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COMPLETIONS OF ORDERED SETS

BY
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Introduction. Completions of categories were studied by Lambek in [3], using the contravariant Hom functor to embed a small category C into the functor category (C^*, S), where C^* is the opposite category of C, and S is the category of sets. Three completions of C were considered; the completion (C^*, S), the full subcategory $(C^*, C)_{\inf} \subseteq (C^*, S)$ whose objects consist of all inf-preserving functors, and the full sub-category $B \subseteq (C^*, S)_{\inf}$ consisting of all subobjects of products of representable functors of the form $\mathrm{Hom}_C(—, C)$, C an object of C.

If a quasi-ordered set A is viewed as a category in the usual way, the Hom sets become objects of the complete category 2, and there is a natural embedding $h: A \to (A^*, 2)$. This allows us to form completions of A analogous to those in [3] by replacing S by 2. The completions we obtain in this way are also order-theoretic completions, and the purpose of this paper is to compare these completions with those given by Banaschewski in [1]. In particular, we find that the completion $(A^*, 2)$ is the largest sup-dense completion of A. The completion $(A^*, 2)_{\inf}$ cannot in general be compared with the ideal completion of A; however, if $(A^*, 2)_\omega$ denotes those order-preserving functions which preserve finite infima, then the ideal completion of A is contained in $(A^*, 2)_\omega$. If A is a lattice, then $(A^*, 2)_\omega$ is the ideal completion of A. Finally, the Dedekind completion of A is the subset $B \subseteq (A^*, 2)$ consisting of all (categorical) products of functions of the form $h(a)$, where $h: A \to (A^*, 2)$ and $a \in A$.

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Preliminaries. A *quasi-ordered set* (abbreviated q.o. set) is a pair $(A, \leq)$ where A is a set and $\leq$ is a reflexive, transitive binary relation on A. If $\leq$ is also anti-symmetric, $(A, \leq)$ is called a *partially ordered set* (p.o. set). When we say that a set A is a q.o. set, we mean of course that there is a binary relation $\leq$ on A such that $(A, \leq)$ is a q.o. set. If A and B are q.o. sets, a function $f: A \to B$ is called *order-preserving* if $a \leq a'$ in A implies $f(a) \leq f(a')$ in B. An order-preserving function $f: A \to B$ is called an *embedding* if $a \leq a'$ in A if and only if $f(a) \leq f(a')$ in B, and a

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strong embedding if in addition f is a monomorphism. Observe that if $f: A \to B$ is an embedding and B is a p.o. set, then f is a strong embedding if and only if A is a p.o. set. For any q.o. set A there is an embedding $u: A \to P(A)$, where $P(A)$ is the power set of A with inclusion order, and $u(a) = \{x \in A \mid x \leq a\}$. A *completion* of a q.o. set A is a pair (C, e) where C is a complete p.o. set (i.e. every subset of C has a supremum) and $e: A \to C$ is an embedding. A subset B of a p.o. set C is said to be *sup-dense* in C if every element of C is the supremum of some subset of B. A completion (C, e) of A is called sup-dense if the image of A under e is sup-dense in C. Two completions (C, e) and (C', e') of A are called *(order)-isomorphic over* A if there exists an (order)-isomorphism $\varphi: C \to C'$ such that $\varphi \circ e = e'$.

A *closure system* on a set A is a nonempty family of subsets of A which is closed under arbitrary intersections. A closure system is called *inductive* if it is closed under unions of nonempty chains, and *completely additive* if it is closed under arbitrary unions. For any q.o. set A, let $D(A)$, $I(A)$ and $L(A)$ denote respectively the smallest closure system, the smallest inductive closure system, and the smallest completely additive closure system, which contains $u[A]$, the image of A under $u: A \to P(A)$. Banaschewski showed in [1] that if A is a p.o. set, then $D(A)$ is the Dedekind completion of A; if A is a lattice, then $I(A)$ is the ideal completion of A, while for any p.o. set A, $L(A)$ may be characterized uniquely up to isomorphism over A as the largest sup-dense completion of A, in the sense that if (C, e) is any other sup-dense completion of A, there exists an embedding $\phi: C \to L(A)$ such that $\phi \circ e = u$. A *closure completion* of a q.o. set A is a closure system on A with the property that $\Gamma(\{a\}) = u(a)$ for all $a \in A$, where Γ is the associated closure operator. The smallest closure completion of A is $D(A)$, and the largest is $L(A)$.

Any q.o. set A may be viewed as a category whose objects are the elements of A, and where

$$\mathrm{Hom}_A(a, a') = \begin{cases} \{\phi\} & \text{if } a \leq a' \\ \phi & \text{otherwise.} \end{cases}$$

The opposite category of a q.o. set A is the dual q.o. set, denoted by A^*. A functor between two q.o. sets viewed as categories is simply an order-preserving function. The categorical notions of complete category, sup-preserving functor and sup-dense functor (as defined in [3]), when applied to q.o. sets and order-preserving functions, are exactly the usual order-theoretic concepts.

Completions of ordered sets. If A is a q.o. set, let $(A^*, 2)$ denote the set of order-preserving functions from A^* to 2. For $f, g \in (A^*, 2)$, define $f \leq g$ if $f(a) \leq g(a)$ for all $a \in A$. Then $(A^*, 2)$ is a p.o. set. Moreover, $(A^*, 2)$ is complete since 2 is complete; explicitly, if $(f_i)_{i \in I}$ is any family of order-preserving functions from A^* to 2,

$$\left(\bigvee_{i \in I} f_i\right)(a) = \bigvee_{i \in I} f_i(a).$$

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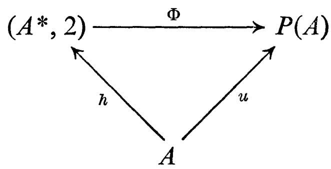
The fact that $\mathrm{Hom}_A(a, a')$ is an object of the p.o. set 2 induces an embedding $h: A \to (A^*, 2)$, where $(h(a))(a') = \mathrm{Hom}_A(a', a)$. Thus $((A^*, 2), h)$ is a completion of A.

PROPOSITION 1. *For any p.o. set A, $((A^*, 2), h)$ is the largest sup-dense completion of A.*

Proof. It suffices to show that $(A^*, 2)$ is isomorphic over A to $L(A)$, the largest closure completion of A. Define $\Phi: (A^*, 2) \to P(A)$ by $\Phi(f) = \{x \in A \mid f(x) = 1\}$ where $f: A^* \to 2$. Φ is clearly an embedding, and the following diagram commutes. We claim that Φ actually maps into $L(A)$.



Thus we must show that if all $f \in (A^*, 2)$, $\Phi(f) = \bigcup_{f(x)=1} u(x)$. This follows by observing that $f(x) = 1$ and $y \leq x$ in A implies $f(y) = 1$. We will complete the proof by showing that Φ has an inverse. Define $\Psi: L(A) \to (A^*, 2)$ by

$$(\Psi(B))(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B \end{cases}$$

where $B \in L(A)$ and $x \in A$. We must verify that $\Psi(B) \in (A^*, 2)$. Since $B \in L(A)$, $B = \bigcup_{b \in B} ub$. If $x \leq y$ in A^*, and $\Psi(B)(x) = 1$, then $y \in ux \subseteq B$ so $\Psi(B)(y) = 1$ and $\Psi(B) \in (A^*, 2)$. Finally, we observe that $\Phi\Psi = 1_{L(A)}$, $\Psi\Phi = 1_{(A^*,2)}$ and deduce that $(A^*, 2)$ and $L(A)$ are order-isomorphic over A. ||

In analogy with [3], let $(A^*, 2)_{\mathrm{inf}}$ denote the p.o .subset of $(A^*, 2)$ consisting of those functions which preserve infima, i.e. which take suprema in A into infima in 2. It is easily verified that $(A^*, 2)_{\mathrm{inf}}$ is complete, and that for any $a \in A$, $h(a): A^* \to 2$ is inf-preserving. Using [1, Lemma 1], we see that $\Phi[(A^*, 2)_{\mathrm{inf}}]$ is a closure completion of A, and since a closure completion is a sup-dense completion, we conclude that $(A^*, 2)_{\mathrm{inf}}$ is a sup-dense completion of A. The elements of $\Phi[(A^*, 2)_{\mathrm{inf}}]$ are easily seen to be those subsets C of A which belong to $L(A)$ and which have the following property: if $S \subseteq C$ and sup S exists then sup $S \in C$. Let us call those elements of $L(A)$ which have this property *conditionally sup-closed*. Observe that $\Phi[(A^*, 2)_{\mathrm{inf}}]$ is not necessarily inductive. For example, let A be any chain with largest element a such that $\sup(A \backslash \{a\}) = a$. Then $(ux)_{x<a}$ forms a chain in $\Phi[(A^*, 2)_{\mathrm{inf}}]$ with union $A \backslash \{a\}$, but clearly $A \backslash \{a\}$ is not conditionally sup-closed, and $\Phi[(A^*, 2)_{\mathrm{inf}}]$ fails to be inductive.

To obtain a closure system which is inductive, we consider the p.o. set $(A^*, 2)_\omega \subseteq (A^*, 2)$ consisting of all those functions from A^* to 2 which preserve finite infima. Under the isomorphism $\Phi: (A^*, 2) \to L(A)$, we see that $(A^*, 2)_\omega$ corresponds to the set of all those $C \in L(A)$ which are conditionally finite-sup-closed, i.e. those $C \in L(A)$ which have the property that if $S \subseteq C$, S finite, and sup S exists then sup $S \in C$. Since the conditionally finite-sup-closed members of $L(A)$ are obviously closed under unions of chains, we conclude that $\Phi[(A^*, 2)_\omega]$ is an inductive closure completion of A. Since $I(A)$ is the smallest inductive closure completion of A, we have $I(A) \subseteq \Phi[(A^*, 2)_\omega]$. Generally this inclusion is strict; for example, if A is a totally unordered set with more than two elements, $I(A) = \{ua \mid a \in A\} \cup \{\phi, A\}$ while $\Phi[(A^*, 2)] = P(A)$. However, for a lattice we can replace the inclusion by equality.

PROPOSITION 2. *If A is a lattice, then the lattice $(A^*, 2)_\omega$ of finite-inf-preserving functions from A^* to 2 is isomorphic over A to the ideal completion $I(A)$ of A.*

Proof. We claim that the isomorphism $\Phi:(A^*, 2) \rightarrow L(A)$, when restricted to $(A^*, 2)_\omega$, maps onto $I(A)$. Let $f \in (A^*, 2)_\omega$. If $\Phi(f) = \phi$, then f is the function with constant value 0. It follows that A has no smallest element, and hence $\phi \in I(A)$. [1, p. 129, footnote 1.] If $x, y \in \Phi(f) \neq \phi$, then $f(x \vee y) = f(x) \wedge f(y) = 1$ and $(x \vee y) \in \Phi(f)$. If $x \in \Phi(f)$ and $y \leq x$ in A, then $1 = f(x) \leq f(y)$ so $y \in \Phi(f)$. Hence $\Phi: (A^*, 2)_\omega \rightarrow I(A)$, and Φ is surjective, since if I is any ideal of A and we define f_I to have value 1 on I and 0 elsewhere, it is easily checked that $f_I \in (A^*, 2)_\omega$ and $\Phi(f_I) = I$. ‖

Finally we obtain a categorical characterization of the Dedekind completion of a p.o. set.

PROPOSITION 3. *The Dedekind completion $D(A)$ of a p.o. set A is isomorphic over A to the p.o. set $B \subseteq (A^*, 2)$ consisting of all (categorical) products in $(A^*, 2)$ of functions of the form $h(a)$, where $a \in A$ and $h: A \rightarrow (A^*, 2)$.*

Proof. Products in a p.o. set regarded as a category are order-theoretic infima, and infima in $(A^*, 2)$ correspond under the isomorphism $\Phi: (A^*, 2) \rightarrow L(A)$ to intersections in $L(A)$. Since $\Phi(h(a)) = u(a)$ for any $a \in A$, $\Phi[B]$ is the family of arbitrary intersections of members of $u[A]$, which is the Dedekind completion of A. [1, p. 119]. ‖

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