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Migdal–Kadanoff recursion relations

In the chapter on mean field theory, we saw that with large space-time dimensionality, lattice gauge theories should exhibit first-order phase transitions. In contrast, in two dimensions the gauge systems reduce to simple one-dimensional spin chains (problem 1 of chapter 9) which exhibit no thermodynamical singularities. It is of crucial importance to know at what intermediate dimension the phase structure ceases to be non-trivial. Indeed, we want four space-time dimensions to be at or below this critical dimensionality for non-Abelian gauge groups so that we can use strong coupling techniques for the study of the confinement problem.

A similar question arises in spin models, where with large, such as three, dimensions ferromagnetic transitions occur whereas in one dimension long-range order is impossible without long-range forces. In the case of magnetic systems, there are rigorous theorems (Peierls, 1935; Mermin and Wagner, 1966; Coleman, 1973) which severely restrict the possible ordering in two dimensions for theories with a continuous symmetry group. The massless spin waves associated with a ferromagnetic state develop severe infrared singularities which disorder the system. Without a decoupling of the long-wavelength spin wave, which occurs in free field theory and with the $U(1)$ symmetry, these models cannot exhibit a massless phase. This gives us a compelling reason to believe that two is the critical dimensionality for spin systems with nearest-neighbor interactions. Indeed, ferromagnetism should not occur in these models when the symmetry is non-Abelian and the spin waves do not decouple. Unfortunately, we do not have such strong arguments for gauge theories.

From a renormalization group point of view, the phase structure of a theory appears when we compare the system on lattices with different spacing. As discussed in chapter 12, we have a second-order transition at a fixed point where physics does not change when we alter either the scale of measurement or the lattice spacing. The Migdal–Kadanoff recursion relations represent a simple approximate method for comparing theories with different lattice spacings (Migdal, 1975*a*, *b*; Kadanoff, 1976; 1977).

The virtue of this technique is that it provides a simple method for obtaining an approximate renormalization group function. Its primary drawback lies in the difficulty of assessing the severity of the approximations involved. The procedure becomes exact in one or two dimensions for spin or gauge models, respectively. In contrast to mean field methods, the recursion should become less accurate as the dimension is increased. A particularly desirable feature of the method is that it correctly predicts two as the critical dimensionality for magnetic systems.

When applied to gauge theories in d dimensions, this approximate recursion gives precisely the same relation as for a spin model in $d/2$ dimensions. This immediately implies that four is the critical dimensionality for continuous gauge groups. In this sense, the absence of a phase transition in a four-dimensional non-Abelian gauge theory corresponds directly to the absence of ferromagnetism in two dimensions. Indeed, before the application of Monte Carlo methods to gauge systems, this was the strongest evidence for quark confinement in the standard gauge model of the strong interactions.

The d to $d/2$ correspondence between gauge and spin models predicts a similar structure for the $U(1)$ model of electrodynamics and the 'planar' or 'XY' model in two dimensions. In recent years the latter model has been the subject of considerable interest in solid state physics. It exhibits an infinite-order phase transition to a weak coupling phase with correlation functions which fall for large separations as a power of distance. This power is a continuously varying function of the coupling (Kosterlitz and Thouless, 1973). For the gauge theory of electrodynamics, this is consistent with the existence of a massless photon phase, which was mentioned in the last chapter. The renormalized electric charge is expected to be a continuously varying function of the bare coupling.

Although the Migdal–Kadanoff relations appear to correctly predict the critical dimensionality and the existence of at least some transitions, it can misidentify their nature. The Z_2 gauge model in four dimensions is predicted to be similar to the two-dimensional Ising model, whereas the former has a strong first-order transition and the latter, second order. For the $U(1)$ models discussed in the previous paragraph, the gauge model appears to be second order (Lautrup and Nauenberg, 1980*a*; DeGrand and Toussaint, 1980; Bhanot, 1981; Moriarty and Pietarinen, 1982) while the spin transition is of infinite order (Kosterlitz and Thouless, 1973).

We begin our detailed discussion with a demonstration of the technique on a trivial example, the one-dimensional Ising model. We consider a chain of N spins, each from the set $Z_2 = \{1, -1\}$. These interact through a nearest-neighbor coupling and give the partition function

$$Z = \sum_{\{s_i\}} \exp\left(\sum_{i=1}^{N} (\beta_0 + \beta_1 s_i s_{i+1})\right). \tag{17.1}$$

As in the last chapter, we find it convenient to keep the normalization β_0 as a free parameter even if it is irrelevant to the thermodynamic singularities. For simplicity we treat the system as periodic, $s_{N+1} = s_1$. We now go to a transfer matrix formalism and write

$$Z = \mathrm{Tr}\,(T^N). \tag{17.2}$$

Here T is the two-by-two matrix describing the interaction between neighboring spins

$$T = e^{\beta_0}\begin{pmatrix} e^{\beta_1} & e^{-\beta_1} \\ e^{-\beta_1} & e^{\beta_1} \end{pmatrix}. \tag{17.3}$$

In this example, the Migdal–Kadanoff relation merely represents an initial 'decimation' or sum over every other spin. In terms of the transfer matrix, we write (consider N even)

$$Z = \mathrm{Tr}\,(T'^{N/2}), \tag{17.4}$$

where

$$T' = T^2 = e^{\beta_0'}\begin{pmatrix} e^{\beta_1'} & e^{-\beta_1'} \\ e^{-\beta_1'} & e^{\beta_1'} \end{pmatrix}. \tag{17.5}$$

The new couplings are given by

$$\beta_0' = 2\beta_0 + \tfrac{1}{2}\log(4\cosh(2\beta_1)), \tag{17.6}$$

$$\beta_1' = \tfrac{1}{2}\log(\cosh(2\beta_1)). \tag{17.7}$$

Thus the theory with parameters β_0 and β_1 has the same physics as the model on a lattice of twice the spacing but with different couplings β_0' and β_1'.

We can use this relation to take the lattice spacing to zero and obtain a continuum limit. This process requires repeatedly adjusting the couplings so as to maintain a constant correlation length in physical rather than lattice units. The parameter β_0 merely represents a zero point energy which cannot cause thermodynamic singularities. Thus we should concentrate on the physically relevant variable β_1. Equation (17.7) relates β_1 with cutoff a to its value with a lattice spacing of $2a$.

$$\beta_1(2a) = \tfrac{1}{2}\log(\cosh(2\beta_1(a))). \tag{17.8}$$

In a continuum limit β_1 must go to a fixed point of this recursion relation. The two fixed points in this one-dimensional model are at $\beta_1 = 0$ and $\beta_1 = \infty$. The former of these is ultraviolet repulsive and the latter ultraviolet attractive in the sense discussed in chapter 12. This theory is asymptotically free in as much as we must go to infinite β_1 or zero 'temperature' for the continuum limit. This is the same behavior conjectured for the four-dimensional gauge theory of the strong interactions.

For further analysis, it is convenient to diagonalize T and find its eigenvalues. This is done directly via the character expansion of the exponentiated action in terms of the variables b_n of the last chapter.

$$T_{s,s'} = b_0 + b_1 ss', \tag{17.9}$$

where

$$\left.\begin{aligned} b_0 &= e^{\beta_0}\cosh(\beta_1), \\ b_1 &= e^{\beta_0}\sinh(\beta_1). \end{aligned}\right\} \tag{17.10}$$

Up to a factor of two, these variables are the eigenvalues of T. The orthogonality of the characters now implies

$$(T^2)_{ss'} = 2(b_0^2 + b_1^2 ss'). \tag{17.11}$$

The factor of two arises because we have not normalized the sums over spins. Normally we set $\int dg = 1$, but here we have taken $\sum_s 1 = 2$. The recursion relation for the variables b_i is thus a simple power

$$b_i' = 2b_i^2. \tag{17.12}$$

This generalizes to all groups. A decimation is a generalized convolution and becomes simple in the transform space of the characters. If the spins are considered as elements g of some group, we consider the partition function

$$Z = \int \left(\prod_i dg_i\right) \exp\left(\sum_i S_L(g_i g_{i+1}^{-1})\right), \tag{17.13}$$

where S_L is the contribution to the action from a single link. We now expand the nearest-neighbor interaction in characters of the irreducible representations of the group

$$\exp(S_L(g)) = \sum_R b_R \chi_R(g) = \exp\left(\sum_R \beta_R \chi_R(g)\right). \tag{17.14}$$

A decimation in this model will utilize the orthogonality

$$\int dg \chi_R^*(g) \chi_{R'}(gg') = d_R^{-1} \delta_{RR'} \chi_R(g'), \tag{17.15}$$

where d_R is the dimension of the matrices in the representation (problem 3 of chapter 8). This gives

$$b_R' = d_R^{-1} b_R^2. \tag{17.16}$$

We assume for a physical theory that b_R is real and that for every representation its conjugate occurs in eq. (17.14) with an equal coefficient.

Up to this point the recursion relations have been exact. In going to higher dimensions, approximations become necessary. Consider the two-dimensional Ising model with partition function

$$Z = \sum_{\{s_i\}} \exp\left(\sum_{\{ij\}} (\beta_0 + \beta_1 s_i s_j)\right), \tag{17.17}$$

where the variables are again from Z_2 and the sum in the exponential is over all nearest neighbor pairs of sites on an N-by-N two-dimensional square lattice. We would like to begin with a decimation or sum over every other site in the x direction, for example those variables on sites with odd x coordinate. In figure 17.1 we show a portion of the lattice and label by

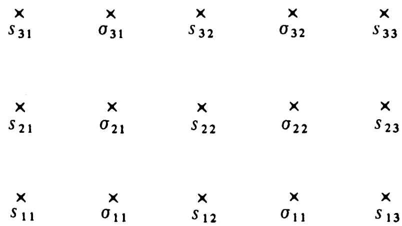

Fig. 17.1. A portion of a two-dimensional lattice. We wish to sum over the spins labeled by σ.

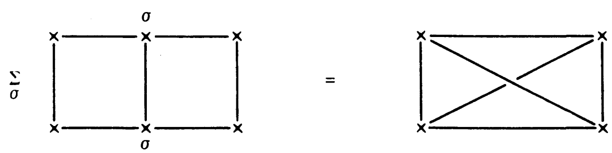

Fig. 17.2. A decimation generates non-nearest-neighbor couplings such as the diagonal bonds illustrated here.

σ those sites we wish to sum over. This should leave a model with action depending only on the remaining sites, labeled by s in the figure. Although in principle this can be done exactly, a complication arises because the new theory will in general involve non-nearest-neighbor interactions. This is schematically shown in figure 17.2. Indeed, the decimation will introduce couplings between spins on adjacent unsummed x rows but arbitrarily separated in the y direction. To keep the recursion relations manageable, some truncation is necessary. The simple Migdal–Kadanoff procedure eliminates these long distance interactions via the trick of bond moving.

The non-local couplings arise because the σ variables are coupled in the y direction. If these troublesome couplings were not present, then the sum over the σ would reproduce the one-dimensional recursion on the x direction bonds. The Migdal–Kadanoff approximation consists of neglecting the y couplings of the σ variables and compensating for them by increasing the strength of the y couplings between the unsummed s

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variables. Effectively the bonds between the σ are moved one site over to give a double strength y bond between sites with even x coordinate. This bond moving is illustrated in figure 17.3. On the decimated lattice the new y coupling becomes

$$\beta_i^y \to 2\beta_i^y, \tag{17.18}$$

where the superscript indicates the direction associated with the coupling. The x bonds now receive the one-dimensional recursion from eqs (17.6) and (17.7)

$$\beta_0^x \to 2\beta_0 + \tfrac{1}{2}\log(4\cosh(2\beta_1^x)), \tag{17.19}$$

$$\beta_1^x \to \tfrac{1}{2}\log(\cosh(2\beta_1^x)). \tag{17.20}$$

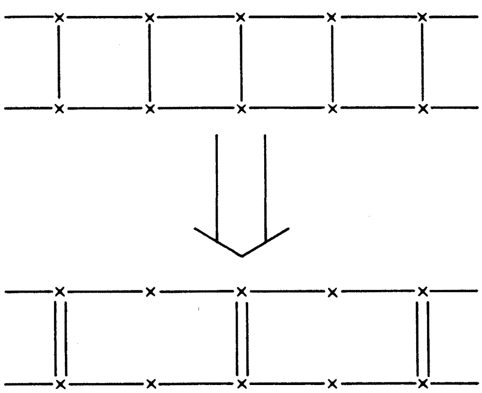

Fig. 17.3. Moving bonds to reduce the decimation to the one-dimensional case.

The next step is to repeat the decimation in the y direction and modify the couplings with x and y interchanged in the above equations. The net result is

$$\beta_0^y \to 4\beta_0^y + \tfrac{1}{2}\log(4\cosh(4\beta_1^y)), \tag{17.21}$$

$$\beta_1^y \to \tfrac{1}{2}\log(\cosh(4\beta_1^y)), \tag{17.22}$$

$$\beta_0^x \to 4\beta_0^x + \log(4\cosh(2\beta_1^x)), \tag{17.23}$$

$$\beta_1^x \to \log(\cosh(2\beta_1^x)). \tag{17.24}$$

Unfortunately the sequential decimation has lost the symmetry of the theory under interchange of the x and y axes. We will repair this momentarily but first note that this recursion does have a fixed point at the asymmetric couplings

$$\beta_1^x = 2\beta_1^y = 0.609\,377\,863\ldots \tag{17.25}$$

To recover the xy symmetry of the model we consider the respective decimations in an infinitesimal manner. Thus, we first perform the change of the lattice spacing in the x direction not by a factor of two, but by a factor of $(1+\Delta)$. The corresponding change in the x couplings is most apparent with the variables b_R, which become raised to the $(1+\Delta)$ power. For the general model, eq. (17.16) is replaced by

$$b'_R = d_R^{-\Delta} b_R^{1+\Delta} = b_R + b_R \log(b_R/d_R)\,\Delta + O(\Delta^2). \qquad (17.26)$$

For the y bond moving we increase the strength of the β_R^y by a factor of $(1+\Delta)$. Keeping only the leading terms in Δ, we repeat the decimation in the y direction. Both the x and y couplings change equally and become

$$\beta'_R = \beta_R + (\beta_R + \sum_{R'} (\partial\beta_R/\partial b_{R'})\, b_{R'} \log(b_{R'}/d_{R'}))\,\Delta. \qquad (17.27)$$

As the change in the lattice spacing is $a\Delta$, we obtain the Migdal–Kadanoff approximation to the renormalization group function

$$a\,\mathrm{d}\beta_R/\mathrm{d}a = \beta_R + \sum_{R'} (\partial\beta_R/\partial b_{R'})\, b_{R'} \log(b_{R'}/d_{R'}). \qquad (17.28)$$

The generalization to d dimensions is immediate; a decimation in any direction requires bond moving in all the $d-1$ orthogonal coordinates. The final formula is

$$a\,\mathrm{d}\beta_R/\mathrm{d}a = (d-1)\,\beta_R + \sum_{R'} (\partial\beta_R/\partial b_{R'})\, b_{R'} \log(b_{R'}/d_{R'}). \qquad (17.29)$$

Applied to the variable β_1 in the Ising model, this reduces to

$$a\,\mathrm{d}\beta_1/\mathrm{d}a = (d-1)\,\beta_1 + \sinh(\beta_1)\cosh(\beta_1)\log(\tanh(\beta_1)). \qquad (17.30)$$

At $d=2$, this has a fixed point at

$$\beta_1 = \tfrac{1}{2}\log(1+2^{\frac{1}{2}}), \qquad (17.31)$$

which remarkably is the exact result, as predicted by duality. As d goes to unity, the fixed point goes to infinity. Using the asymptotic forms for the hyperbolic functions, we obtain for the fixed point

$$\beta_1 = \tfrac{1}{2}(d-1)^{-1} + O(\mathrm{e}^{-2\beta_1}). \qquad (17.32)$$

Thus we say that unity represents the critical dimensionality for the Ising model.

For a model based on a continuous group, the recursion relations predict two as the critical dimensionality. Physically this follows because at weak coupling the exponentiated action strongly peaks near the identity element of the group and approximates a Gaussian in the group parameter space. The decimation in the x direction convolutes these Gaussians, increasing their width by a factor of $(1+\Delta)$. In contrast, the bond moving decreases the widths of the Gaussians on orthogonal bonds by a factor $(1+\Delta)^{-1}$. For precisely two dimensions these operations are done equally on all bonds and the leading effects cancel.

To see this in more detail, it is convenient to introduce a new variable

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which simplifies the form of the recursion relation. We define

$$f_R = \log(b_R/(d_R b_0)) \\ = \log(d_R^{-1}(\partial/\partial\beta_R)\log b_0), \tag{17.33}$$

where b_0 is the b parameter for the singlet representation

$$b_0 = \int \mathrm{d}g \exp(S_L(g)). \tag{17.34}$$

The variable f_R has several nice properties. First, it is readily calculable from b_0 alone. Second, being a function of the ratio of two b parameters, it is independent of the overall normalization represented in β_0. Finally, the recurrence relation assumes a 'linearized' form

$$a\,\mathrm{d}f_R/\mathrm{d}a = (1+(d-1)\sum_{R'}\beta_{R'}\partial/\partial\beta_{R'})f_R. \tag{17.35}$$

We now consider the weak coupling limit of a truncated action with only a single coupling β representing the fundamental group representation. The parameter b_0 is then

$$b_0 = \int \mathrm{d}g\, \mathrm{e}^{\beta\,\mathrm{Re\,Tr}\,(g)}. \tag{17.36}$$

From this we define the corresponding f variable

$$f = \log(d_F^{-1}(\partial/\partial\beta)\log(b_0)), \tag{17.37}$$

where d_F is the dimension of the group matrix g. The recursion relation for f is

$$a(\mathrm{d}/\mathrm{d}a)f = (1+(d-1)\beta\partial/\partial\beta)f. \tag{17.38}$$

If we now let β become large, the integral in eq. (17.36) receives its dominant contribution from g near the identity, where we write

$$g = \mathrm{e}^{\mathrm{i}\omega\cdot\lambda}, \tag{17.39}$$

$$\mathrm{Re\,Tr}\,(g) = d_F - \tfrac{1}{4}\omega^2 + O(\omega^4). \tag{17.40}$$

Here λ^α are the group generators, of which there are n_g. Straightforward Gaussian integration gives

$$f = n_g/(2\beta d_F) + O(\beta^{-2}). \tag{17.41}$$

Inserting this into eq. (17.38) gives

$$a(\mathrm{d}/\mathrm{d}a)\beta = (d-2)\beta + O(1), \tag{17.42}$$

which shows the critical nature of two dimensions. The $O(1)$ term depends on the details of the quartic term in the action as well as the group measure. This sensitivity presumably is a signal of overextension of the approximations in the recursion relations; nevertheless, Kadanoff (1977) has given heuristic arguments which suggest a negative sign for this correction in two-dimensional non-Abelian theories. This supports the perturbative prediction of asymptotic freedom in these models (Polyakov, 1975).

With this machinery in hand, the generalization to gauge theories is direct. Indeed, this extension is almost trivial with the choice of an appropriate gauge. In d dimensions, to do a decimation along, say, the x axis, it is natural to work in the axial gauge where the links along that axis are all set to the identity. We then have a set of coupled one-dimensional chains of spins, as discussed in the chapter on gauge fixing. Suppose we now take those plaquettes which are transverse to the decimation direction and move those with odd x coordinate back one site. Then we can integrate the variables with odd x coordinate. In this procedure, the decimation on the plaquettes parallel to the x axis is precisely that of the one-dimensional spin system. After the decimation in one direction, we undo the gauge fixing along that axis and repeat the entire process along another. Continuing to make an infinitesimal decimation in every direction gives the Migdal–Kadanoff approximation for the renormalization group function in a gauge theory

$$a(\mathrm{d}/\mathrm{d}a)\,\beta_R = (d-2)\,\beta_R + 2\sum_{R'} (\partial\beta_R/\partial b_{R'})\,b_{R'}\log b_{R'}. \qquad (17.43)$$

The parameters β_R and b_R are defined in analogy with eq. (17.14) with S_L replaced by $S_\square$ and $g_i g_{i+1}^*$ replaced by $U_\square$. The factor $d-2$ in the first term arises because for any plaquette we perform bond moving for the $d-2$ dimensions orthogonal to that plaquette, while the other two dimensions in the plane of the plaquette give the factor of two in the second term.

This is the result advertised at the beginning of this chapter. Up to an overall factor of two, the recursion relation is identical to that for the spin system in $d/2$ dimensions. The correspondence appears because the spin interaction is along one-dimensional bonds while the gauge interaction utilizes two-dimensional plaquettes. The important prediction is that the critical dimensionalities in the gauge theory are twice those of the spin models. Thus our four-dimensional world represents a critical case for continuous gauge groups.

Problems

1. As d goes to infinity, what happens to the fixed point of eq. (17.30)? Which should be more reliable, this prediction or that of mean field theory?

2. Consider an action with a quartic term

$$S_L(g) = (d_F + \tfrac{1}{4}\omega^2 + C\omega^4 + O(\omega^6))$$

in the analysis leading to eq. (17.42). Find the $O(1)$ terms in the latter equation in terms of the parameters C and the integration measure $\mathrm{d}g = \mathrm{d}^{n_g}\omega\,(J_0 + J_1\,\omega^2 + O(\omega^4))$.

3. Consider the general Z_P model discussed in the last chapter. Show that the processes of bond moving and decimation interchange under duality. Thus the infinitesimal recursion relation respects the duality symmetry and gives the exact result in eq. (17.31).

4. What does the Migdal–Kadanoff relation predict for the behavior of the correlation length near the critical point of the two-dimensional Ising model?