, h$, for some $u \in U \backslash P_U$, $h \in P_U$. Now, by Lemma 6.5, $u = kt$, where $k \in P_U$ and $t = t_1 \dots t_n$ for some $t_1, \dots, t_n \in X_0$. Here $u^+ \leq k$, so $u = u^+ t$. In the case $e' \mathbb{R}\, u$, $u = e't \in D$ and so $h = u^* \in D$. In the case $e' \mathbb{L}\, u$, $h = u^+ \leq t_1^+ \in P'$. Since $t_1^+ \mathbb{D}\, e' \,\mathbb{D}\, h$, by $\mathbb{D}$-majorization we obtain $h = t_1^+ \in P'$.

By induction, it follows that the subset D of T is an entire $\mathbb{D}$-class, that is, by Result 5.3, a $\mathbb{J}$-class, of U. Referring to Section 5, denote by F the r-principal factor associated with D. Also by Result 5.3, U is an R-subdirect product of its r-principal factors. Thus F belongs to the R-variety generated by U.

We may regard F as $D \cup \{0\}$. Since U is strict, F is a (strongly) connected restriction semigroup and so, by Result 5.1, if $t_1, t_2 \in D$, then $t_1 t_2 \in D$ if and only if $t_1^* = t_2^+$; similarly, $(t_1\phi)(t_2\phi) \neq 0$ in S if and only if $(t_1\phi)^* = (t_2\phi)^+$, and so if and only if $t_1^* = t_2^+$. Therefore the extension of ϕ to F, mapping 0 to 0, is an R-homomorphism upon S, completing the proof. □

The next result, complemented by Proposition 7.9, completes the proof of the first statement of Theorem 7.1.

Corollary 7.11. *Let* $\mathbf{W}$ *be a variety of restriction semigroups in the interval* $[(\mathbf{B_2})_R, \mathbf{B}_R]$. *Then* $\mathbf{W} = (\langle \mathbf{W} \rangle_{LR})^R \cap \mathbf{B}_R$.

Proof. One inclusion is obvious. The case $\mathbf{W} = (\mathbf{B_2})_R$ follows from Lemma 7.2, so suppose otherwise. If $\mathbf{W} \subset \mathbf{mon(T)}$, then by Result 6.8, $\mathbf{W} = \mathbf{B}_R \cap \mathbf{mon(T)}$. Now the proof of the corresponding result for $[(\mathbf{B_0})_R, (\mathbf{B_0})_R \vee \mathbf{M}]$ in Proposition 7.9 is straightforwardly modified, using instead the first part of Proposition 7.5.

In the general case $\mathbf{W}$ and, therefore, $(\langle \mathbf{W} \rangle_{LR})^R \cap \mathbf{B}_R$, is not contained in $\mathbf{mon(T)}$. By Proposition 6.10, $(\langle \mathbf{W} \rangle_{LR})^R \cap \mathbf{B}_R$ is generated by its strongly connected members. Let S be such a semigroup. Now S is the direct limit of its finitely generated restriction subsemigroups. While not every such subsemigroup T will be strongly connected, each has only a finite semilattice of projections, by Corollary 6.6, and so can be extended to a finitely generated, strongly connected subsemigroup by adding a new generator, chosen from S, for each pair (e, f) of nonzero projections of T such that $\mathbb{R}_e \cap \mathbb{L}_f$ is empty. Thus in fact $(\langle \mathbf{W} \rangle_{LR})^R \cap \mathbf{B}_R$ is generated by its finitely generated, strongly connected members. If S is such a member, it LR-divides a member of the R-variety $\mathbf{W}$ and so, by Lemma 7.10, it belongs to $\mathbf{W}$, as required. □

Once the first statement of the theorem has been proved, the second follows immediately from the two equations $(\mathbf{B} \cap \mathbf{mon(N)})^R \cap \mathbf{B}_R = \mathbf{B}_R \cap \mathbf{mon(N)}$ and $(((\mathbf{B_0})_R \vee \mathbf{N})_{LR})^R = \mathbf{B_0} \vee \mathbf{N}$.

8. Pseudovarieties of strict left restriction semigroups

This section is the one-sided analogue of [9, Section 6]. It was shown there that the pseudovarietal analogues of the material cited in Section 6 hold in their entirety, since finiteness is preserved by all the relevant processes. Refer to Section 4 for an introduction to the topic of this section and basic connections between pseudovarieties and varieties in this context.

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Recall from the final paragraphs of Section 4 that the lattice $\mathcal{L}(F\mathbf{B})$ of pseudovarieties of strict left restriction semigroups is the disjoint union of the lattice $\mathcal{L}(F(\mathbf{D} \vee \mathbf{M}))$ and the interval $[F\mathbf{B_0}, F\mathbf{B}]$. Result 4.14 decomposes the former as shown in (the pseudovarietal analogue of) Figure 1. The second is the disjoint union of the intervals $[F\mathbf{B_0}, F\mathbf{B_0} \vee F\mathbf{M}]$ and $[F\mathbf{B_2}, F\mathbf{B_2} \vee F\mathbf{M} = F\mathbf{B}]$.

It remains to show that the pseudovarietal analogue of Theorem 7.1 holds. While this could be achieved by demonstrating that finiteness is preserved throughout the proofs in Section 7, it can in fact be deduced from the varietal theorem by a combination of general arguments and the finiteness arguments in Corollaries 4.12 and 6.6, as we now show.

The analogue of the second part of Theorem 7.1 follows similarly and so we do not bother to state it here. Note that throughout this section the subscript LR refers to the pseudovariety of left restriction semigroups so generated.

Theorem 8.1. *The maps* $\mathbf{V} \to \mathbf{V}^R \cap F\mathbf{B}_R$ *and* $\mathbf{W} \to \langle \mathbf{W} \rangle_{LR}$ *are mutually inverse order isomorphisms between the interval* $[F\mathbf{B_0}, F\mathbf{B}]$ *in the lattice of pseudovarieties of left restriction semigroups and the corresponding interval* $[F(\mathbf{B_0})_R, F\mathbf{B}_R]$ *in the lattice of pseudovarieties of restriction semigroups.*

Proof. That the maps take their respective domains into each other follows from the pseudovarietal analogue of Lemma 7.2, which is immediate from the results of Section 4.

We next show the analogue of Corollary 7.7: that if $\mathbf{V}$ is a pseudovariety of left restriction semigroups in the interval $[F\mathbf{B_0}, F\mathbf{B}]$, then $\mathbf{V} = \langle \mathbf{V}^R \cap F\mathbf{B}_R \rangle_{LR}$. One inclusion is clear.

Let $S \in \mathbf{V}$ and let $\mathbf{U}$ be the *variety* generated by S. It may be assumed that $B_0 \in \mathbf{U}$, for if not we may replace S by $S \times B_0 \in \mathbf{V}$. Since S is finite, $\mathbf{U}$ is locally finite and $F\mathbf{U}$ is the pseudovariety generated by S. Applying Corollary 7.7, S belongs to the *variety* of left restriction semigroups generated by $\mathbf{U}^R \cap \mathbf{B}_R$ and so there exists a strict restriction semigroup U in $\mathbf{U}^R$, an LR-subsemigroup T and an LR-homomorphism of T upon S. Now T may clearly be assumed finitely generated as a left restriction semigroup; and we may take for U the restriction semigroup generated as such by the generators of T.

By Corollary 6.6, U is also finitely generated as a left restriction semigroup, so by local finiteness of $\mathbf{U}$, T is finite. Thus $S \in \langle F\mathbf{U}^R \cap F\mathbf{B}_R \rangle_{LR} = \langle \mathbf{V}^R \cap F\mathbf{B}_R \rangle_{LR}$.

It remains to prove the combined analogues of Proposition 7.9 and Corollary 7.11: if $\mathbf{W}$ is a pseudovariety of restriction semigroups in the interval $[F(\mathbf{B_0})_R, F\mathbf{B}_R]$, then $\mathbf{W} = \langle \mathbf{W}^R \rangle_{LR} \cap F\mathbf{B}_R$.

Again, one direction is clear. Let $S \in \langle \mathbf{W}^R \rangle_{LR} \cap F\mathbf{B}_R$. Thus there exists a finite strict restriction semigroup $U \in \mathbf{W}$, an LR-subsemigroup T and an LR-homomorphism of T upon S. Let $\mathbf{U}$ be the *variety* of restriction semigroups generated by U. It may be assumed that $B_0 \in \mathbf{U}$, for, similarly to the above, if not we may replace U by the finite restriction semigroup $U \times B_0 \in \mathbf{W}$. Now S belongs to the LR-variety generated by $\mathbf{U}$ and so, applying Proposition 7.9 and Corollary 7.11, $S \in \mathbf{U}$. But $\mathbf{U}$ is locally

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finite and so S belongs to the pseudovariety generated by U, which is included in $\mathbf{W}$, as required. □

Therefore the diagram of the lattice of pseudovarieties of strict left restriction semigroups is again that in Figure 1, subject to suitably modified notation. Finally it may be noted that all the relevant category-theoretic theorems from [13] respect finiteness, yielding from the following a description of the interval $[F\mathbf{B_2}, F\mathbf{B}]$ in the lattice of pseudovarieties of finite *left* strict restriction semigroups.

Result 8.2 [9, Theorem 6.4]. *The analogues of the isomorphisms $\mathcal{R}$ and C in Theorem 6.7 are mutually inverse isomorphisms between the lattice of pseudovarieties of finite categories and the interval $[F(\mathbf{B_2})_R, F\mathbf{B}_R]$ in the lattice of pseudovarieties of finite strict restriction semigroups.*

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