Nagoya Math. J., **236** (2019), 214–250
DOI 10.1017/nmj.2018.34

HIGHER IDELES AND CLASS FIELD THEORY

MORITZ KERZ AND YIGENG ZHAO

Abstract. We use higher ideles and duality theorems to develop a universal approach to higher dimensional class field theory.

CONTENTS

Received March 31, 2018. Revised August 21, 2018. Accepted August 23, 2018.
2010 Mathematics subject classification. Primary 11G45; Secondary 14F20, 14F35, 11R37, 14G17.
The authors are supported by the DFG through CRC 1085 *Higher Invariants* (Universität Regensburg).

§1. Introduction

In higher dimensional class field theory one tries to describe the abelian fundamental group of a scheme X of arithmetic interest in terms of idelic or cycle theoretic data on X. More precisely, assume that X is regular and connected and fix a modulus data, that is, an effective divisor D on X. Let $\pi_1^{\mathrm{ab}}(X, D)$ be the abelian fundamental group classifying étale coverings with ramification bounded by D. One defines an idele class group $C(X, D)$ which is a quotient of the idele group

$$I(U \subset X) := \bigoplus_{P \in \mathcal{P}} K^M_{d(P)}(k(P))$$

by a modulus subgroup depending on D and certain reciprocity relations. Here $P \in \mathcal{P}$ runs through some set of chains of prime ideals and $k(P)$ is a generalized form of Henselian local residue field at the chain P; see Section 2.1 and [Ker11].

One then constructs a residue map

$$\rho : C(X, D) \to \pi_1^{\mathrm{ab}}(X, D)$$

which we show to be an isomorphism after tensoring with $\mathbb{Z}/n\mathbb{Z}$ $(n > 0)$ in the following situations:

(i) X is a smooth proper variety over a finite field, recovering (with simpler proof) the main result of [KS86] for varieties; see Section 3.
(ii) X is an (equal characteristic) complete regular local ring with finite residue field, recovering in case $\dim(X) = 2$ results of [Sai87], recovering in case n is invertible on X results of [Sat09] and completing our understanding in case X is of equal characteristic p and n is a power of p; see Section 4.
(iii) X is a smooth proper scheme over an (equal characteristic) complete discrete valuation ring with finite residue field, recovering results of Bloch and Saito, see [Sai85], for $\dim(X) = 2$ and results of [For15] for n invertible on X and completing our understanding in case X is of characteristic p and n is a power of p; see Section 5.

Here is an outline of our universal strategy to all three cases of the reciprocity isomorphism ρ in higher dimensional class field theory listed above:

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Step 1: Show that $C(X, D)$ is isomorphic to a Nisnevich cohomology group of relative Milnor K-sheaf $\mathcal{K}^M_{X,D}$, for example, in case (i) above one has an isomorphism

$$C(X, D) \cong H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D}),$$

where $d = \dim(X)$.

Step 2: Show that the Nisnevich cohomology of the relative Milnor K-sheaf with finite coefficients is isomorphic to a certain analogous étale cohomology group, for example, in case (i) and for $n = p^m$ a power of the characteristic p of the base field one has an isomorphism

$$H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D}/n) \cong H^d(X_{\mathrm{\acute{e}t}}, W_m\Omega^d_{X|D,\log})$$

where $W_m\Omega^d_{X|D,\log}$ is a relative de Rham–Witt sheaf. This isomorphism is established by comparing coniveau spectral sequences and observing that based on cohomological dimension arguments there is just one additional potentially nonvanishing row in the spectral sequence in the étale situation, which however disappears at the end by known cases of the Kato conjecture.

Step 3: Arithmetic duality tells us that the étale cohomology group from Step 2 is isomorphic to an abelian étale fundamental group, for example, in the special case as in Step 2 the profinite group $\lim_D H^d(X_{\mathrm{\acute{e}t}}, W_m\Omega^d_{X|D,\log})$, where D runs through all effective divisors with a fixed support $X \setminus U$, is Pontryagin dual to the (discrete) cohomology group $H^1(U_{\mathrm{\acute{e}t}}, \mathbb{Z}/n\mathbb{Z})$.

§2. Higher ideles and Milnor K-sheaves

2.1 Higher ideles

Let X be an integral Noetherian scheme with a dimension function d. Recall that a dimension function on a scheme X is a set theoretic function $d \colon X \to \mathbb{Z}$ such that:

(i) for all $x \in X$, $d(x) \geqslant 0$;
(ii) for $x, y \in X$ with $y \in \overline{\{x\}}$ of codimension one, $d(x) = d(y) + 1$, where $\overline{\{x\}}$ denotes the closure of $\{x\}$ in X.

We also denote $d = d(\eta)$, where η is the generic point of X. Let d_m be the minimal of the integers $d(x)$ for $x \in X$. For an effective Weil divisor D of X, we denote $U = X \setminus D$.

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DEFINITION 2.1.1.

(i) A chain on X is a sequence of points $P=(p_0,p_1,\ldots,p_s)$ of X such that
$$\overline{\{p_0\}}\subset\overline{\{p_1\}}\subset\cdots\subset\overline{\{p_s\}}.$$

(ii) A Parshin chain on X is a chain $P=(p_0,p_1,\ldots,p_s)$ on X such that $d(p_i)=i+d_m$, for $0\leqslant i\leqslant s$.

(iii) A Parshin chain on the pair $(U\subset X)$ is a Parshin chain $P=(p_0,p_1,\ldots,p_s)$ on X such that $p_i\in D$ for $0\leqslant i<s$ and such that $p_s\in U$.

(iv) The dimension $d(P)$ of a chain $P=(p_0,p_1,\ldots,p_s)$ is defined to be $d(p_s)$.

(v) A Q-chain on $(U\subset X)$ is defined as a chain $P=(p_0,\ldots,p_{s-2},p_s)$ on X for $1\leqslant s\leqslant d$, such that $d(p_i)=i+d_m$ for $i\in\{0,1,\ldots,s-2,s\}$, $p_i\in D$ for $0\leqslant i\leqslant s-2$ and $p_s\in U$.

We also recall the definition of Milnor K-theory.

DEFINITION 2.1.2.

(i) For a commutative unital ring R, the Milnor K-ring $K^M_\bullet(R)$ of R is the graded ring $T(R^\times)/I$, where I is the ideal of the tensor algebra $T(R^\times)$ over $R^\times$ generated by elements $a\otimes(1-a)$ with $a,1-a\in R^\times$. The image of $a_1\otimes\cdots\otimes a_r$ in $K^M_r(R)$ is denoted by $\{a_1,\ldots,a_r\}$.

(ii) If R is a discrete valuation ring with quotient field K and maximal ideal $\mathfrak{m}\subset R$ we define $K^M_r(K,n)\subset K^M_r(K)$ be the subgroup generated by $\{1+\mathfrak{m}^n,K^\times,\ldots,K^\times\}$ for an integer $n\geqslant 0$.

DEFINITION 2.1.3. Let $P=(p_0,\ldots,p_s)$ be a chain on X.

(i) We define the ring $\mathcal{O}^h_{X,P}$, which is a finite product of Henselian local rings, as follows: If $s=0$ set $\mathcal{O}^h_{X,P}=\mathcal{O}^h_{X,p_0}$. If $s>0$ assume that $\mathcal{O}^h_{X,P'}$ has been defined for chains of the form $P'=(p_0,\ldots,p_{s-1})$. Denote $R=\mathcal{O}^h_{X,P'}$, let T be the finite set of prime ideals of R lying over p_s. Then we define
$$\mathcal{O}^h_{X,P}:=\prod_{\mathfrak{p}\in T}R^h_{\mathfrak{p}}.$$

(ii) For a chain $P=(p_0,\ldots,p_s)$ on X we let $k(P)$ be the finite product of the residue fields of $\mathcal{O}^h_{X,P}$. If $s\geqslant 1$ each of these residue fields has a natural discrete valuation such that the product of their rings of integers is equal to the normalization of $\mathcal{O}^h_{X,P'}/p_s$, where $P' = (p_0, \ldots, p_{s-1})$.

Let $\mathcal{P}$ be the set of Parshin chains on the pair $(U \subset X)$, and let $\mathcal{Q}$ be the set of Q-chains on $(U \subset X)$. For a Parshin chain $P = (p_0, \ldots, p_{d-d_m}) \in \mathcal{P}$ of dimension d we denote by $D(P)$ the multiplicity of the prime divisor $\overline{\{p_{d-d_m-1}\}}$ in D.

DEFINITION 2.1.4.

(i) The idele class group of $(U \subset X)$ is defined as

$$I(U \subset X) := \bigoplus_{P \in \mathcal{P}} K^M_{d(P)}(k(P)),$$

and endow this group with the topology generated by the open subgroups

$$\bigoplus_{\substack{P \in \mathcal{P} \\ d(P)=d}} K^M_d(k(P), D(P)) \subset I(U \subset X),$$

where D runs through all effective Weil divisors with support $X \setminus U$.

(ii) The idele group of X relative to the fixed effective divisor D with complement U is defined as

$$I(X, D) := \operatorname{Coker}\Bigg(\bigoplus_{\substack{P \in \mathcal{P} \\ d(P)=d}} K^M_d(k(P), D(P)) \to I(U \subset X) \Bigg).$$

(iii) The idele class group $C(U \subset X)$ is

$$C(U \subset X) := \operatorname{Coker}\Bigg(\bigoplus_{P \in \mathcal{Q}} K^M_{d(P)}(k(P)) \xrightarrow{Q} I(U \subset X) \Bigg),$$

where Q is defined to be the sum of all $Q^{P' \to P}$ for $P' = (p_0, \ldots, p_{s-2}, p) \in \mathcal{Q}$ and $P = (p_0, \ldots, p_{s-2}, p_{s-1}, p_s) \in \mathcal{P}$:

- if $p_{s-1} \in D$, then $Q^{P' \to P}$ is the natural map $K^M_{d(P')}(k(P')) \to K^M_{d(P)}(k(P))$ induced on Milnor K-groups by the ring homomorphism $k(P') \to k(P)$;
- if $p_{s-1} \in U$, then $Q^{P' \to P}$ is the residue symbol $K^M_{d(P')}(k(P')) \to K^M_{d(P'')}(k(P''))$ where $P'' = (p_0, \ldots, p_{s-1})$.

(iv) The idele class group $C(X, D)$ of X relative to the effective divisor D is defined as

$$C(X, D) := \operatorname{Coker}\left(\bigoplus_{P \in \mathcal{Q}} K^M_{d(P)}(k(P)) \xrightarrow{Q} I(X, D)\right).$$

2.2 Milnor K-sheaves

Let X be an integral scheme. Recall the Milnor K-sheaf $\mathcal{K}^M_*$ is defined as the Nisnevich sheafification of the presheaf on affine scheme $\operatorname{Spec}(A)$ given as follows:

$$A \mapsto K^M_\bullet(A) = \bigoplus_{i \in \mathbb{N}} \underbrace{(A^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A^\times)}_{i \text{ times}} / I,$$

where I is the two-sided ideal of the tensor algebra generated by the elements $a \otimes (1-a)$ with $a, 1-a \in A^\times$. This sheaf is closely related to a p-primary sheaf if X is of characteristic $p \geqslant 0$, so-called logarithmic de Rham–Witt sheaf $W_m\Omega^r_{X,\log}$ on the small Nisnevich (resp. étale) site, which is a subsheaf of $W_m\Omega^r_X$ (cf. [Ill79]) Nisnevich (resp. étale) locally generated by $d\log[x_1]_m \wedge \cdots \wedge d\log[x_r]_m$ with $x_i \in \mathcal{O}_X^\times$ for all i, $d\log[x]_m := d[x]_m/[x]_m$ and $[x]_m$ is the Teichmüller representative of x in $W_m\mathcal{O}_X$.

These notations can be generalized to a relative situation with respect to a divisor. Let $i\colon D \hookrightarrow X$ be an effective divisor with its complement $j\colon U := X \setminus D \hookrightarrow X$.

Definition 2.2.1. Let $r \in \mathbb{N}$. We define:

(i) [RS18, Definition 2.4] the relative Milnor K-sheaf $\mathcal{K}^M_{r,X|D}$ on the small Nisnevich (resp. étale) site is defined to be the subsheaf of $j_*\mathcal{K}^M_{r,U}$ Nisnevich (resp. étale) locally generated by $\{x_1, \ldots, x_r\}$ with $x_1 \in \ker(\mathcal{O}_X^\times \to \mathcal{O}_D^\times)$ and $x_i \in \mathcal{O}_U^\times$ for all i. Note that if X is a regular scheme over a field, then $\mathcal{K}^M_{r,X|D} \subset \mathcal{K}^M_{r,X}$ by the known Gersten conjecture [Ker09] (see also [RS18, Corollary 2.9]).

(ii) [JSZ18, Definition 1.1.1] in the case that X is of characteristic $p \geqslant 0$, the relative logarithmic de Rham–Witt sheaf $W_m\Omega^r_{X|D,\log}$ on the small Nisnevich (resp. étale) site is the subsheaf of $j_*W_m\Omega^r_{U,\log}$ Nisnevich (resp. étale) locally generated by $d\log[x_1]_m \wedge \cdots \wedge d\log[x_r]_m$ with $x_1 \in \ker(\mathcal{O}_X^\times \to \mathcal{O}_D^\times)$ and $x_i \in \mathcal{O}_U^\times$ for all i. Similar to the relative Milnor K-group, we also have $W_m\Omega^r_{X|D,\log} \subset W_m\Omega^r_{X,\log}$ in the case that X is a regular scheme.

We will show relations between them in a local case, and then we may use these results in different settings. In the following, we fix the notation as follows: Let R be a Henselian regular local ring of characteristic $p>0$ with the residue field k. We assume that k is finite. Let D be an effective divisor such that $C:=\mathrm{Supp}(D)$ is a simple normal crossing divisor on $X:=\mathrm{Spec}(R)$. Let $\{D_\lambda\}_{\lambda\in\Lambda}$ be the (regular) irreducible components of D, and let $i_\lambda: D_\lambda \hookrightarrow X$ be the natural map.

THEOREM 2.2.2. *The* $d\log$ *map induces an isomorphism of Nisnevich sheaves on* X_{Nis}

$$\begin{aligned} d\log[-]: \mathcal{K}^M_{r,X|D}/(p^m\mathcal{K}^M_{r,X}\cap \mathcal{K}^M_{r,X|D}) &\xrightarrow{\cong} W_m\Omega^r_{X|D,\log} \\ \{x_1,\ldots,x_r\} &\mapsto d\log[x_1]_m\wedge\cdots\wedge d\log[x_r]_m. \end{aligned}$$

Proof. The assertion follows directly by the following commutative diagram

$$\begin{tikzcd} \mathcal{K}^M_{r,X|D}/(p^m\mathcal{K}^M_{r,X}\cap \mathcal{K}^M_{r,X|D}) \arrow[r, hook] \arrow[d, "d\log"'] & \mathcal{K}^M_{r,X}/p^m \arrow[d, "d\log"', "\cong"] \\ W_m\Omega^r_{X|D,\log} \arrow[r, hook] & W_m\Omega^r_{X,\log} \end{tikzcd}$$

where the right vertical map is an isomorphism by Bloch–Gabber–Kato theorem [BK86] and Gersten resolutions of $\epsilon_*\mathcal{K}^M_{r,X}$ and $\epsilon_*W_m\Omega^r_{X,\log}$ from [Ker09, GS88]; here $\epsilon\colon X_{\mathrm{Nis}}\to X_{\mathrm{Zar}}$ is the canonical map. ∎

In order to study the structure of the relative logarithmic de Rham–Witt sheaves, we introduce some notions here. We endow $\mathbb{N}^\Lambda$ with a semiorder by

$$\underline{n}:=(n_\lambda)_{\lambda\in\Lambda}\geqslant \underline{n'}:=(n'_\lambda)_{\lambda\in\Lambda}\quad \text{if } n_\lambda\geqslant n'_\lambda \text{ for all } \lambda\in\Lambda.$$

For $\underline{n}=(n_\lambda)_{\lambda\in\Lambda}\in\mathbb{N}^\Lambda$ let

$$D_{\underline{n}}=\sum_{\lambda\in\Lambda} n_\lambda D_\lambda$$

be the associated divisor. For $\nu\in\Lambda$ we set $\delta_\nu=(0,\ldots,1,\ldots,0)\in\mathbb{N}^\Lambda$, where 1 is on the νth place, and we define the following Nisnevich sheaves for $r\geqslant 1$

$$\begin{gathered} \mathrm{gr}^{\underline{n},\nu}\mathcal{K}^M_{r,X} := \mathcal{K}^M_{r,X|D_{\underline{n}}}/\mathcal{K}^M_{r,X|D_{\underline{n}+\delta_\nu}}; \\ \mathrm{gr}^{\underline{n},\nu}W_m\Omega^r_{X,\log} := W_m\Omega^r_{X|D_{\underline{n}},\log}/W_m\Omega^r_{X|D_{\underline{n}+\delta_\nu},\log}. \end{gathered}$$

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PROPOSITION 2.2.3. [RS18, Proposition 2.10] *Let* $\underline{n} = (n_\lambda)_{\lambda\in\Lambda} \in \mathbb{N}^\Lambda$, *and let* $\nu \in \Lambda, r \geqslant 1$. *Assume* $n_\nu = 0$ *and set*

$$D_{\nu,\underline{n}} := \sum_{\lambda\in\Lambda\setminus\{\nu\}} n_\lambda(D_\lambda \cap D_\nu).$$

Then there is a natural isomorphism of Nisnevich sheaves

$$\mathrm{gr}^{\underline{n},\nu}\mathcal{K}^M_{r,X} \xrightarrow{\cong} i_{\nu,*}\mathcal{K}^M_{r,D_\nu|D_{\nu,\underline{n}}}.$$

Proof. The argument in [RS18] works verbatim for our case. ∎

THEOREM 2.2.4. *If D is reduced, then $d\log$ induces an isomorphism of Nisnevich sheaves*

$$\begin{aligned} d\log[-] : \mathcal{K}^M_{r,X|D}/p^m &\xrightarrow{\cong} W_m\Omega^r_{X|D,\log} \\ \{x_1, \ldots, x_r\} &\mapsto d\log[x_1]_m \wedge \cdots \wedge d\log[x_r]_m. \end{aligned}$$

Proof. By the commutative diagram

$$\begin{tikzcd} \mathcal{K}^M_{r,X|D}/p^m \arrow[r] \arrow[d, "d\log"'] & \mathcal{K}^M_{r,X}/p^m \arrow[d, "d\log"', "\cong"] \\ W_m\Omega^r_{X|D,\log} \arrow[r, hook] & W_m\Omega^r_{X,\log} \end{tikzcd}$$

it is enough to show that $\mathcal{K}^M_{r,X|D}/p^m \hookrightarrow \mathcal{K}^M_{r,X}/p^m$. On the other hand, we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}^M_{r,X|D} \arrow[r] \arrow[d, "p^m"] & \mathcal{K}^M_{r,X} \arrow[r] \arrow[d, "p^m"] & \mathcal{K}^M_{r,X}/\mathcal{K}^M_{r,X|D} \arrow[r] \arrow[d, "p^m"] & 0 \\ 0 \arrow[r] & \mathcal{K}^M_{r,X|D} \arrow[r] & \mathcal{K}^M_{r,X} \arrow[r] & \mathcal{K}^M_{r,X}/\mathcal{K}^M_{r,X|D} \arrow[r] & 0 \end{tikzcd}$$

Combining the fact [GL00, Theorem 8.1] and the Gersten resolution [Ker09], we know that $\mathcal{K}^M_{r,X}$ is p-torsion free. Therefore the middle vertical map is injective, so is the first vertical map. By the snake lemme, it is sufficient to check that the third vertical map $p^m : \mathcal{K}^M_{r,X}/\mathcal{K}^M_{r,X|D} \to \mathcal{K}^M_{r,X}/\mathcal{K}^M_{r,X|D}$ is injective. This follows from the above Proposition 2.2.3, by noting that

$\mathcal{K}^M_{r,X}/\mathcal{K}^M_{r,X|D}$ is a successive extension of sheaves $\mathrm{gr}^{\underline{n},\nu}\mathcal{K}^M_{r,X}$ and the map $p^m : i_{\nu,*}\mathcal{K}^M_{r,D_\nu|D_{\nu,\underline{n}}} \to i_{\nu,*}\mathcal{K}^M_{r,D_\nu|D_{\nu,\underline{n}}}$ is injective (similar to the injectivity of the first vertical map in the above diagram). We remark that the assumption in Proposition 2.2.3 is satisfied, since D is reduced. ∎

PROPOSITION 2.2.5. [JSZ18, Proposition 1.1.9] *Let X, D be as above. Then we have:*

(i) $W_m\Omega^d_{X,\log} = W_m\Omega^d_{X|D_{\mathrm{red}},\log}$;
(ii) *for $\underline{n} \geqslant \underline{1}$, the quotient $\mathrm{gr}^{\underline{n},\nu}W_m\Omega^r_{X,\log}$ is a coherent $\mathcal{O}^{p^e}_{D_\nu}$-module, for some $e \gg 0$.*

Proof. In the case that $d = 1$ (i.e., R is a discrete valuation ring), the assertions have been given in [BK86, (4.7), (4.8)]. For general d, in [JSZ18], the graded pieces have been studied in the case that R is the Henselization of a local ring of a smooth scheme over k. But note that the argument also works in our setting. We only need to show (i). By Theorem 2.2.4, we see that, for $\underline{n} < \underline{1}$,

$$\mathrm{gr}^{\underline{n},\nu}\mathcal{K}^M_{d,X}/p^m \cong i_{\nu,*}\mathcal{K}^M_{d,D_\nu|D_{\nu,\underline{n}}}/p^m = i_{\nu,*}W_m\Omega^d_{D_\nu|D_{\nu,\underline{n}},\log} = 0,$$

where the vanishing is by dimension. ∎

§3. Class field theory for proper varieties over finite fields

In this section we reprove the main results of the class field theory of smooth proper varieties over finite fields with ramification along divisors D, which originally are due to Kato and Saito [KS86].

Let X be a smooth proper variety of dimension d over a finite field k, let D be an effective divisor such that $C := \mathrm{Supp}(D)$ is a simple normal crossing divisor on X, and let $j : U := X - C \hookrightarrow X$ be the complement of C. Let $\{D_\lambda\}_{\lambda\in\Lambda}$ be the (smooth) irreducible components of D, and let $i_\lambda : D_\lambda \hookrightarrow X$ be the natural map. We use the dimension function $d(x) = \dim(\overline{\{x\}})$ for $x \in X$. We also denote by $X_r := \{x \in X \mid d(x) = r\}$ the set of points of dimension r of X and $X^r := X_{d-r}$ the set of points of codimension r of X.

3.1 Idele class groups

The K-theoretic class group $H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})$ is introduced by Kato and Saito in [KS86], and they also give an idelic description of the dual of this class group. In [Ker11], we give a direct description of this class group, and prove the following theorem.

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THEOREM 3.1.1. [Ker11, Theorem 8.4] *There exists a unique isomorphism*

$$\rho_{X,D}\colon C(X,D) \cong H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})$$

such that the following triangle commutes

$$\begin{tikzcd} & \bigoplus_{x\in X_0} \mathbb{Z} \arrow[dl, "\imath"'] \arrow[dr, "\imath_{\mathrm{Nis}}"] & \\ C(X,D) \arrow[rr, "\rho_{X,D}"] & & H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D}) \end{tikzcd}$$

where $\imath$ is the obvious map, and $\imath_{\mathrm{Nis}}$ is the map from [KS86, Theorem 2.5].

3.2 The ℓ-primary part

In this subsection, we study the group $H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})/\ell^m$, and compare it with $H^{2d}(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d})$.

The coniveau spectral sequence for an abelian étale (resp. Nisnevich) sheaf $\mathcal{F}$ on $X_{\text{ét}}$ (resp. X_{Nis}) writes

$$E^{p,q}_{1,\text{ét}}(\mathcal{F}) := \bigoplus_{x\in X^p} H^{p+q}_x(X_{\text{ét}}, \mathcal{F}) \Longrightarrow H^{p+q}(X_{\text{ét}}, \mathcal{F})$$

$$E^{p,q}_{1,\mathrm{Nis}}(\mathcal{F}) := \bigoplus_{x\in X^p} H^{p+q}_x(X_{\mathrm{Nis}}, \mathcal{F}) \Longrightarrow H^{p+q}(X_{\mathrm{Nis}}, \mathcal{F}),$$

where X^p is the set of points of codimension p of X. Note that the degeneration of the coniveau spectral sequence due to cohomological dimension (cf. [KS86, 1.2.5]) for $\mathcal{K}^M_{d,X|D}$ on X_{Nis} gives rise to a short exact sequence

$$\bigoplus_{x\in X^{d-1}} H^{d-1}_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D}) \to \bigoplus_{x\in X^d} H^d_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})$$

$$\to H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D}) \to 0. \tag{3.2.1}$$

We now study the coniveau spectral sequence for $j_!\mu_{\ell^m}^{\otimes d}$ on $X_{\text{ét}}$.

PROPOSITION 3.2.1. *Let X be a smooth (not necessarily proper) variety over a finite field of dimension d. For any $x \in X^a$, we have*

$$H^{a+d+1}_x(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) = H^{a+d+1}_x(X_{\text{ét}}, \mu_{\ell^m}^{\otimes d}),$$

that is, $E^{\bullet,d+1}_{1,\text{ét}}(j_!\mu_{\ell^m}^{\otimes d}) = E^{\bullet,d+1}_{1,\text{ét}}(\mu_{\ell^m}^{\otimes d})$. In particular, we have $E^{d-2,d+1}_{2,\text{ét}}(j_!\mu_{\ell^m}^{\otimes d})$ $= E^{d-1,d+1}_{2,\text{ét}}(j_!\mu_{\ell^m}^{\otimes d}) = 0$.

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Proof. We prove the first claim by induction on the codimension a. For $x \in X^a$, we denote by $X_x = \mathrm{Spec}(\mathcal{O}^h_{X,x})$ the Henselization of X at x, and $Y_x = X_x \setminus \{x\}$. If $a = 1$, then any divisor of X_x must have support in the closed point $\{x\}$. Therefore

$$j_! \mu_{\ell^m}^{\otimes d}|_{Y_x} = \mu_{\ell^m}^{\otimes d}|_{Y_x}$$

by the definition of $j_!$. Using the localization exact sequences twice, we obtain

$$\begin{aligned} H_x^{d+2}(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) &\cong H^{d+1}(Y_{x,\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) = H^{d+1}(Y_{x,\text{ét}}, \mu_{\ell^m}^{\otimes d}) \\ &\cong H_x^{d+2}(X_{\text{ét}}, \mu_{\ell^m}^{\otimes d}), \end{aligned}$$

where the first isomorphism is due to $j_!\mu_{\ell^m}^{\otimes d}|_x = 0$, and the second isomorphism is by the vanishing $H^{d+2}(X_{x,\text{ét}}, \mu_{\ell^m}^{\otimes d}) \cong H^{d+2}(x_{\text{ét}}, \mu_{\ell^m}^{\otimes d}) = 0 = H^{d+1}(x_{\text{ét}}, \mu_{\ell^m}^{\otimes d}) \cong H^{d+1}(X_{x,\text{ét}}, \mu_{\ell^m}^{\otimes d})$, where we use the fact that $\mathrm{cd}_\ell(x) \leqslant d + 1 - \mathrm{codim}_X(x)$ (cf. [Sat09, Lemma 4.2(1)]).

For general codimension $a > 1$, the coniveau spectral sequence on Y_x and cohomological vanishing give us an exact sequence

$$\begin{aligned} \bigoplus_{y\in Y_x^{a-2}} H_y^{a+d-1}(Y_{x,\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) &\to \bigoplus_{y\in Y_x^{a-1}} H_y^{a+d}(Y_{x,\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \\ &\to H^{a+d}(Y_{x,\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \to 0. \end{aligned} \tag{3.2.2}$$

On the other hand, the localization exact sequence for $j_!\mu_{\ell^m}^{\otimes d}$ on X_x tells us

$$H^{a+d}(Y_{x,\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \cong H_x^{a+d+1}(X_{x,\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}), \tag{3.2.3}$$

Indeed due to $\mathrm{cd}_\ell(x) \leqslant d + 1 - \mathrm{codim}_X(x)$ we have

$$H^{a+d}(X_{x,\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) = 0 = H^{a+d+1}(X_{x,\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}).$$

Combining these facts, we get the following diagram with exact rows

$$\begin{tikzcd} \bigoplus_{y\in Y_x^{a-2}} H_y^{a+d-1}(Y_{x,\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \arrow[r] \arrow[d] & \bigoplus_{y\in Y_x^{a-1}} H_y^{a+d}(Y_{x,\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \arrow[r] \arrow[d] & H_x^{a+d+1}(X_{x,\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \arrow[r] \arrow[d] & 0 \\ \bigoplus_{y\in X_x^{a-2}} H_y^{a+d-1}(X_{x,\text{ét}}, \mu_{\ell^m}^{\otimes d}) \arrow[r] & \bigoplus_{y\in X_x^{a-1}} H_y^{a+d}(X_{x,\text{ét}}, \mu_{\ell^m}^{\otimes d}) \arrow[r] & H_x^{a+d+1}(X_{x,\text{ét}}, \mu_{\ell^m}^{\otimes d}) \arrow[r] & 0 \end{tikzcd}$$

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The first two vertical maps are isomorphisms by induction. Hence the third vertical arrow is also an isomorphism. Thanks to [JSS14, Theorem 3.5.1], we see that the complex $E^{\bullet,d+1}_{1,\text{ét}}(\mu_{\ell^m}^{\otimes d})$ is the Kato complex of $\mu_{\ell^m}^{\otimes d}$ (cf. [KS12, (0.2)]) up to a sign. By the known Kato conjecture on vanishing of cohomology groups of this complex at places $d-1$ and $d-2$ (cf. [KS12, Theorem 8.1]) we obtain the second part of Proposition 3.2.1. □

COROLLARY 3.2.2. *We have the following exact sequence*

$$\bigoplus_{x\in X^{d-1}} H^{2d-1}_x(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \to \bigoplus_{x\in X^d} H^{2d}_x(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \to H^{2d}(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \to 0.$$

Proof. By the above proposition, we have $E^{d,d}_{2,\text{ét}}(j_!\mu_{\ell^m}^{\otimes d}) = H^{2d}(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d})$. □

Using the Galois symbol maps and induction on codimension, Sato constructs the localized Chern class map and proves the following theorem.

THEOREM 3.2.3. [Sat09, Theorem 1.2 and Section 3] *For any $x \in X^a$, there exists a canonical surjective map*

$$\mathrm{cl}^{d,\mathrm{loc}}_{X,D,x,\ell^m} : H^a_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})/\ell^m \twoheadrightarrow H^{d+a}_x(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}),$$

which is called the localized Chern class map. Moreover, if $x \in X^d$, the localized Chern class map

$$\mathrm{cl}^{d,\mathrm{loc}}_{X,D,x,\ell^m} : H^d_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})/\ell^m \xrightarrow{\cong} H^{2d}_x(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d})$$

is bijective.

COROLLARY 3.2.4. *There is a canonical isomorphism*

$$H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})/\ell^m \cong H^{2d}(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}).$$

Proof. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\bigoplus_{x\in X^{d-1}} H^{d-1}_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})/\ell^m \arrow[r] \arrow[d, "\mathrm{cl}^{d,\mathrm{loc}}_{X,D,x,\ell^m}"] & \bigoplus_{x\in X^d} H^d_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})/\ell^m \arrow[r] \arrow[d, "\mathrm{cl}^{d,\mathrm{loc}}_{X,D,x,\ell^m}", "\cong"'] & H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})/\ell^m \arrow[r] \arrow[d, dashed] & 0 \\
\bigoplus_{x\in X^{d-1}} H^{2d-1}_x(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \arrow[r] & \bigoplus_{x\in X^d} H^{2d}_x(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \arrow[r] & H^{2d}(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \arrow[r] & 0,
\end{tikzcd}$$

where the first exact row follows from the exact sequence (3.2.1) by tensoring with $\mathbb{Z}/\ell^m\mathbb{Z}$, the second is Corollary 3.2.2. By Theorem 3.2.3 the first vertical arrow is surjective and the second is bijective. Then the assertion follows from an easy diagram chasing. $\square$

THEOREM 3.2.5. [Sai89, Lemma 2.9] *There is a perfect pairing of finite $\mathbb{Z}/\ell^m\mathbb{Z}$-modules*

$$H^i(U_{\text{ét}}, \mu_{\ell^m}^{\otimes r}) \times H^{2d+1-i}(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d-r}) \to H^{2d+1}(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d}) \xrightarrow{\cong} \mathbb{Z}/\ell^m\mathbb{Z}.$$

In particular, in case $i = 1, r = 0$, we obtain

$$H^d(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d})/\ell^m \cong \pi_1^{\text{ab}}(U)/\ell^m. \tag{3.2.4}$$

In summary:

COROLLARY 3.2.6. *We obtain canonical isomorphisms*

$$C(X, D)/\ell^m \overset{\rho_{X,D}}{\cong} H^d(X_{\text{Nis}}, \mathcal{K}^M_{d,X|D})/\ell^m \cong \pi_1^{\text{ab}}(U)/\ell^m.$$

3.3 The p-primary part

In this subsection we want to compare the group $H^d(X_{\text{Nis}}, \mathcal{K}^M_{d,X|D})/p^m$ with the group $H^d(X_{\text{ét}}, W_m\Omega^d_{X|D,\log})$.

The coniveau spectral sequence for a p-primary étale (resp. Nisnevich) sheaf $\mathcal{F}$ on $X_{\text{ét}}$ (resp. X_{Nis}) writes

$$E_{1,\text{ét}}^{p,q}(\mathcal{F}) := \bigoplus_{x \in X^p} H_x^{p+q}(X_{\text{ét}}, \mathcal{F}) \Longrightarrow H^{p+q}(X_{\text{ét}}, \mathcal{F})$$
$$E_{1,\text{Nis}}^{p,q}(\mathcal{F}) := \bigoplus_{x \in X^p} H_x^{p+q}(X_{\text{Nis}}, \mathcal{F}) \Longrightarrow H^{p+q}(X_{\text{Nis}}, \mathcal{F}).$$

We know that $E_{1,\text{ét}}^{p,q}(\mathcal{F}) = 0$ if $q > 1$ or $p > d$, and $E_{1,\text{Nis}}^{p,q}(\mathcal{F}) = 0$ if $q > 0$ or $p > d$.

THEOREM 3.3.1. *The canonical map*

$$H^d(X_{\text{Nis}}, W_m\Omega^d_{X|D,\log}) \xrightarrow{\cong} H^d(X_{\text{ét}}, W_m\Omega^d_{X|D,\log})$$

is an isomorphism.

Proof. By the coniveau spectral sequences, it follows from the following two propositions. $\square$

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PROPOSITION 3.3.2. *Let X be a smooth (not necessarily proper) variety over a finite field of dimension d. The map $E_{1,\text{ét}}^{\bullet,1}(W_m\Omega^d_{X|D,\log}) \xrightarrow{\cong} E_{1,\text{ét}}^{\bullet,1}(W_m\Omega^d_{X,\log})$ is an isomorphism of complexes. Therefore we have $E_{2,\text{ét}}^{d-1,1}(W_m\Omega^d_{X|D,\log}) = E_{2,\text{ét}}^{d-2,1}(W_m\Omega^d_{X|D,\log}) = 0$.*

Proof. For $x \in X^a$, we denote by $X_x := \operatorname{Spec}(\mathcal{O}^h_{X,x})$ the Henselization of X at x, and $Y_x := X_x \setminus \{x\}$. We want to prove that

$$H^{a+1}_x(X, W_m\Omega^d_{X|D,\log}) \cong H^{a+1}_x(X, W_m\Omega^d_{X,\log}).$$

If $a = 1$, then any divisor of X_x must have support in the closed point $\{x\}$. Therefore, we have

$$W_m\Omega^d_{X|D,\log}|_{Y_x} = W_m\Omega^d_{X,\log}|_{Y_x}$$

by the definition of $W_m\Omega^d_{X|D,\log}$. Using the localization exact sequences twice, we obtain

$$\begin{tikzcd}
H^1(X_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) \arrow[r] \arrow[d] & H^1(Y_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) \arrow[r] \arrow[d, equal] & H^2_x(X_{\text{ét}}, W_m\Omega^d_{X|D,\log}) \arrow[r] \arrow[d] & 0 \\
H^1(X_{x,\text{ét}}, W_m\Omega^d_{X,\log}) \arrow[r] & H^1(Y_{x,\text{ét}}, W_m\Omega^d_{X,\log}) \arrow[r] & H^2_x(X_{\text{ét}}, W_m\Omega^d_{X,\log}) \arrow[r] & 0
\end{tikzcd}$$

We claim that the first vertical arrow is surjective: Indeed, we have the exact sequence

$$\begin{aligned} H^1(X_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) &\to H^1(X_{x,\text{ét}}, W_m\Omega^d_{X,\log}) \\ &\to H^1(X_{\text{ét}}, W_m\Omega^d_{X,\log}/W_m\Omega^d_{X|D,\log}), \end{aligned}$$

where $H^1(X_{\text{ét}}, W_m\Omega^d_{X,\log}/W_m\Omega^d_{X|D,\log}) = 0$ since this sheaf is a successive extension of coherent sheaves by Proposition 2.2.5. We conclude that the third vertical map in the previous commutative diagram is an isomorphism.

For general codimension $a > 1$, we prove this by induction. The coniveau spectral sequence on Y_x gives us the exact sequence

$$\begin{aligned} \bigoplus_{y \in Y_x^{a-2}} H^{a-1}_y(Y_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) &\to \bigoplus_{y \in Y_x^{a-1}} H^a_y(Y_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) \\ &\to H^a(Y_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) \to 0. \end{aligned} \tag{3.3.1}$$

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On the other hand, the localization exact sequence for $W_m\Omega^d_{X|D,\log}$ on X_x tells us

$$H^a(Y_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) \cong H^{a+1}_x(X_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}), \tag{3.3.2}$$

since we know that $H^{a+1}(X_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) \cong H^{a+1}(x_{\text{ét}}, W_m\Omega^d_{X|D,\log}) = 0$ and similarly $H^a(X_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) \cong H^a(x_{\text{ét}}, W_m\Omega^d_{X|D,\log}) = 0$. Combining these facts, we get the following diagram with exact rows:

$$\begin{tikzcd}
\bigoplus_{y\in Y_x^{a-2}} H^{a-1}_y(Y_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) \arrow[r] \arrow[d] & \bigoplus_{y\in Y_x^{a-1}} H^{a}_y(Y_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) \arrow[r] \arrow[d] & H^{a+1}_x(X_{x,\text{ét}}, W_m\Omega^d_{X|D,\log}) \arrow[r] \arrow[d] & 0 \\
\bigoplus_{y\in X_x^{a-2}} H^{a-1}_y(X_{x,\text{ét}}, W_m\Omega^d_{X,\log}) \arrow[r] & \bigoplus_{y\in X_x^{a-1}} H^{a}_y(X_{x,\text{ét}}, W_m\Omega^d_{X,\log}) \arrow[r] & H^{a+1}_x(X_{x,\text{ét}}, W_m\Omega^d_{X,\log}) \arrow[r] & 0
\end{tikzcd}$$

The first two vertical maps are isomorphisms by induction. Hence the third vertical arrow is also an isomorphism. Thanks to [JSS14, Theorem 4.11.1], we see that the complex $E^{\bullet,1}_{1,\text{ét}}(W_m\Omega^d_{X,\log})$ is the Kato complex of $W_m\Omega^d_{X,\log}$ (cf. [KS12, (0.2)]) up to a sign. By the known Kato conjecture on vanishing of the cohomology groups of this complex at places $d-1$ and $d-2$ (cf. [JS03]), we obtain the second part of Proposition 3.3.2. ∎

Proposition 3.3.3. *Let X be a smooth (not necessarily proper) over a finite field k of dimension d. For any $x \in X^a$, the canonical map*

$$H^a_x(X_{\text{Nis}}, W_m\Omega^d_{X|D,\log}) \to H^a_x(X_{\text{ét}}, W_m\Omega^d_{X|D,\log}) \tag{3.3.3}$$

is an isomorphism.

That is, there is a natural isomorphism of complexes

$$E^{\bullet,0}_{1,\text{Nis}}(W_m\Omega^d_{X|D,\log}) \xrightarrow{\cong} E^{\bullet,0}_{1,\text{ét}}(W_m\Omega^d_{X|D,\log})$$

Proof. To prove this, we use Proposition 2.2.5(ii). We reduced to the case that D is reduced, since the quotient $W_m\Omega^d_{X|D}/W_m\Omega^d_{X|D_{\text{red}}}$ on X_{Nis} is a successive extension of coherent sheaves, for which the étale and Nisnevich cohomology groups are the same. By Proposition 2.2.5(i), it is equivalent to show that the canonical map

$$H^a_x(X_{\text{Nis}}, W_m\Omega^d_{X,\log}) \xrightarrow{\cong} H^a_x(X_{\text{ét}}, W_m\Omega^d_{X,\log})$$

is an isomorphism. This is true since both are isomorphic to $K^M_{d-a}(k(x))/p^m = W_m\Omega^{d-a}_{x,\log}$ by purity [Mil86, Proposition 2.1] and the known Gersten conjecture [GS88]. ∎

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COROLLARY 3.3.4. *There is a canonical isomorphism*

$$H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})/p^m \cong H^d(X_{\text{ét}}, W_m\Omega^d_{X|D,\log}).$$

Proof. First we have

$$\begin{aligned} H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})/p^m &\cong H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D}/p^m) \\ &\cong H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D}/p^m\mathcal{K}^M_{d,X} \cap \mathcal{K}^M_{d,X|D}), \end{aligned}$$

where the first isomorphism is due to the fact that the Nisnevich cohomological dimension of X is d, and the second follows from the observation that the support of $p^m\mathcal{K}^M_{d,X} \cap \mathcal{K}^M_{d,X|D}/p^m\mathcal{K}^M_{d,X|D}$ is contained in D, which is of dimension $d-1$.

By Theorems 2.2.2 and 3.3.1, hence we have

$$H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})/p^m \cong H^d(X_{\mathrm{Nis}}, W_m\Omega^d_{X|D,\log}) \cong H^d(X_{\text{ét}}, W_m\Omega^d_{X|D,\log}). \quad \square$$

COROLLARY 3.3.5. *Let D_1, D_2 be two effective divisors on X whose supports are simple normal crossing divisors. Assume $D_1 \geqslant D_2$. Then the canonical map*

$$H^d(X_{\text{ét}}, W_m\Omega^d_{X|D_1,\log}) \to H^d(X_{\text{ét}}, W_m\Omega^d_{X|D_2,\log})$$

is surjective.

Proof. Note that we have the following exact sequence on X_{Nis}

$$0 \to \mathcal{K}^M_{d,X|D_1} \to \mathcal{K}^M_{d,X|D_2} \to \mathcal{K}^M_{d,X|D_2}/\mathcal{K}^M_{d,X|D_1} \to 0,$$

but the Nisnevich sheaf $\mathcal{K}^M_{d,X|D_2}/\mathcal{K}^M_{d,X|D_1}$ is supported in D_2, which is of dimension $d-1$. Hence the associated long exact sequence implies that

$$H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D_1}) \to H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D_2})$$

is surjective. Therefore the claim follows from Corollary 3.3.4. $\square$

Now, we recall the duality theorem of the relative logarithmic de Rham–Witt sheaves.

THEOREM 3.3.6. [JSZ18, Theorem 4.1.4] *Let X, U, D be as before. For $i \in \mathbb{N}, r \in \mathbb{N}$, there are natural perfect pairings of topological groups*

$$\begin{aligned} &H^i(U_{\text{ét}}, W_m\Omega^r_{U,\log}) \times \varprojlim_{\substack{E \\ \mathrm{Supp}(E)\subset X\setminus U}} H^{d+1-i}(X_{\text{ét}}, W_m\Omega^{d-r}_{X|E,\log}) \\ &\to H^{d+1}(X_{\text{ét}}, W_m\Omega^d_{X,\log}) \xrightarrow{\mathrm{Tr}} \mathbb{Z}/p^m\mathbb{Z}, \end{aligned}$$

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where the first group is endowed with discrete topology, the second is endowed with profinite topology, and the limit with respect to all effective divisor E with $\mathrm{Supp}(E) \subset X \setminus U$.

In particular, for $i = 1$ and $r = 0$ we get isomorphisms

$$\varprojlim_E H^d(X_{\text{ét}}, W_m\Omega^d_{X|E,\log}) \xrightarrow{\cong} H^1(U_{\text{ét}}, \mathbb{Z}/p^m\mathbb{Z})^\vee \cong \pi_1^{ab}(U)/p^m,$$

and

$$H^1(U_{\text{ét}}, \mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\cong} \varinjlim_E H^d(X_{\text{ét}}, W_m\Omega^d_{X|E,\log})^\vee,$$

where $A^\vee$ is the Pontryagin dual of a topological abelian group A. These isomorphisms can be used to define a measure of ramification for étale abelian covers of U whose degree divides p^m.

Definition 3.3.7. For our divisor D, we define

$$\mathrm{Fil}_D H^1(U_{\text{ét}}, \mathbb{Z}/p^m\mathbb{Z}) := H^d(X_{\text{ét}}, W_m\Omega^d_{X|D,\log})^\vee.$$

Dually we define

$$\pi_1^{\mathrm{ab}}(X, D)/p^m := \mathrm{Hom}(\mathrm{Fil}_D H^1(U_{\text{ét}}, \mathbb{Z}/p^m\mathbb{Z}), \mathbb{Z}/p^m\mathbb{Z}).$$

The group $\pi_1^{\mathrm{ab}}(X, D)/p^m$ is a quotient of $\pi_1^{\mathrm{ab}}(U)/p^m$, which can be thought of as classifying abelian étale coverings of U whose degree divides p^m with ramification bounded by D.

Corollary 3.3.8. *We have canonical isomorphisms*

$$C(X, D)/p^m \cong H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})/p^m \xrightarrow{\cong} \pi_1^{\mathrm{ab}}(X, D)/p^m.$$

Proof. This is a consequence of Theorem 3.3.6 and Corollary 3.3.4. □

3.4 Class field theory via ideles

Theorem 3.4.1. (Logarithmic version of wildly ramified class field theory) *For any integer n, there exists a canonical isomorphism*

$$\rho_{X,D,n} \colon C(X, D)/n \xrightarrow{\cong} \pi_1^{\mathrm{ab}}(X, D)/n,$$

such that the following triangle commutes

$$\begin{tikzcd} & \bigoplus_{x\in X_0} \mathbb{Z} \arrow[dl, "\imath"'] \arrow[dr, "\rho_U"] & \\ C(X, D)/n \arrow[rr, "\rho_{X,D}/n"] & & \pi_1^{\mathrm{ab}}(U)/n \end{tikzcd}$$

where the right diagonal map ρ_U sends 1 *at the point x to the Frobenius* Frob_x. *In particular, $\rho_{X,D,n}$ induces an isomorphism*

$$\varprojlim_{D,n} C(X, D)/n \cong \pi_1^{\mathrm{ab}}(U). \tag{3.4.1}$$

Proof. For $n = p^m$, this follows from Corollary 3.3.8 and Theorem 3.1.1 directly. For n prime to p, this is Corollary 3.2.6. □

REMARK 3.4.2. The wildly ramified class field theory in [KS16], where we work with the relative Chow group of zero cycles instead of the idelic class group, comprises Theorem 3.4.1.

§4. Class field theory for complete local rings over $\mathbb{F}_q$

Let $(A, \mathfrak{m})$ be a complete regular local ring of dimension d and of characteristic $p > 0$, and let $k := A/\mathfrak{m}$ be the residue field. We assume that k is finite. We denote $X = \mathrm{Spec}(A)$, $x = \mathfrak{m} \in X$. Let D be an effective divisor with $\mathrm{Supp}(D)$ is a simple normal crossing divisor, let $U = X \setminus D$ be its complement. Set $X' = X \setminus \{x\}$, $D' = D \setminus \{x\}$. We use the dimension function on X (hence also induces one on X') by $d(x) = \dim(\overline{\{x\}})$.

4.1 Grothendieck's local duality

We know that the sheaf Ω_X^d is a dualizing sheaf of X. There exists a natural homomorphism called the residue homomorphism [KCD08, Section 5]:

$$\mathrm{res}\colon H_x^d(X, \Omega_X^d) \to k.$$

By compositing with the trace map $\mathrm{Tr}_{k/\mathbb{F}_p}\colon k \to \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, we get the map

$$\mathrm{Tr}_{k/\mathbb{F}_p} \circ \mathrm{res}\colon H_x^d(X, \Omega_X^d) \to \mathbb{Z}/p\mathbb{Z}.$$

For any finite A-module M, the Yoneda pairing and the above trace map give us a canonical pairing

$$H_x^i(X, M) \times \mathrm{Ext}_X^{d-i}(M, \Omega_X^d) \to \mathbb{Z}/p\mathbb{Z}. \tag{4.1.1}$$

THEOREM 4.1.1. (Grothendieck local duality [GH67]) *For each integer $i \geqslant 0$, the pairing (4.1.1) induces the isomorphisms*

$$\mathrm{Ext}_A^{d-i}(M, \Omega_X^d) \cong \mathrm{Hom}_{\mathbb{Z}/p\mathbb{Z}}(H_x^i(X, M), \mathbb{Z}/p\mathbb{Z}),$$
$$H_x^i(X, M) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Ext}_A^{d-i}(M, \Omega_X^d), \mathbb{Z}/p\mathbb{Z}),$$

where $\mathrm{Hom}_{\mathrm{cont}}$ *denotes the set of continuous homomorphisms with respect to $\mathfrak{m}$-adic topology on Ext group.*

In particular, if M is a locally free A-module, we obtain the isomorphisms

$$H^{d-i}(M^t) \cong \mathrm{Hom}_{\mathbb{Z}/p\mathbb{Z}}(H^i_x(X, M), \mathbb{Z}/p\mathbb{Z}), \tag{4.1.2}$$

where $M^t := \mathrm{Hom}_A(M, \Omega^d_X)$ is the dual A-module, and

$$H^i_x(X, M) \cong \mathrm{Hom}_{\mathrm{cont}}(H^{d-i}(M^t), \mathbb{Z}/p\mathbb{Z}). \tag{4.1.3}$$

Note that, for a locally free A-module M, we have [GH67]

$$H^i_x(X, M) = 0 \quad \text{if } i \neq d. \tag{4.1.4}$$

4.2 Duality theorems

The purity result of Shiho [Shi07, Theorem 3.2] tells us that there exists a canonical isomorphism

$$\mathrm{Tr}\colon H^{d+1}_x(X_{\text{ét}}, W_m\Omega^d_{X,\log}) \xrightarrow{\cong} H^1(x, \mathbb{Z}/p^m\mathbb{Z}) \cong \mathbb{Z}/p^m\mathbb{Z}. \tag{4.2.1}$$

Using the same method as in [Zha16], we obtain a map

$$\Phi^{i,r}_m\colon H^i(U_{\text{ét}}, W_m\Omega^r_{U,\log}) \to \varinjlim_E \mathrm{Hom}_{\mathbb{Z}/p^n\mathbb{Z}} \times (H^{d+1-i}_x(X_{\text{ét}}, W_m\Omega^{d-r}_{X|E,\log}), H^{d+1}_x(X_{\text{ét}}, W_m\Omega^d_{X,\log})).$$

If we endow $H^i(U_{\text{ét}}, W_m\Omega^r_{U,\log})$ with the discrete topology and endow the inverse limit $\varprojlim_E H^{d+1-i}_x(X_{\text{ét}}, W_m\Omega^{d-r}_{X|E,\log})$ with the profinite topology, where E runs over the set of effective divisors with support on $X \setminus U$, then the (continuous) map $\Phi^{i,r}_m$ and the trace map (4.2.1) induce a pairing of topological abelian groups:

$$H^i(U_{\text{ét}}, W_m\Omega^r_{U,\log}) \times \varprojlim_E H^{d+1-i}_x(X_{\text{ét}}, W_m\Omega^{d-r}_{X|E,\log}) \to \mathbb{Z}/p^m\mathbb{Z}. \tag{4.2.2}$$

Using Pontryagin duality, we see that $\Phi^{i,r}_m$ is an isomorphism if and only if the pairing (4.2.2) is a perfect pairing of topological abelian groups for the respective i, m, r.

THEOREM 4.2.1. *For any integers $r \geqslant 0, m \geqslant 1$, the maps $\Phi^{i,r}_m$ are isomorphisms.*

Proof. We are reduced to the case $m=1$ by induction on m and the following two exact sequences on the small étale site

$$0 \to W_{m-1}\Omega^r_{U,\log} \xrightarrow{\cdot p} W_m\Omega^r_{U,\log} \xrightarrow{R} \Omega^r_{U,\log} \to 0$$

and

$$0 \to W_{m-1}\Omega^{d-r}_{X|[E/p],\log} \xrightarrow{\cdot p} W_m\Omega^{d-r}_{X|E,\log} \xrightarrow{R} \Omega^{d-r}_{X|E,\log} \to 0,$$

where $[E/p]=\sum_{\lambda\in\Lambda}[n_\lambda/p]D_\lambda$ if $D=\sum_{\lambda\in\Lambda} n_\lambda D_\lambda$; here $[n/p]=\min\{n' \in \mathbb{Z} | pn' \geqslant n\}$, and the exactness of the second complex follows from [JSZ18, Theorem 1.1.6].

Using the relation between logarithmic forms and differential forms ([Ill79, 0, Corollary 2.1.18] and [JSZ18, Theorem 1.2.1]), we see that the assertion for $i \neq 0, 1$ follows from the vanishing (4.1.4) directly. We have the following diagram with exact rows

$$\begin{tikzcd}
H^0(U_{\mathrm{et}}, \Omega^r_{U,\log}) \arrow[r, hook] \arrow[d] & H^0(U, Z\Omega^r_U) \arrow[r] \arrow[d] & H^0(U, \Omega^r_U) \arrow[r, two heads] \arrow[d] & H^1(U_{\mathrm{et}}, \Omega^r_{X,\log}) \arrow[d] \\
\varinjlim_E H^{d+1}_x(X_{\mathrm{\acute{e}t}}, \Omega^{d-r}_{X|E,\log})^* \arrow[r, hook] & \varinjlim_E H^d_x(X_{\mathrm{\acute{e}t}}, \Omega^{d-r}_{X|E}/d\Omega^{d-r-1}_{X|E})^* \arrow[r] & \varinjlim_E H^d_x(X_{\mathrm{\acute{e}t}}, \Omega^{d-r}_{X|E})^* \arrow[r, two heads] & \varinjlim_E H^d_x(X_{\mathrm{\acute{e}t}}, \Omega^{d-r}_{X|E,\log})^*
\end{tikzcd}$$

where $A^* := \mathrm{Hom}_{\mathbb{Z}/p\mathbb{Z}}(A, \mathbb{Z}/p\mathbb{Z})$ for an abelian group A,

$$\Omega^{d-r}_{X|E} := \Omega^{d-r}_X(\log E_{\mathrm{red}}) \otimes \mathcal{O}_X(-E),$$

and $d\Omega^{d-r-1}_{X|E} := \mathrm{Image}(d\colon \Omega^{d-r-1}_{X|E} \to \Omega^{d-r}_X)$, and $Z\Omega^r_U := \mathrm{Ker}(d\colon \Omega^r_U \to \Omega^{r+1}_U)$.

The proof is same as the proof in [JSZ18, Zha16], we quickly recall the argument: since $j\colon U \to X$ is affine, we may rewrite $H^0(U, \Omega^r_U)$ as $\varinjlim_E H^0(X, \Omega^i_X(\log E_{\mathrm{red}}) \otimes \mathcal{O}_X(E))$. Then we use Theorem 4.1.1 for sheaves $\Omega^i_X(\log E_{\mathrm{red}})(-E)$ to conclude that the second and the third vertical arrows are isomorphisms. Hence the assertion follows. ∎

For $r=0, i=1$, we get

$$H^1(U_{\mathrm{\acute{e}t}}, \mathbb{Z}/p^m\mathbb{Z}) \cong \varinjlim_E \mathrm{Hom}(H^d_x(X_{\mathrm{\acute{e}t}}, W_m\Omega^d_{X|E,\log}), \mathbb{Z}/p^m\mathbb{Z}).$$

Similar to Corollary 3.3.5, the transition maps are surjective in the projective system, for our divisor D we define

$$\mathrm{Fil}_D H^1(U_{\mathrm{\acute{e}t}}, \mathbb{Z}/p^m\mathbb{Z}) := \mathrm{Hom}(H^d_x(X_{\mathrm{\acute{e}t}}, W_m\Omega^d_{X|D,\log}), \mathbb{Z}/p^m\mathbb{Z});$$

by Pontryagin duality, we also define

$$\pi_1^{\mathrm{ab}}(X,D)/p^m := \mathrm{Hom}(\mathrm{Fil}_D H^1(U_{\text{ét}},\mathbb{Z}/p^m\mathbb{Z}),\mathbb{Z}/p^m\mathbb{Z}).$$

Theorem 4.2.1 gives us an isomorphism

$$H^d_x(X_{\text{ét}}, W_m\Omega^d_{X|D,\log}) \xrightarrow{\cong} \pi_1^{\mathrm{ab}}(X,D)/p^m.$$

PROPOSITION 4.2.2. *We have*

$$H^d_x(X_{\mathrm{Nis}}, W_m\Omega^d_{X|D,\log}) \cong H^d_x(X_{\text{ét}}, W_m\Omega^d_{X|D,\log}).$$

Proof. This is similar to the argument in the proof of Proposition 3.3.3. Only the last step, to claim

$$H^a_x(X_{\mathrm{Nis}}, W_m\Omega^d_{X,\log}) \xrightarrow{\cong} H^a_x(X_{\text{ét}}, W_m\Omega^d_{X,\log})$$

is an isomorphism, uses different results. In this case, it is an isomorphism since both are isomorphic to $K^M_{d-a}(k(x))/p^m = W_m\Omega^{d-a}_{x,\log}$ by purity [Shi07, Theorem 3.2] and the known Gersten conjecture [Ker09]. ∎

4.3 Class field theory via ideles

For a complete regular local ring A of dimension d of characteristic $p>0$, and X, X', U, D, D' as before. An idelic description of $H^d_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})$ is given by the following theorem.

THEOREM 4.3.1. [Ker11, Theorem 8.2] *There exists an isomorphism*

$$C(X',D') \cong H^d_x(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D}).$$

In summary, the class field theory of Henselian regular local ring over $\mathbb{F}_p$ can be reformulated as follows:

COROLLARY 4.3.2. *There is a canonical isomorphism*

$$C(X',D')/p^m \xrightarrow{\cong} \pi_1^{\mathrm{ab}}(X,D)/p^m.$$

REMARK 4.3.3. The case $d=2$ has been studied in [Sai87]. The case $d=3$ has been investigated in [Mat02] using a slightly different class group. The ℓ-primary analog has been studied by Sato in [Sat09].

https://doi.org/10.1017/nmj.2018.34 Published online by Cambridge University Press

§5. Class field theory for schemes over discrete valuation rings

Let R be a Henselian discrete valuation ring with fraction field K, and let k be its residue field of characteristic $p > 0$ which we assume to be finite. We fix an uniformizer π of R. We use the notation as in the following diagram:

$$\begin{tikzcd}
X_s \arrow[r, hook, "i"] \arrow[d, "f_s"] & X \arrow[d, "f"] & X_\eta \arrow[l, hook', "j"'] \arrow[d, "f_\eta"] \\
s = \mathrm{Spec}(k) \arrow[r, hook, "i_s"] & B = \mathrm{Spec}(R) & \eta = \mathrm{Spec}(K) \arrow[l, hook', "j_\eta"']
\end{tikzcd}$$

where f is a flat projective of fiber dimension d. We assume that X is a regular scheme with smooth generic fiber X_η such that the reduced special fiber $X_{s,\mathrm{red}}$ is a simple normal crossing divisor. Let $\jmath\colon U \hookrightarrow X$ be an open subscheme contained in the generic fiber such that $X \setminus U$ is the support of a simple normal crossing divisor D.

5.1 Idele class group

We want to give an idelic description of the class group $H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X|D})$. We use the dimension function $d(x) = \dim(\overline{\{x\}})$ on X.

Definition 5.1.1.

(i) A Q^o-chain on $(U \subset X)$ is a Q-chain

$$P = (p_0, \ldots, p_{s-2}, p_s)$$

on $(U \subset X)$ such that $s \geqslant 2$. We denote the set of Q^o-chain on $(U \subset X)$ by $\mathcal{Q}^o$.

(ii) The idele class group $C(U \subset X; X_s)$ is

$$C(U \subset X; X_s) := \mathrm{Coker}\Bigg(\bigoplus_{P \in \mathcal{Q}^o} K^M_{d(P)}(k(P)) \oplus \bigoplus_{y \in U^{d-1}_\eta} K^M_2(k(y)) \xrightarrow{Q} I(U \subset X) \Bigg);$$

(iii) The idele class group $C(X, D; X_s)$ of X relative to the effective divisor D is defined as

$$C(X, D; X_s) := \operatorname{Coker}\Big(\bigoplus_{P\in \mathcal{Q}^o} K^M_{d(P)}(k(P)) \oplus \bigoplus_{y\in U_\eta^{d-1}} K^M_2(k(y)) \xrightarrow{Q} I(X, D)\Big).$$

THEOREM 5.1.2.

(i) *There exists a canonical isomorphism*

$$C(X, D; X_s) \cong H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D}).$$

(ii) $H^{d+1}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D}) = 0.$

Proof. Let $\mathcal{F}$ be the Nisnevich sheaf $\mathcal{K}^M_{d+1,X|D}$. We start with part (i). We have seen that the degeneration of the coniveau spectral sequence

$$E^{p,q}_{1,\mathrm{Nis}}(\mathcal{F}) := \bigoplus_{x\in X^p} H^{p+q}_x(X_{\mathrm{Nis}}, \mathcal{F}) \Longrightarrow H^{p+q}(X_{\mathrm{Nis}}, \mathcal{F})$$

implies

$$H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{F}) = \operatorname{Coker}\Big(\bigoplus_{x\in X_1\cap X_s} H^d_x(X_{\mathrm{Nis}}, \mathcal{F}) \to \bigoplus_{x\in X_0} H^{d+1}_x(X_{\mathrm{Nis}}, \mathcal{F})\Big). \tag{5.1.1}$$

By definition and [Ker11, Theorem 8.2] we obtain an isomorphism

$$C(X, D; X_s) \cong \operatorname{Coker}\Big(\bigoplus_{y\in U_\eta^{d-1}} K^M_2(k(y)) \to \bigoplus_{x\in X_0} H^{d+1}_x(X_{\mathrm{Nis}}, \mathcal{F})\Big). \tag{5.1.2}$$

It is sufficient to observe that the canonical map

$$\bigoplus_{y\in U_\eta^{d-1}} K^M_2(k(y)) \to \bigoplus_{x\in X_1\cap X_s} H^d_x(X_{\mathrm{Nis}}, \mathcal{F})$$

is surjective; see [Ker11, Section 6]. This finishes the proof of part (i).

For part (ii) we use the isomorphism

$$H^{d+1}(X_{\mathrm{Nis}}, \mathcal{F}) = \operatorname{Coker}\Big(\bigoplus_{x\in X_1} H^d_x(X_{\mathrm{Nis}}, \mathcal{F}) \to \bigoplus_{x\in X_0} H^{d+1}_x(X_{\mathrm{Nis}}, \mathcal{F})\Big)$$

and the surjectivity of

$$\bigoplus_{x\in X_1\cap X_\eta} K_1^M(k(x)) \to \bigoplus_{x\in X_0} H_x^{d+1}(X_{\mathrm{Nis}}, \mathcal{F});$$

see [Ker11, Section 6]. ∎

Note that the generic fiber X_η is a smooth variety over the local field K. Its class field theory has been studied in several cases, for example, the case $d=1$ is well understood by the work of Bloch and Saito; see [Sai85, Hir16]. In [For15], Forré determines the kernel of the reciprocity map in unramified ℓ-adic class field theory in the higher dimension case.

Definition 5.1.3. Assume $\mathrm{Supp}(D)\supset X_s$, we denote $D_\eta = D\times_X X_\eta$, and define

$$\widehat{SK}_1(U) := \varprojlim_D C(X, D; X_s) = \varprojlim_E H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|E}),$$

where the limit is over all effective divisors E with support $X\setminus U$.

$$SK_1(X_\eta, D_\eta) := H^d(X_{\eta,\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D}).$$

Remark 5.1.4.

(i) We have seen that, by the degeneration of the coniveau spectral sequence, the group $SK_1(X_\eta, D_\eta) = H^d(X_{\eta,\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D})$ is isomorphic to

$$\begin{aligned}&\mathrm{coker}\Bigg(\bigoplus_{y\in(X_\eta)_1} H_y^{d-1}(X_{\eta,\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D})\\ &\xrightarrow{\partial} \bigoplus_{x\in(X_\eta)_0} H_x^{d}(X_{\eta,\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D})\Bigg).\end{aligned} \tag{5.1.3}$$

Using the methods from [Ker11] it is easy to write down an idelic description of this group, for example, if $D_\eta=0$ then $SK_1(X_\eta,0)= SK_1(X_\eta)$ where $SK_1(X_\eta)$ is defined as

$$\mathrm{coker}\Bigg(\bigoplus_{y\in(X_\eta)_1} K_2^M(\kappa(y)) \xrightarrow{\partial} \bigoplus_{x\in(X_\eta)_0} \kappa(x)^\times\Bigg).$$

(ii) If $d=1$ and $\operatorname{Supp}(D)=X_s$, then $\widehat{SK}_1(U)=\widehat{SK}_1(X_\eta)$, which has been defined in [KS83] via the idelic method.

(iii) By Theorem 5.1.2 we get a canonical surjection

$$SK_1(X_\eta, D_\eta) \to C(X, D; X_s).$$

We do not know, whether this map is an isomorphism in general, but Theorem 5.3.7 suggests that it is so at least after tensoring with $\mathbb{Z}/n\mathbb{Z}$ for any integer $n>0$.

5.2 Kato complexes on simple normal crossing varieties

We recall notations and theorems in [JS03]. Let Y be a proper simple normal crossing variety over the finite field k of dimension d, and let $Y_1, \ldots, Y_N$ be its smooth irreducible components. Let

$$Y_{i_1,\ldots,i_s} := Y_{i_1} \times_Y \cdots \times_Y Y_{i_s}$$

be the scheme-theoretic intersection of $Y_{i_1}, \ldots, Y_{i_s}$, and denote

$$Y^{[s]} := \coprod_{1\leqslant i_1<\cdots<i_s\leqslant N} Y_{i_1,\ldots,i_s}$$

for the disjoint union of the s-fold intersections of the Y_i, for any $s>0$. Since Y is simple, all $Y^{[s]}$ are smooth of dimension $d-s+1$. The immersions $Y_{i_1,\ldots,i_s} \hookrightarrow Y$ and $Y_{i_1,\ldots,i_s} \hookrightarrow Y_{i_1,\ldots,\hat{i_\nu},\ldots,i_s}$ induce canonical maps

$$i^{[s]}\colon Y^{[s]} \to Y, \qquad \delta_\nu\colon Y^{[s]} \to Y^{[s-1]}.$$

For integer $n>0, i\geqslant 0$ we define the following étale sheaves on Y:

(i) If $p\nmid n$, then let $\mathbb{Z}/n\mathbb{Z}(i) := \mu_{n,Y}^{\otimes i}$ be the ith tensor power over $\mathbb{Z}/n\mathbb{Z}$ of the sheaf of nth roots of unity.

(ii) If $n=mp^r, r\geqslant 0$ with $p\nmid m$, then let

$$\mathbb{Z}/n\mathbb{Z}(i) := \nu_{r,Y}^i[-i] \oplus \mu_{m,Y}^{\otimes i}$$

where $\nu_{r,Y}^i(U) := \ker(\partial : \bigoplus_{x\in U^0} W_r\Omega^i_{x,\log} \to \bigoplus_{x\in U^1} W_r\Omega^{i-1}_{x,\log})$ for $U\subset Y$ open. Note that $\nu_{r,Y}^d = W_r\Omega^d_{Y,\log}$ if Y is smooth [Sat07, 1.3.2].

The Kato complex $C^{1,0}(Y,\mathbb{Z}/n\mathbb{Z}(d))$ is defined to be the complex:

$$\bigoplus_{y\in Y^0} H^{d+1}(y,\mathbb{Z}/n\mathbb{Z}(d)) \to \bigoplus_{y\in Y^1} H^{d}(y,\mathbb{Z}/n\mathbb{Z}(d-1)) \to \cdots$$
$$\cdots \to \bigoplus_{y\in Y^a} H^{d-a+1}(y,\mathbb{Z}/n\mathbb{Z}(d-a)) \to \cdots \to \bigoplus_{y\in Y^d} H^{1}(y,\mathbb{Z}/n\mathbb{Z}),$$

where $\mathbb{Z}/n\mathbb{Z}(i)$ is defined as above for the residue field of Y at y, and put the term $\bigoplus_{y\in Y^a}$ in degree $a-d$ as an object in derived category. Similarly, for each s, on $Y^{[s]}$ we define the complex $C^{1,0}(Y^{[s]},\mathbb{Z}/n\mathbb{Z}(d-s+1))$, and moreover we define the complex $C(Y^\bullet,\mathbb{Z}/n\mathbb{Z})$ as

$$\cdots \to (\mathbb{Z}/n\mathbb{Z})^{\pi_0(Y^{[s+1]})} \xrightarrow{d_s} (\mathbb{Z}/n\mathbb{Z})^{\pi_0(Y^{[s]})} \cdots \to (\mathbb{Z}/n\mathbb{Z})^{\pi_0(Y^{[1]})},$$

where $\pi_0(Z)$ is the set of connected components of a scheme Z, the last term of this complex is placed in degree 0, and the differential d_s is $\sum_{\nu=1}^{s+1}(-1)^{\nu+1}(\delta_\nu)_*$.

THEOREM 5.2.1. [JS03, Proposition 3.6 and Theorem 3.9]

(i) *There is a spectral sequence*

$$E^1_{s,t}(Y^\bullet,\mathbb{Z}/n\mathbb{Z}) = H_t(C^{1,0}(Y^{[s+1]},\mathbb{Z}/n\mathbb{Z}(d-s)))$$
$$\Rightarrow H_{s+t}(C^{1,0}(Y,\mathbb{Z}/n\mathbb{Z}(d)))$$

in which the differentials $d^1_{s,t}=\sum_{\nu=1}^{s+1}(-1)^{\nu+1}(\delta_\nu)_*$.

(ii) *We have* $E^1_{s,t}(Y,\mathbb{Z}/n\mathbb{Z})=0$ *if* $t<0$, *and hence there are canonical edge morphisms*

$$e_a^{\mathcal{Y},p^m}\colon H_a(C^{1,0}(Y,\mathbb{Z}/n\mathbb{Z}(d)))\to E^2_{a,0}(Y^\bullet,\mathbb{Z}/n\mathbb{Z}).$$

(iii) *The trace map induces a canonical isomorphism*

$$\mathrm{tr}\colon E^2_{a,0}(Y^\bullet,\mathbb{Z}/n\mathbb{Z})\to H_a(C(Y^\bullet,\mathbb{Z}/n\mathbb{Z})).$$

(iv) *The composite of edge and trace morphisms gives us a canonical map*

$$\gamma_a^{Y,p^m}\colon H_a(C^{1,0}(Y,\mathbb{Z}/n\mathbb{Z}(d)))\to H_a(C(Y^\bullet,\mathbb{Z}/n\mathbb{Z})),$$

which is an isomorphism if $0\leqslant a\leqslant 4$.

REMARK 5.2.2. In the following, we need the cases $a=1$ and $a=2$, which will give us an explicit description of E_2-terms of certain coniveau spectral sequences.

5.3 The ℓ-primary part

Let ℓ be a prime number and $\ell \neq p$. The cup product induces the following morphism

$$R\jmath_*\mu_{\ell^m,U}^{\otimes r} \to R\jmath_*\mathscr{H}\mathrm{om}_U(\mu_{\ell^m,U}^{\otimes d+1-r}, \mu_{\ell^m,U}^{\otimes d+1}).$$

As $\mu_{\ell^m,U}^{\otimes d+1} = \jmath^*\mu_{\ell^m,X}^{\otimes d+1}$ the adjoint pair $(\jmath_!, \jmath^*)$ gives an isomorphism

$$R\jmath_*R\mathscr{H}\mathrm{om}_U(\mu_{\ell^m,U}^{\otimes d+1-r}, \mu_{\ell^m,U}^{\otimes d+1}) = R\mathscr{H}\mathrm{om}_X(\jmath_!\mu_{\ell^m,U}^{\otimes d+1-r}, \mu_{\ell^m,X}^{\otimes d+1}).$$

Using the adjoint pair $(i_*, Ri^!)$ and these two maps above, we obtain a pairing on $X_{\text{ét}}$:

$$i^*R\jmath_*\mu_{\ell^m,U}^{\otimes r} \otimes^L Ri^!\jmath_!\mu_{\ell^m,U}^{\otimes d+1-r} \to Ri^!\mu_{\ell^m,X}^{\otimes d+1}. \tag{5.3.1}$$

Therefore a pairing of cohomology groups:

$$H^i(U_{\text{ét}}, \mu_{\ell^m,U}^{\otimes r}) \times H^j_{X_s}(X_{\text{ét}}, \jmath_!\mu_{\ell^m,U}^{\otimes d+1-r}) \to H^{i+j}_{X_s}(X_{\text{ét}}, \mu_{\ell^m,X}^{\otimes d+1}). \tag{5.3.2}$$

We have the following duality theorem; see [Gei10, Theorem 7.5].

THEOREM 5.3.1.

(i) *There is a canonical isomorphism, so-called the trace map,*

$$\mathrm{Tr}\colon H^{2d+3}_{X_s}(X_{\text{ét}}, \mu_{\ell^m,X}^{\otimes d+1}) \xrightarrow{\cong} \mathbb{Z}/\ell^m\mathbb{Z}$$

(ii) *The trace map Tr and the pair (5.3.2) induce a perfect pairing of finite groups*

$$H^i(U_{\text{ét}}, \mu_{\ell^m,U}^{\otimes r}) \times H^{2d+3-i}_{X_s}(X_{\text{ét}}, \jmath_!\mu_{\ell^m,U}^{\otimes d+1-r})$$
$$\to H^{2d+3}_{X_s}(X_{\text{ét}}, \mu_{\ell^m,X}^{\otimes d+1}) \xrightarrow{\mathrm{Tr}} \mathbb{Z}/\ell^m\mathbb{Z}.$$

For $r = 0$, $i = 1$, we obtain

$$H^1(U_{\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z}) \cong \mathrm{Hom}(H^{2d+2}_{X_s}(X_{\text{ét}}, \jmath_!\mu_{\ell^m}^{\otimes d+1}), \mathbb{Z}/\ell^m\mathbb{Z}),$$

and by Pontryagin duality

$$H^{2d+2}_{X_s}(X_{\text{ét}}, \jmath_!\mu_{\ell^m}^{\otimes d+1}) \cong \pi_1^{\mathrm{ab}}(U)/\ell^m. \tag{5.3.3}$$

For any abelian sheaf $\mathcal{F}$ on X_{Nis} or $X_{\text{ét}}$, we have the following two coniveau spectral sequences:

$$E^{p,q}_{1,\text{ét}}(\mathcal{F}) := \bigoplus_{x\in X^p\cap X_s} H^{p+q}_x(X_{\text{ét}}, \mathcal{F}) \Longrightarrow H^{p+q}_{X_s}(X_{\text{ét}}, \mathcal{F}),$$

$$E^{p,q}_{1,\mathrm{Nis}}(\mathcal{F}) := \bigoplus_{x\in X^p\cap X_s} H^{p+q}_x(X_{\mathrm{Nis}}, \mathcal{F}) \Longrightarrow H^{p+q}_{X_s}(X_{\mathrm{Nis}}, \mathcal{F}).$$

PROPOSITION 5.3.2.

(i) $E^{\bullet,d+2}_{1,\acute{e}t}(\jmath_!\mu^{\otimes d+1}_{\ell^m,U}) \cong E^{\bullet,d+2}_{1,\acute{e}t}(\mu^{\otimes d+1}_{\ell^m,X})$.

(ii) *The local Chern class map induces a surjection* $E^{\bullet,0}_{1,\mathrm{Nis}}(\mathcal{K}^M_{d+1,X|D})/\ell^m \twoheadrightarrow E^{\bullet,d+1}_{1,\acute{e}t}(\jmath_!\mu^{\otimes d+1}_{\ell^m,U})$ *and an isomorphism*

$$E^{d+1,0}_{1,\mathrm{Nis}}(\mathcal{K}^M_{d+1,X|D})/\ell^m \cong E^{d+1,d+1}_{1,\acute{e}t}(\jmath_!\mu^{\otimes d+1}_{\ell^m,U}).$$

Proof. The argument is analogous to that in Section 3.2. More precisely, part (i) corresponds to Proposition 3.2.1 and part (ii) corresponds to Theorem 3.2.3. ∎

COROLLARY 5.3.3. *There are canonical isomorphisms*

$$H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D})/\ell^m \cong E^{d+1,0}_{2,\mathrm{Nis}}(\mathcal{K}^M_{d+1,X|D})/\ell^m \cong E^{d+1,d+1}_{2,\acute{e}t}(\jmath_!\mu^{\otimes d+1}_{\ell^m,U}).$$

Proof. The degenerating coniveau spectral sequence on X_{Nis} gives the first isomorphism. The second isomorphism results from the same argument as in Corollary 3.2.4 using Proposition 5.3.2(ii). ∎

By purity the complex $E^{\bullet,d+2}_{1,\acute{e}t}(\mu^{\otimes d+1}_{\ell^m,X})$ is isomorphic to the complex Kato complex $C^{1,0}(X_s, \mathbb{Z}/\ell^m\mathbb{Z}(d))$ from Section 5.2 (up to a shift), that is, to

$$\bigoplus_{y\in X_s^0} H^{d+1}(y, \mathbb{Z}/\ell^m\mathbb{Z}(d)) \to \bigoplus_{y\in X_s^1} H^{d}(y, \mathbb{Z}/\ell^m\mathbb{Z}(d-1)) \to \cdots$$
$$\cdots \to \bigoplus_{y\in X_s^a} H^{d-a+1}(y, \mathbb{Z}/\ell^m\mathbb{Z}(d-a)) \to \cdots \to \bigoplus_{y\in X_s^d} H^{1}(y, \mathbb{Z}/\ell^m\mathbb{Z}),$$

where we set the last term in degree 0 as an object in the derived category.

THEOREM 5.3.4. *The canonical morphism*

$$H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D})/\ell^m \to H^{2d+2}_{X_s}(X_{\acute{e}t}, \jmath_!\mu^{\otimes d+1}_{\ell^m})$$

fits into an exact sequence

$$H_2(C(X_s^\bullet, \mathbb{Z}/\ell^m\mathbb{Z})) \to H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D})/\ell^m \to H^{2d+2}_{X_s}(X_{\acute{e}t}, \jmath_!\mu^{\otimes d+1}_{\ell^m})$$
$$\to H_1(C(X_s^\bullet, \mathbb{Z}/\ell^m\mathbb{Z})) \to 0.$$

Proof. By the coniveau spectral sequence for $\mathcal{F} = \jmath_!\mu^{\otimes d+1}_{\ell^m,U}$ on $X_{\acute{e}t}$, we have an exact sequence:

$$E^{d-1,d+2}_{2,\acute{e}t}(\mathcal{F}) \to E^{d+1,d+1}_{2,\acute{e}t}(\mathcal{F}) \to H^{2d+2}_{X_s}(X_{\acute{e}t}, \mathcal{F}) \to E^{d,d+2}_{2,\acute{e}t}(\mathcal{F}) \to 0.$$

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Using Proposition 5.3.2, we have

$$\begin{aligned}E_{2,\text{ét}}^{d+1,d+1}(\mathcal{F}) &= E_{2,\text{Nis}}^{d+1,0}(\mathcal{K}^M_{d+1,X|D}/\ell^m) = H^{d+1}_{X_s}(X_{\text{Nis}}, \mathcal{K}^M_{d+1,X|D}/\ell^m)\\ &= H^{d+1}_{X_s}(X_{\text{Nis}}, \mathcal{K}^M_{d+1,X|D})/\ell^m.\end{aligned}$$

Moreover combining with Theorem 5.2.1, we obtain

$$\begin{aligned}E_{2,\text{ét}}^{d-1,d+2}(\mathcal{F}) &= E_{2,\text{ét}}^{d-1,d+2}(\mu_{\ell^m,X}^{\otimes d+1}) = H_2(C(X_s^\bullet, \mathbb{Z}/\ell^m\mathbb{Z}));\\ E_{2,\text{ét}}^{d,d+2}(\mathcal{F}) &= E_{2,\text{ét}}^{d,d+2}(\mu_{\ell^m,X}^{\otimes d+1}) = H_1(C(X_s^\bullet, \mathbb{Z}/\ell^m\mathbb{Z})).\end{aligned}$$ ∎

In summary, combining Theorems 5.3.4 and 5.1.2 with the identification (5.3.3), we reformulate the ℓ-primary part of class field theory in this setting as follows.

THEOREM 5.3.5. *There is a canonical map*

$$\rho_{X,D}\colon C(X,D;X_s)/\ell^m \to \pi_1^{\text{ab}}(U)/\ell^m,$$

which fits into an exact sequence of finite groups

$$\begin{aligned}H_2(C(X_s^\bullet, \mathbb{Z}/\ell^m\mathbb{Z})) \to C(X,D;X_s)/\ell^m &\to \pi_1^{\text{ab}}(U)/\ell^m\\ &\to H_1(C(X_s^\bullet, \mathbb{Z}/\ell^m\mathbb{Z})) \to 0.\end{aligned}$$

Equivalently, there is an exact sequence:

$$\begin{aligned}H_2(C(X_s^\bullet, \mathbb{Z}/\ell^m\mathbb{Z})) \to \widehat{SK}_1(U)/\ell^m &\to \pi_1^{\text{ab}}(U)/\ell^m\\ &\to H_1(C(X_s^\bullet, \mathbb{Z}/\ell^m\mathbb{Z})) \to 0.\end{aligned}\tag{5.3.4}$$

Proof. The map is defined by the following diagram

$$\begin{tikzcd} C(X,D;X_s)/\ell^m \arrow[r, "\cong"] \arrow[d, dashed, "\rho_{X,D}"'] & H^{d+1}_{X_s}(X_{\text{Nis}}, \mathcal{K}^M_{d+1,X|D})/\ell^m \arrow[d] \\ \pi_1^{\text{ab}}(U)/\ell^m & H^{2d+2}_{X_s}(X_{\text{ét}}, j_!\mu_{\ell^m}^{\otimes d+1}) \arrow[l, "\cong"'] \end{tikzcd}$$

So the first exact sequence is a direct consequence of Theorem 5.3.4. The second exact sequence results from the fact that

$$\widehat{SK}_1(U)/\ell^m = H^{d+1}_{X_s}(X_{\text{Nis}}, \mathcal{K}^M_{d+1,X|D})/\ell^m \tag{5.3.5}$$

for any D with $\text{Supp}(D) = X \setminus U$. Indeed, we denote by $D_0 = X \setminus U$ the reduced divisor, it suffices to show the following claim. ∎

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Claim 5.3.6. We have

$$\Big(\varprojlim_D H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D})\Big) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^m\mathbb{Z} \xrightarrow{\cong} H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D_0})/\ell^m.$$

Proof of Claim. The canonical surjective map

$$\varphi_D\colon H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D}) \to H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D_0})$$

fits into the exact sequence
(5.3.6)

$$0 \longrightarrow \ker(\varphi_D) \longrightarrow H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D}) \xrightarrow{\varphi_D} H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D_0}) \longrightarrow 0$$

Applying $\varprojlim_D$ to the above exact sequence, we obtain an exact sequence
(5.3.7)

$$0 \longrightarrow \varprojlim_D \ker(\varphi_D) \longrightarrow \varprojlim_D H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D}) \longrightarrow H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D_0}) \longrightarrow 0.$$

By the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{K}^M_{d+1,X|D} \to \mathcal{K}^M_{d+1,X|D_0} \to \mathcal{K}^M_{d+1,X|D_0}/\mathcal{K}^M_{d+1,X|D} \to 0,$$

we see that $H^d_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D_0}/\mathcal{K}^M_{d+1,X|D}) \twoheadrightarrow \ker(\varphi_D)$ is surjective. Proposition 2.2.5(ii) tells us that $H^d_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D}/\mathcal{K}^M_{d+1,X|D_0})$ is p-primary torsion group, therefore in particular $\ker(\varphi_D)$ is a $\mathbb{Z}_{(p)}$-module, so is the inverse limit $\varprojlim_D \ker(\varphi_D)$. It follows that

$$\mathbb{Z}/\ell^m\mathbb{Z} \otimes_{\mathbb{Z}} \varprojlim_D \ker(\varphi_D) = 0.$$

Tensoring the exact sequence (5.3.7) with $\mathbb{Z}/\ell^m\mathbb{Z}$, we obtain the claim. $\square$

In the case that $\mathrm{Supp}(D) = X_s$, we have the following diagram:

$$\begin{tikzcd}
 & SK_1(X_\eta)/\ell^m \arrow[r, "\rho_{X_\eta}"] \arrow[d, "\phi"] & \pi_1^{\mathrm{ab}}(X_\eta)/\ell^m \arrow[d, equal] & & \\
H_2(C(X_s^\bullet, \mathbb{Z}/\ell^m\mathbb{Z})) \arrow[r] & \widehat{SK}_1(X_\eta)/\ell^m \arrow[r, "\rho_{X,X_s}"] & \pi_1^{\mathrm{ab}}(X_\eta)/\ell^m \arrow[r] & H_1(C(X_s^\bullet, \mathbb{Z}/\ell^m\mathbb{Z})) \arrow[r] & 0
\end{tikzcd}$$

where the last row is the exact sequence (5.3.4), the morphism ρ_{X_η} is the reciprocity map of variety over the local field K (cf. [KS83]),

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and the map ϕ is induced by the connection map $H^d(X_\eta, \mathcal{K}^M_{d+1,X_\eta}) \to H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D})$.

In the remainder of this subsection, we explain why our new approach recovers the known result for varieties over local fields (cf. [For15]) in the good reduction case.

THEOREM 5.3.7. *If* $\mathrm{Supp}(D) = X_s$ *is smooth, then the map* $\phi\colon SK_1(X_\eta)/\ell^m \to \widehat{SK}_1(X_\eta)/\ell^m$ *is an isomorphism.*

To prove this theorem, we may further assume that $D = X_s$, since the multiplicity of D has no contribution to $\widehat{SK}_1(X_\eta)/\ell^m$. To simplify our notations, we denote $\Lambda(i)_Y := \mathbb{Z}/\ell^m\mathbb{Z} \otimes \mathbb{Z}(i)_Y$ for a scheme Y and $i \in \mathbb{Z}$, where $\mathbb{Z}(i)$ is Bloch's cycle complex on the small Nisnevich site (cf. [Gei04]).

We can define the restriction map $r_i\colon \Lambda(i)_X \to i_*\Lambda(i)_{X_s}$ as the composition

$$\Lambda(i)_X \to j_*\Lambda(i)_{X_\eta} \xrightarrow{\cdot\pi} j_*\Lambda(i+1)_{X_\eta}[1] \to i_*\Lambda(i)_{X_s},$$

where the middle arrow is given by multiplication by π, and the last arrow is the localization map.

Let

$$\Lambda(i)_{X|X_s} := \mathrm{hofib}(r_i\colon \Lambda(i)_X \to i_*\Lambda(i)_{X_s})$$

be the homotopy fiber of r_i. By rigidity [Gei04, Theorem 1.2(3)] we get an isomorphism $j_!\Lambda(i)_{X_\eta} \cong \Lambda(i)_{X|X_s}$. Notice that we also have an analogous isomorphism $j_!\mathcal{K}^M_{i,X_\eta}/\ell^m \cong \mathcal{K}^M_{i,X|X_s}/\ell^m$. So we conclude:

PROPOSITION 5.3.8. *There is a canonical isomorphism*

$$\mathcal{K}^M_{i,X|X_s}/\ell^m \cong \mathcal{H}^i(\Lambda(i)_{X|X_s})$$

and $\mathcal{H}^j(\Lambda(i)_{X|X_s}) = 0$ *for* $j > i$.

Note that Proposition 5.3.8 implies that the canonical map

$$H^{2d+2}_{X_s}(X_{\mathrm{Nis}}, \Lambda(d+1)_{X|X_s}) \xrightarrow{\cong} H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|X_s})/\ell^m \tag{5.3.8}$$

is an isomorphism.

To finish the proof of Theorem 5.3.7, we also need the following result:

PROPOSITION 5.3.9. *The group* $H^{2d+1}(X_{\mathrm{Nis}}, \Lambda(d+1)_{X|X_s}) = 0$.

Proof. By the definition of $\Lambda(d+1)_{X|X_s}$, there is a long exact sequence

$$\begin{aligned} &H^{2d}(X_{\mathrm{Nis}}, \Lambda(d+1)_X) \xrightarrow{\alpha} H^{2d}(X_{s,\mathrm{Nis}}, \Lambda(d+1)_{X_s}) \\ &\quad \to H^{2d+1}(X_{\mathrm{Nis}}, \Lambda(d+1)_{X|X_s}) \\ &\quad \to H^{2d+1}(X_{\mathrm{Nis}}, \Lambda(d+1)_X) \xrightarrow{\beta} H^{2d+1}(X_{s,\mathrm{Nis}}, \Lambda(d+1)_{X_s}). \end{aligned}$$

It suffices to show that α is surjective and β is injective. In fact, using the relation between motivic cohomology and higher Chow groups, we will show that both α and β are isomorphisms. More precisely, the fact that α is an isomorphism follows from the diagram:

$$\begin{tikzcd} H^{2d}(X_{\mathrm{Nis}}, \Lambda(d+1)_X) \arrow[r, equal] \arrow[d, "\alpha"] & CH^{d+1}(X, 2; \mathbb{Z}/\ell^m\mathbb{Z}) \arrow[r, "\cong"] & H^{2d}(X_{\text{ét}}, \mu_{\ell^m}^{\otimes d+1}) \arrow[d, "\cong"] \\ H^{2d}(X_{s,\mathrm{Nis}}, \Lambda(d+1)_{X_s}) \arrow[r, equal] & CH^{d+1}(X_s, 2; \mathbb{Z}/\ell^m\mathbb{Z}) \arrow[r, "\cong"] & H^{2d}(X_{s,\text{ét}}, \mu_{\ell^m}^{\otimes d+1}) \end{tikzcd}$$

where the equalities in the rows are the definitions of higher Chow groups with coefficients in $\mathbb{Z}/\ell^m\mathbb{Z}$ (cf. [GL01]), the two horizontal arrows are isomorphisms by the known Kato conjecture [KS12, Theorem 9.3], and the right vertical is the proper base change theorem (SGA$4\frac{1}{2}$, [Del77, Arcata IV]). The assertion for β is similar:

$$\begin{tikzcd} H^{2d+1}(X_{\mathrm{Nis}}, \Lambda(d+1)_X) \arrow[r, equal] \arrow[d, "\beta"] & CH^{d+1}(X, 1; \mathbb{Z}/\ell^m\mathbb{Z}) \arrow[r, "\cong"] & H^{2d+1}(X_{\text{ét}}, \mu_{\ell^m}^{\otimes d+1}) \arrow[d, "\cong"] \\ H^{2d+1}(X_{s,\mathrm{Nis}}, \Lambda(d+1)_{X_s}) \arrow[r, equal] & CH^{d+1}(X_s, 2; \mathbb{Z}/\ell^m\mathbb{Z}) \arrow[r, "\cong"] & H^{2d+1}(X_{s,\text{ét}}, \mu_{\ell^m}^{\otimes d+1}). \end{tikzcd}$$ ∎

Proof of Theorem 5.3.7. The assertion follows directly from the diagram:

$$\begin{tikzcd} H^{2d+1}(X_{\mathrm{Nis}}, \Lambda(d+1)_{X|X_s}) \arrow[r] \arrow[d, equal, "\textit{Proposition } 4.3.9"] & H^{2d+1}(X_{\eta,\mathrm{Nis}}, \Lambda(d+1)_{X_\eta}) \arrow[r, two heads] \arrow[d, "\cong"] & H^{2d+2}_{X_s}(X_{\mathrm{Nis}}, \Lambda(d+1)_{X|X_s}) \arrow[d, "\cong"] \\ 0 & SK_1(X_\eta)/\ell^m \arrow[r, "\phi"] & \widehat{SK}_1(X_\eta)/\ell^m \end{tikzcd}$$

where the first row is the exact localization sequence, note that $j^*\Lambda(d+1)_{X|X_s} = \Lambda(d+1)_{X_\eta}$. The first vertical isomorphism is given by (5.3.8) and the second vertical isomorphism is given by Proposition 5.3.8 and (5.3.5). ∎

5.4 The p-primary part: equicharacteristic

Due to the lack of ramified duality in the mixed characteristic case for p-primary sheaves, we only treat the case that $R=\mathbb{F}_q[[t]]$ in this subsection and assume X_s is reduced. In [Zha16], we proved the following duality theorem for the relative logarithmic de Rham–Witt sheaves in this setting.

THEOREM 5.4.1. [Zha16, Theorem 3.4.2] *Let $R=\mathbb{F}_q[[t]]$. There is a perfect pairing of topological abelian groups*

$$H^i(U_{\text{ét}}, W_m\Omega^r_{U,\log}) \times \varprojlim_E H^{d+2-i}_{X_s}(X_{\text{ét}}, W_m\Omega^{d+1-r}_{X|E,\log})$$
$$\to H^{d+2}_{X_s}(X_{\text{ét}}, W_m\Omega^{d+1}_{X,\log}) \xrightarrow{\text{Tr}} \mathbb{Z}/p^m\mathbb{Z},$$

where the inverse limit runs over the set of effective divisors D such that $\text{Supp}(D) \subset X-U$. *The first group is endowed with the discrete topology, and the second is with profinite topology.*

For $r=0$, $i=1$, we get

$$H^1(U_{\text{ét}}, \mathbb{Z}/p^m\mathbb{Z}) \cong \varinjlim_E \text{Hom}(H^{d+1}_{X_s}(X_{\text{ét}}, W_m\Omega^{d+1}_{X|E,\log}), \mathbb{Z}/p^m\mathbb{Z}).$$

Similar to Corollary 3.3.5, the transition maps are surjective in the projective limit, for our divisor D we define

$$\text{Fil}_D H^1(U_{\text{ét}}, \mathbb{Z}/p^m\mathbb{Z}) := \text{Hom}(H^{d+1}_{X_s}(X_{\text{ét}}, W_m\Omega^{d+1}_{X|D,\log}), \mathbb{Z}/p^m\mathbb{Z});$$

by Pontryagin duality, we also define

$$\pi_1^{\text{ab}}(X,D)/p^m := \text{Hom}(\text{Fil}_D H^1(U_{\text{ét}}, \mathbb{Z}/p^m\mathbb{Z}), \mathbb{Z}/p^m\mathbb{Z}).$$

Therefore Theorem 5.4.1 gives us an isomorphism

$$H^{d+1}_{X_s}(X_{\text{ét}}, W_m\Omega^{d+1}_{X|D,\log}) \xrightarrow{\cong} \pi_1^{\text{ab}}(X,D)/p^m.$$

As before we want to compare the group $H^{d+1}_{X_s}(X_{\text{Nis}}, W_m\Omega^{d+1}_{X|D,\log})$ with $H^{d+1}_{X_s}(X_{\text{ét}}, W_m\Omega^{d+1}_{X|D,\log})$, by using the coniveau spectral sequence.

For any abelian sheaf $\mathcal{F}$ on X_{Nis} or $X_{\text{ét}}$, we have the following two coniveau spectral sequences:

$$E^{p,q}_{1,\text{ét}}(\mathcal{F}) := \bigoplus_{x\in X^p\cap X_s} H^{p+q}_x(X_{\text{ét}}, \mathcal{F}) \Longrightarrow H^{p+q}_{X_s}(X_{\text{ét}}, \mathcal{F})$$
$$E^{p,q}_{1,\text{Nis}}(\mathcal{F}) := \bigoplus_{x\in X^p\cap X_s} H^{p+q}_x(X_{\text{Nis}}, \mathcal{F}) \Longrightarrow H^{p+q}_{X_s}(X_{\text{Nis}}, \mathcal{F}).$$

Proposition 5.4.2. *We have the following isomorphisms:*

(i) $E^{\bullet,1}_{1,\text{ét}}(W_m\Omega^{d+1}_{X|D,\log}) \cong E^{\bullet,1}_{1,\text{ét}}(W_m\Omega^{d+1}_{X,\log})$;
(ii) $E^{\bullet,0}_{1,\text{Nis}}(W_m\Omega^{d+1}_{X|D,\log}) \cong E^{\bullet,0}_{1,\text{ét}}(W_m\Omega^{d+1}_{X|D,\log})$.

Proof. This is a local question. The first claim follows by the same argument as in Proposition 3.3.2, and the second as in Proposition 3.3.3. ∎

By purity [Shi07] the complex $E^{\bullet,1}_{1,\text{ét}}(W_m\Omega^{d+1}_{X,\log})$ is isomorphic to the Kato complex $C^{1,0}(X_s, \mathbb{Z}/p^m\mathbb{Z}(d))$ (up to a shift), that is, to

$$\bigoplus_{y\in X_s^0} H_y^{d+1}(X_{s,\text{ét}}, \mathbb{Z}/p^m\mathbb{Z}(d)) \to \bigoplus_{y\in X_s^1} H_y^{d+2}(X_{s,\text{ét}}, \mathbb{Z}/p^m\mathbb{Z}(d)) \to \cdots$$
$$\cdots \to \bigoplus_{y\in X_s^a} H_y^{d+a+1}(X_{s,\text{ét}}, \mathbb{Z}/p^m\mathbb{Z}(d)) \to \cdots$$
$$\to \bigoplus_{y\in X_s^d} H_y^{2d+1}(X_{s,\text{ét}}, \mathbb{Z}/p^m\mathbb{Z}(d)),$$

where $\mathbb{Z}/p^m\mathbb{Z}(d) = \nu^d_{m,X_s}[-d]$ and where the last term is placed in degree 0.

Theorem 5.4.3. *The canonical map*

$$H^{d+1}_{X_s}(X_{\text{Nis}}, W_m\Omega^{d+1}_{X|D,\log}) \to H^{d+1}_{X_s}(X_{\text{ét}}, W_m\Omega^{d+1}_{X|D,\log})$$

fits into an exact sequence of finite groups

$$H_2(C(X_s^\bullet, \mathbb{Z}/p^m\mathbb{Z})) \to H^{d+1}_{X_s}(X_{\text{Nis}}, W_m\Omega^{d+1}_{X|D,\log})$$
$$\to H^{d+1}_{X_s}(X_{\text{ét}}, W_m\Omega^{d+1}_{X|D,\log}) \to H_1(C(X_s^\bullet, \mathbb{Z}/p^m\mathbb{Z})) \to 0.$$

Proof. By the coniveau spectral sequence for $\mathcal{F} = W_m\Omega^{d+1}_{X|D,\log}$ on $X_{\text{ét}}$, we have the following exact sequence

$$E^{d-1,1}_{2,\text{ét}}(\mathcal{F}) \to E^{d+1,0}_{2,\text{ét}}(\mathcal{F}) \to H^{d+1}_{X_s}(X_{\text{ét}}, \mathcal{F}) \to E^{d,1}_{2,\text{ét}}(\mathcal{F}) \to 0.$$

By Proposition 5.4.2, we have

$$E^{d+1,0}_{2,\text{ét}}(\mathcal{F}) = E^{d+1,0}_{2,\text{Nis}}(\mathcal{F}) = H^{d+1}_{X_s}(X_{\text{Nis}}, \mathcal{F}).$$

Moreover combining with Theorem 5.2.1, we obtain

$$E^{d-1,1}_{2,\text{ét}}(W_m\Omega^{d+1}_{X|D,\log}) = E^{d-1,1}_{2,\text{ét}}(W_m\Omega^{d+1}_{X,\log}) = H_2(C(X_s^\bullet, \mathbb{Z}/p^m\mathbb{Z}));$$
$$E^{d,1}_{2,\text{ét}}(W_m\Omega^{d+1}_{X|D,\log}) = E^{d,1}_{2,\text{ét}}(W_m\Omega^{d+1}_{X,\log}) = H_1(C(X_s^\bullet, \mathbb{Z}/p^m\mathbb{Z})).$$ ∎

REMARK 5.4.4. In particular, if X has good reduction, then

$$H^{d+1}_{X_s}(X_{\mathrm{Nis}}, W_m\Omega^{d+1}_{X|D,\log}) \cong H^{d+1}_{X_s}(X_{\text{ét}}, W_m\Omega^{d+1}_{X|D,\log}).$$

The p-primary part of class field theory in this setting can be reformulated as follows:

THEOREM 5.4.5. *There is a canonical map*

$$\rho_{X,D} \colon C(X, D; X_s)/p^m \to \pi_1^{\mathrm{ab}}(X, D)/p^m,$$

which fits into an exact sequence of finite groups

$$\begin{aligned} H_2(C(X_s^\bullet, \mathbb{Z}/p^m\mathbb{Z})) \to C(X, D; X_s)/p^m &\to \pi_1^{\mathrm{ab}}(X, D)/p^m \\ &\to H_1(C(X_s^\bullet, \mathbb{Z}/p^m\mathbb{Z})) \to 0. \end{aligned}$$

In particular, we have

$$\begin{aligned} H_2(C(X_s^\bullet, \mathbb{Z}/p^m\mathbb{Z})) \to \varprojlim_D(C(X, D; X_s)/p^m) &\to \pi_1^{\mathrm{ab}}(U)/p^m \\ &\to H_1(C(X_s^\bullet, \mathbb{Z}/p^m\mathbb{Z})) \to 0. \end{aligned}$$

Proof. The map is defined by the following composition:

$$\begin{tikzcd} C(X, D; X_s)/p^m \arrow[r, "\cong"] \arrow[dr, dashed, "\rho_{X,D}"'] & H^{d+1}_{X_s}(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1,X|D})/p^m \arrow[r, "\cong"] & H^{d+1}_{X_s}(X_{\mathrm{Nis}}, W_m\Omega^{d+1}_{X|D,\log}) \arrow[d] \\ & \pi_1^{\mathrm{ab}}(X, D)/p^m & H^{d+1}_{X_s}(X_{\text{ét}}, W_m\Omega^{d+1}_{X|D,\log}) \arrow[l, "\cong"'] \end{tikzcd}$$

where the second isomorphism in the upper row is obtained in analogy to the proof of Corollary 3.3.4. Theorem 5.4.5 now is a consequence of Theorems 5.4.3, 5.1.2 and 5.4.1. ∎

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Moritz Kerz
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
Germany

moritz.kerz@mathematik.uni-regensburg.de

Yigeng Zhao
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
Germany

yigeng.zhao@mathematik.uni-regensburg.de

https://doi.org/10.1017/nmj.2018.34 Published online by Cambridge University Press