N-ARY TRANSFORMATIONS OF SEQUENCES

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1. Let $T(a_1, a_2, \ldots, a_{n-1})$ be the n-ary transformation which takes the sequence $\{s_i\}$, $(i=0, 1, \ldots)$, into the sequence $\{s_i'\}$ where

$$s_i' = \sum_{r=1}^{n} a_r s_{i+1-r}, \quad (i=0, 1, 2, \ldots),$$

with $s_m=0$ when m is a negative integer, and where $a_1, \ldots, a_n$ are real numbers with sum unity. In a previous note (**1**) conditions were found on a and β for the ternary transformation $T(a, \beta)$ to be equivalent to convergence. A method is given here for treating the similar problem for the general n-ary transformation.

2. It is clear that, in the notation of (**1**), the $T(a_1, a_2, \ldots, a_{n-1})$ transformation is a Nörlund transformation with

$$M=n-1, \quad p_r=\begin{cases} a_{r+1} (r=0 \text{ to } n-1), \\ 0 \quad (r \geqslant n), \end{cases}$$

$$P_r=1 (r \geqslant n), \quad p(x)=\sum_{r=1}^{n} a_r x^{r-1}.$$

Then, as in the particular case $n=3$, the n-ary transformation is equivalent to convergence if and only if all the zeros of

$$\phi(x)=\sum_{r=1}^{n} a_r x^{n-r}$$

lie in the region $|x|<1$. The set of points in the $(a_1, \ldots, a_{n-1})$-space for which this is the case may easily be determined as shown in the next paragraph by using the following results quoted from (**2**):

Exercise 10.2. A polynomial $g(z)=z^n+g_1 z^{n-1}+\ldots+g_n$, with real coefficients, has all its roots in the half-plane $Rl\, z<0$ if and only if the determinants

$$H_k=\begin{vmatrix} g_1 & g_3 & g_5 & \cdots & g_{2k-1} \\ 1 & g_2 & g_4 & \cdots & g_{2k-2} \\ 0 & g_1 & g_3 & \cdots & g_{2k-3} \\ 0 & 1 & g_2 & \cdots & g_{2k-4} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & g_k \end{vmatrix}, \quad \begin{array}{l} k=1, 2, \ldots, n, \\ (g_r=0 \text{ if } r>n) \end{array}$$

are all positive.

Exercise 10.3. The transformation

$$z=r\frac{1+w}{1-w}, \quad r>0,$$

maps the half-plane $Rl\, w<0$ into the circular region $|z|<r$.

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3. The Quaternary Transformation $T(\alpha,\beta,\gamma)$.

Let S denote the set of points (α, β, γ) for which the $T(\alpha, \beta, \gamma)$ transformation is equivalent to convergence.

When $\alpha=0$, $T(\alpha, \beta, \gamma)$ reduces to the ternary transformation $T(\beta, \gamma)$ and the results of (1) show then that the portion of S lying in the plane $\alpha=0$ is made up of the point (0, 0, 0), the segment $\gamma>\frac{1}{2}$ of the γ-axis, and the region for which both $2\beta+\gamma>1$ and $\gamma<\frac{1}{2}$.

If $\alpha\neq 0$ then, on putting

$$x=\frac{1+w}{1-w},$$

the equation $\phi(x)=0$ becomes $(2\alpha+2\gamma-1)f(w)=0$ where

$$f(w)=w^3+\frac{3-4\beta-4\gamma}{2\alpha+2\gamma-1}w^2+\frac{6\alpha+4\beta+2\gamma-3}{2\alpha+2\gamma-1}w+\frac{1}{2\alpha+2\gamma-1}.$$

When $2\alpha+2\gamma=1$ the equation $\phi(x)=0$ has a root $x=-1$. Hence, by exercise 10.3, we require $f(w)=0$ to have all its roots in the half-plane $R1\,w<0$. The necessary and sufficient conditions

$$g_1>0, \quad \begin{vmatrix} g_1 & g_3 \\ 1 & g_2 \end{vmatrix}>0, \quad \text{and} \quad \begin{vmatrix} g_1 & g_3 & 0 \\ 1 & g_2 & 0 \\ 0 & g_1 & g_3 \end{vmatrix}>0$$

or

$$g_3>0, \quad g_1>0 \quad \text{and} \quad g_1g_2-g_3>0$$

of exercise 10.2 then become

$$\alpha+\gamma>\tfrac{1}{2}, \quad \beta+\gamma<\tfrac{3}{4} \quad \text{and}$$
$$2\beta^2+\gamma^2+3\alpha\beta+3\beta\gamma+3\gamma\alpha-2\alpha-3\beta-2\gamma+1<0.$$

This last inequality may be written

$$(\sqrt{7}+\tfrac{3}{2})\left\{x'-\frac{28+13\sqrt{7}}{19\sqrt{(28+6\sqrt{7})}}\right\}^2-(\sqrt{7}-\tfrac{3}{2})\left\{y'-\frac{28-13\sqrt{7}}{19\sqrt{(28-6\sqrt{7})}}\right\}^2 <-\frac{1}{\sqrt{19}}\left\{z'-\frac{3\sqrt{19}}{361}\right\}$$

where

$$x'\sqrt{(28+6\sqrt{7})}=3\alpha+(2+\sqrt{7})\beta+(1+\sqrt{7})\gamma,$$
$$y'\sqrt{(28-6\sqrt{7})}=3\alpha+(2-\sqrt{7})\beta+(1-\sqrt{7})\gamma,$$
$$z'\sqrt{19}=\alpha-3\beta+3\gamma,$$

so that it is satisfied to one side of an hyperbolic paraboloid whose vertex is at

$$\left(\frac{54}{361}, \frac{123}{361}, \frac{124}{361}\right).$$

REFERENCES

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(2) H. S. Wall, *Analytic Theory of Continued Fractions*, New York, 1948.

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