

Bivariate Polynomials of Least Deviation from Zero

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Abstract. Bivariate polynomials with a fixed leading term $x^m y^n$, which deviate least from zero in the uniform or L^2 -norm on the unit disk D (resp. a triangle) are given explicitly. A similar problem in L^p , $1 \leq p \leq \infty$, is studied on D in the set of products of linear polynomials.

Introduction

One of the most remarkable properties of the Tchebycheff polynomials of the first kind defined by

$$T_n(x) = \cos n \arccos x, \quad -1 \leq x \leq 1,$$

is that they have minimal uniform norm on $[-1, 1]$ among all polynomials of degree n with the same leading coefficient (which is 2^{n-1}). In other words, T_n deviates least from zero in the described class of polynomials. The Tchebycheff polynomials of the second kind,

$$U_n(x) = \frac{\sin(n+1) \arccos x}{\sin \arccos x}, \quad -1 \leq x \leq 1,$$

have the corresponding property with respect to the L^1 -norm on $[-1, 1]$; note that the leading coefficient of U_n is 2^n . These extremal properties give rise to interesting consequences and important relations for algebraic polynomials (see Borwein-Erdélyi [4], Milovanović-Mitrinović-Rassias [12], Rivlin [20]).

The minimum deviation property of the Tchebycheff polynomials in the univariate case was a motivation for various investigations of the problem of least deviation in the multivariate setting. Of course, since there are many counterparts to a univariate interval even in the bivariate case (e.g., a square, a disk, a triangle or others), and since the leading term of a multivariate polynomial can be specified in many ways, these questions lead to different problems and solutions. In particular, we mention the following work:

Uniform approximation on the square (Sloss [22], Ehlich-Zeller [6], Reimer [18]), on the disk (Gearhart [8], Reimer [17]), on a triangle (Newman-Xu [13]), L^2 -approximation on the square (Šac [21], Rack [15]), L^1 -approximation (Fromm [7], Rack [16], Haußmann-Zeller [9]) and mixed norm approximation (Haußmann-Zeller [10]) on the square.

In the present paper we give explicit representations for bivariate polynomials of least deviation from zero on the *disk*

$$D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

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and on the *triangle*

$$S := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, 1 - x - y \geq 0\}.$$

We consider the class

$$\pi_k(\mathbb{R}^2) = \left\{ \sum_{0 \leq i+j \leq k} a_{i,j} x^i y^j : a_{i,j} \in \mathbb{R} \right\}$$

of bivariate polynomials of total degree $\leq k$. We determine an *error function* $\omega_{m,n}$ of type $\{m, n\}$ on a set $\Omega \subset \mathbb{R}^2$ as a polynomial of the form

$$\omega_{m,n}(x, y) := x^m y^n - p(x, y), \quad p \in \pi_{m+n-1}(\mathbb{R}^2),$$

which deviates least from zero on Ω with respect to a given norm, *i.e.*,

$$\|\omega_{m,n}\| = \text{minimum!}$$

The first two sections are devoted to uniform approximation. In Section 1 we present a construction which allows to generate a family of error functions of type $\{m, n\}$ on the disk D , starting from any given one. In addition to the results described by Gearhart [8] and Reimer [17], our construction provides families of new error functions. In Section 2, error functions for the triangle S are found on the basis of those obtained by Gearhart for the disk. Moreover, we describe an interesting relation between Gearhart's error functions on D and another family of error functions on S due to Newman-Xu [13]. In Section 3 we give the explicit form for the coefficients of the error functions $\omega_{m,n}$ on D with respect to the L^2 -norm. Section 4 treats a nonlinear least deviation problem on D in any L^p -norm ($1 \leq p \leq \infty$) in the set of polynomials from $\pi_n(\mathbb{R}^2)$ which vanish on n lines (*i.e.*, that can be factorized in terms of degree 1).

1 Best Uniform Approximation on the Disk

Although the unit disk D in \mathbb{R}^2 is a quite natural domain, there are only few papers concerning polynomials of least uniform deviation from zero on D . We mention the following results:

Theorem A (Gearhart [8]) For integers $n \geq 0$ and $m \geq 0$ with $m + n \geq 1$ set

$$(1) \quad G_{m,n}(x, y) = \frac{1}{2^{m+n}} (U_m(x)U_n(y) + U_{m-2}(x)U_{n-2}(y)).$$

Then $G_{m,n}$ is an error function of type $\{m, n\}$ on D with respect to the uniform norm. The deviation is

$$\|G_{m,n}\|_{D,\infty} = \frac{1}{2^{m+n-1}}.$$

Here and in what follows, $U_{-1} = 0$, $U_{-2} = -1$.

Another family of polynomials of least uniform deviation from zero on D was discovered by Reimer. He considered the polynomials $R_{\bar{\mu}}$ generated by

$$(2) \quad |\bar{t}|^N T_N \left(\frac{\bar{t} \cdot \bar{x}}{|\bar{t}|} \right) = \sum_{|\bar{\mu}|=N} \bar{t}^{\bar{\mu}} R_{\bar{\mu}}(\bar{x})$$

with $\bar{\mu} = (\mu_1, \dots, \mu_d)$, $\bar{t} = (t_1, \dots, t_d)$, $\bar{x} = (x_1, \dots, x_d)$, where $\bar{t} \cdot \bar{x}$ is the Euclidean inner product. When $d = 1$, then we have $R_{\mu} = T_{\mu}$, which has minimal uniform norm on $[-1, 1]$. For the bivariate case the following result was proved:

Theorem B (Reimer [17]) *Let $R_{(m,n)}$ be defined by (2). Then*

$$(3) \quad R_{m,n}(x, y) := \frac{1}{2^{m+n-1} \binom{m+n}{n}} R_{(m,n)}(x, y)$$

is an error function of type $\{m, n\}$ on D with respect to the uniform norm.

Except for special cases, the polynomials of Gearhart and Reimer do not coincide. However, as pointed out by Gearhart [8, Theorem 2.2], all extremal polynomials of a given type $\{m, n\}$ must agree on ∂D . He also showed that given any error function $p_{m,n}(x, y)$ one can obtain error functions $p_{km,kn}$ ($k \in \mathbb{N}$) by

$$(4) \quad p_{km,kn}(x, y) = \frac{1}{2^{k(m+n)-1}} T_k(2^{m+n-1} p_{m,n}(x, y)).$$

In this section we give a construction which allows us to produce a series of new error functions starting from a given one. This is a variant of the well-known semi-group property of the classical Tchebycheff polynomials T_m (see Rivlin [20, p. 160]).

Theorem 1 *Let $p_{m,n}$ be an error function of type $\{m, n\}$. Then for every $k \in \mathbb{N}$*

$$(5) \quad L_k[p_{m,n}](x, y) := \frac{1}{2^{(k-1)(m+n)}} p_{m,n}(T_k(x), yU_{k-1}(x))$$

is an error function of type $\{k(m+n) - n, n\}$ with respect to the uniform norm.

Proof We start with the observation that $(x, y) \in D$ implies $(T_k(x), yU_{k-1}(x)) \in D$. Indeed, for $(x, y) \in D$ we have

$$T_k^2(x) + y^2 U_{k-1}^2(x) \leq T_k^2(x) + (1 - x^2) U_{k-1}^2(x) = \cos^2 k\varphi + \sin^2 k\varphi = 1,$$

where $x = \cos \varphi$. Now, since $p_{m,n}$ is an error function of type $\{m, n\}$, it follows that

$$(6) \quad \|L_k[p_{m,n}]\|_{D,\infty} \leq \frac{1}{2^{(k-1)(m+n)}} \cdot \|p_{m,n}\|_{D,\infty} = \frac{1}{2^{k(m+n)-1}}.$$

Since the leading term of $L_k[p_{m,n}]$ is $x^{k(m+n)-n} \cdot y^n$ with coefficient 1, Theorem A implies $\|L_k[p_{m,n}]\| \geq 2^{-k(m+n)+1}$, hence equality. ■

As was already mentioned, the error functions from Theorems A and B found by Gearhart and Reimer are different, in general. They necessarily coincide when $m \cdot n = 0$ or $m = n = 1$, in which cases we have uniqueness (Gearhart [8, Theorem 2.2]). It should also be pointed out the following coincidence:

For any non-negative integer m

$$(7) \quad G_{m,1} = R_{m,1}.$$

Proof By the definition of $G_{m,1}$ one finds $G_{m,1}(x, y) = \frac{1}{2^m} y U_m(x)$, while the normalization of $R_{m,1}$ yields

$$R_{m,1}(x, y) = \frac{1}{2^m(m+1)} R_{(m,1)}(x, y),$$

therefore (7) is equivalent to

$$(8) \quad R_{(m,1)}(x, y) = (m+1)yU_m(x),$$

which is easily verified for $m = 0, 1, 2$. Assuming (8) is true for some natural $m \geq 2$, we shall show its validity for $m+1$. Making use of the induction hypothesis, the recurrence relation (Reimer [17, equations (1.5)])

$$R_{(m+1,1)}(x, y) = 2xR_{(m,1)}(x, y) + 2yR_{(m+1,0)}(x, y) - R_{(m-1,1)}(x, y)$$

and the fact that $R_{(n,0)}(x, y) = T_n(x)$ we obtain

$$R_{(m+1,1)}(x, y) = 2xy(m+1)U_m(x) + 2yT_{m+1}(x) - myU_m(x).$$

In view of the identity $(2xT_{m+1}(x))' = (T_{m+2}(x) + T_m(x))'$, this last expression is equal to $(m+2)yU_{m+1}(x)$, and the induction step is completed. ■

It is easy to see that the application of the operators L_k defined above to the error functions $G_{m,1} = R_{m,1}$ does not provide new error functions in each case. Indeed, we have

$$\begin{aligned} L_k[G_{m,1}](x, y) &= L_k \left[\frac{1}{2^m} y U_m(x) \right] \\ &= \frac{1}{2^{(k-1)(m+1)}} \frac{1}{2^m} y U_m(T_k(x)) U_{k-1}(x) \\ &= \frac{1}{2^{k(m+1)-1}} y U_{k(m+1)-1}(x) \\ &= G_{k(m+1)-1,1}(x, y). \end{aligned}$$

But the next examples will show that the operator L_k indeed produces new error functions:

Example 1: Error functions of type $\{4, 2\}$ The Gearhart and the Reimer $\{4, 2\}$ -type error functions are

$$G_{4,2}(x, y) = x^4 y^2 - \frac{1}{4}x^4 - \frac{3}{4}x^2 y^2 + \frac{1}{4}x^2 + \frac{1}{16}y^2 - \frac{1}{32},$$

$$R_{4,2}(x, y) = x^4 y^2 - \frac{1}{10}x^4 - \frac{3}{5}x^2 y^2 + \frac{3}{40}x^2 + \frac{3}{80}y^2 - \frac{1}{160},$$

respectively. Two further $\{4, 2\}$ -type error functions are

$$L_2[R_{1,2}](x, y) \equiv L_2[G_{1,2}](x, y) = x^4 y^2 - \frac{1}{2}x^2 y^2 - \frac{1}{16}x^2 + \frac{1}{32}$$

and

$$L_3[R_{0,2}](x, y) \equiv L_3[G_{0,2}](x, y) = x^4 y^2 - \frac{1}{2}x^2 y^2 + \frac{1}{16}y^2 - \frac{1}{32}$$

(note: the last error function can also be obtained from (4) as $2^{-5}T_2(2^2 R_{2,1}(x, y))$). It turns out that the Reimer error function $R_{4,2}$ is a convex combination of the remaining three polynomials, namely

$$R_{4,2} = \frac{2}{5}G_{4,2} + \frac{2}{5}L_2[R_{1,2}] + \frac{1}{5}L_3[R_{0,2}].$$

Hence $R_{4,2}$ is not a vertex of the set of all $\{4, 2\}$ -type error functions.

Example 2: Error functions of type $\{6, 2\}$ The Gearhart and the Reimer $\{6, 2\}$ -type error functions are

$$R_{6,2}(x, y) = x^6 y^2 - \frac{1}{14}x^6 - \frac{15}{14}x^4 y^2 + \frac{5}{56}x^4 + \frac{15}{56}x^2 y^2 - \frac{3}{112}x^2 - \frac{1}{112}y^2 + \frac{1}{896},$$

$$G_{6,2}(x, y) = x^6 y^2 - \frac{1}{4}x^6 - \frac{5}{4}x^4 y^2 + \frac{3}{8}x^4 + \frac{3}{8}x^2 y^2 - \frac{9}{64}x^2 - \frac{1}{64}y^2 + \frac{1}{128},$$

respectively. To these we add three further $\{6, 2\}$ error functions,

$$L_2[R_{2,2}](x, y) = x^6 y^2 - x^4 y^2 - \frac{1}{24}x^4 + \frac{5}{24}x^2 y^2 + \frac{1}{24}x^2 - \frac{1}{128},$$

$$L_2[G_{2,2}](x, y) = x^6 y^2 - x^4 y^2 - \frac{1}{16}x^4 + \frac{3}{16}x^2 y^2 + \frac{1}{16}x^2 - \frac{1}{128},$$

$$L_4[R_{0,2}](x, y) \equiv L_4[G_{0,2}](x, y) = x^6 y^2 - x^4 y^2 + \frac{1}{4}x^2 y^2 - \frac{1}{128}.$$

The last error function can also be obtained from (4) as $2^{-7}T_2(2^3 R_{3,1}(x, y))$. Note that these new error functions are linearly dependent:

$$L_2[R_{2,2}] = \frac{2}{3}L_2[G_{2,2}] + \frac{1}{3}L_4[R_{0,2}].$$

In the above examples, the error functions produced by the operators L_k are somewhat simpler than the corresponding error functions of Gearhart and Reimer (in particular, they have less non-zero coefficients).

2 Best Uniform Approximation on a Triangle

Here we consider best approximation on the triangle

$$S = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, 1 - x - y \geq 0\}.$$

In this case there is the following interesting

Theorem C (Newman-Xu [13]) For every $p \in \pi_{m+n-1}(\mathbb{R}^2)$,

$$\|x^m y^n - p\|_{S,\infty} \geq \frac{1}{2^{2(m+n)-1}} \|A_{m,n}\|_{S,\infty} = \frac{1}{2^{2(m+n)-1}},$$

where

$$A_{m,n}(x, y) = \begin{cases} T_{m-n}(2x-1)T_n(8xy-1) \\ \quad + 8xy(2x-1)U_{m-n-1}(2x-1)U_{n-1}(8xy-1) & \text{if } m > n, \\ T_m(8xy-1) & \text{if } m = n, \\ T_{n-m}(2y-1)T_m(8xy-1) \\ \quad + 8xy(2y-1)U_{n-m-1}(2y-1)U_{m-1}(8xy-1) & \text{if } m < n. \end{cases}$$

In our terminology, this means that $2^{-2(m+n)+1}A_{m,n}(x, y)$ is an error function of type $\{m, n\}$ on S . The proof of Theorem C in [13] is very tricky. Again, there is no uniqueness of the extremal polynomials, as Newman-Xu show.

There is a simple relation between approximation on the triangle S and on the disk D . It is based on the observation that S can be mapped into D by a polynomial transformation (see Reimer [19]). For easy reference we formulate an immediate consequence of this relation as a separate proposition.

Proposition 2 If $p(x, y)$ is an error function on S of type $\{m, n\}$, then $p(x^2, y^2)$ is an error function of type $\{2m, 2n\}$ on D . Conversely, if $P(x, y)$ is an error function of type $\{2m, 2n\}$ on D such that $P(x, y) = p(x^2, y^2)$, then $p(x, y)$ is an error function of type $\{m, n\}$ on S .

Proof The mapping $\Phi: S \rightarrow D$ defined by

$$S \ni (u, v) \rightarrow (x, y) \in D \quad \text{with } u = x^2, v = y^2$$

gives a one-to-one correspondence between S and $D_+ := D \cap \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. If $p(u, v)$ is an error function of type $\{m, n\}$ on S , then $P(x, y) := p(x^2, y^2)$ is a polynomial of the form

$$x^{2m} y^{2n} + \text{lower degree terms.}$$

It is easily seen that P is an error function of type $\{2m, 2n\}$ on D . To show this, assume that Q is an error function of type $\{2m, 2n\}$ on D , but

$$\|Q\|_{D,\infty} < \|P\|_{D,\infty}.$$

Without loss of generality we may assume that $Q(x, y) = q(x^2, y^2)$, otherwise we can take $(Q(x, y) + Q(-x, y) + Q(x, -y) + Q(-x, -y)) / 4$. Then

$$\|q\|_{S,\infty} = \|Q\|_{D,\infty} < \|P\|_{D,\infty} = \|p\|_{S,\infty},$$

a contradiction. The second part of the claim follows in the same way. ■

Thus, any family of extremal polynomials on D generates a family of extremal polynomials on S and vice versa. An explicit formula can be given for the extremal polynomials on S produced by the Gearhart polynomials $\{G_{2m,2n}\}$. Indeed, we have:

Theorem 3 For any fixed pair (m, n) of non-negative integers the polynomial

$$\begin{aligned} Q_{m,n}(x, y) := & \frac{1}{2^{2(m+n)}} \{ [U_m(2x - 1) + U_{m-1}(2x - 1)] \\ & \cdot [U_n(2y - 1) + U_{n-1}(2y - 1)] \\ & + [U_{m-1}(2x - 1) + U_{m-2}(2x - 1)] \\ & \cdot [U_{n-1}(2y - 1) + U_{n-2}(2y - 1)] \} \end{aligned}$$

is an error function of type $\{m, n\}$ on S .

Proof We use the formula

$$G_{2m,2n}(x, y) = \frac{1}{2^{2m+2n}} [U_{2m}(x)U_{2n}(y) + U_{2m-2}(x)U_{2n-2}(y)]$$

for Gearhart's polynomials of least deviation on D and represent it in terms of x^2 and y^2 using the identity

$$(9) \quad U_{2k}(t) = U_k(2t^2 - 1) + U_{k-1}(2t^2 - 1).$$

Then the replacement $x^2 \rightarrow x, y^2 \rightarrow y$ results in the desired expression for $Q_{m,n}$. ■

Now we shall give the relation between the polynomials $A_{m,n}$ found by Newman-Xu and the polynomials of Gearhart. Consider Gearhart's error function of type $\{m - n, 2n\}$ on D for $m > n$,

$$G_{m-n,2n}(x, y) = \frac{1}{2^{m+n}} [U_{m-n}(x)U_{2n}(y) + U_{m-n-2}(x)U_{2n-2}(y)].$$

We apply the operator L_2 defined in Theorem 1 to $G_{m-n,2n}$ to obtain an error function $V_{2m,2n}$ of type $\{2m, 2n\}$ on D ,

$$V_{2m,2n}(x, y) = \frac{1}{2^{2m+2n}} [U_{m-n}(2x^2 - 1)U_{2n}(2xy) + U_{m-n-2}(2x^2 - 1)U_{2n-2}(2xy)].$$

Using (9) we get the representation

$$\begin{aligned} V_{2m,2n}(x, y) &= \frac{1}{2^{2m+2n}} \{U_{m-n}(2x^2 - 1)[U_n(8x^2y^2 - 1) + U_{n-1}(8x^2y^2 - 1)] \\ &\quad + U_{m-n-2}(2x^2 - 1)[U_{n-1}(8x^2y^2 - 1) + U_{n-2}(8x^2y^2 - 1)]\}. \end{aligned}$$

According to Proposition 2, the replacement $x^2 \rightarrow x, y^2 \rightarrow y$ in $V_{2m,2n}$ yields an error function of type $\{m, n\}$ on the triangle S , which we denote by $W_{m,n}$:

$$\begin{aligned} W_{m,n}(x, y) &= \frac{1}{2^{2m+2n}} \{U_{m-n}(2x - 1)[U_n(8xy - 1) + U_{n-1}(8xy - 1)] \\ &\quad + U_{m-n-2}(2x - 1)[U_{n-1}(8xy - 1) + U_{n-2}(8xy - 1)]\} \\ &=: \frac{1}{2^{2m+2n}} \{I + II\}. \end{aligned}$$

We shall show that $W_{m,n}$ is exactly the Newman-Xu error function $2^{-2m-2n+1}A_{m,n}$ from Theorem C (for $m > n$).

To this end we make use of the formulae

$$(10) \quad U_k(z) = zU_{k-1}(z) + T_k(z),$$

$$(11) \quad U_{k-2}(z) = zU_{k-1}(z) - T_k(z).$$

Applying (10) to I and (11) to II we get

$$\begin{aligned} I &= [(2x - 1)U_{m-n-1}(2x - 1) + T_{m-n}(2x - 1)] \\ &\quad \cdot [8xyU_{n-1}(8xy - 1) + T_n(8xy - 1)], \\ II &= [(2x - 1)U_{m-n-1}(2x - 1) - T_{m-n}(2x - 1)] \\ &\quad \cdot [8xyU_{n-1}(8xy - 1) - T_n(8xy - 1)]. \end{aligned}$$

It is clear now that $I + II = 2A_{m,n}$, hence

$$W_{m,n}(x, y) = \frac{1}{2^{2(m+n)-1}} A_{m,n}(x, y).$$

The case $m < n$ is treated in the same way. The case $m = n$ is even simpler: We take $p_{1,1}(x, y) = xy$ which is the error function of type $\{1, 1\}$ on D , then in view of (4),

$$\frac{1}{2^{4n-1}} T_{2n}(2p_{1,1}(x, y)) = \frac{1}{2^{4n-1}} T_n(8x^2y^2 - 1)$$

is an error function of type $\{2n, 2n\}$ on D . Hence, by Proposition 2,

$$\frac{1}{2^{4n-1}} T_n(8xy - 1)$$

is an error function of type $\{n, n\}$ on S .

3 Polynomials of Least L^2 -Deviation on the Disk

The inner product

$$(f, g) := \iint_D f(x, y)g(x, y) \, dx \, dy$$

on D induces the L^2 -norm

$$\|f\|_{D,2} := \left\{ \iint_D f^2(x, y) \, dx \, dy \right\}^{\frac{1}{2}}.$$

We shall consider the problem of describing the polynomials of the form

$$(12) \quad x^m y^n - Q(x, y), \quad Q \in \pi_{m+n-1}(\mathbb{R}^2),$$

which have least L^2 -norm on the unit disk D . Because of the strict convexity of the ball in L^2 , for a fixed pair (m, n) there is only one polynomial of the form (12) of least deviation from zero on D , say $L_{m,n}$. As is well known, $L_{m,n}$ is characterized by the orthogonality conditions

$$(13) \quad (L_{m,n}, g) = 0 \quad \text{for each } g \in \pi_{m+n-1}(\mathbb{R}^2).$$

On the other hand, it is also known that the Tchebycheff polynomials of the second kind $\{U_k(t)\}$ generate an orthonormal basis in $\pi(\mathbb{R}^2)$. More precisely, let $\theta_{k,j} := \frac{j\pi}{k+1}$ and $\bar{\xi}_{k,j} := (\cos \theta_{k,j}, \sin \theta_{k,j})$ for $k = 0, 1, \dots, m+n, j = 0, 1, \dots, k$. Put

$$U_{k,j}(\bar{x}) := \frac{1}{\sqrt{\pi}} U_k(\bar{x} \cdot \bar{\xi}_{k,j}), \quad \text{where } \bar{x} = (x, y).$$

The system $\{U_{k,j}\}_{k=0, j=0}^{m+n, k}$ constitutes an orthonormal basis in $\pi_{m+n}(\mathbb{R}^2)$, see for example [3, Lemma 6]. Therefore, $L_{m,n}$ can be represented as a linear combination of the polynomials from this system. However, because of the orthogonality (13), only terms of degree $m+n$ will occur in this representation. Precisely, the following theorem holds true:

Theorem 4 *The unique polynomial $L_{m,n}$ of the form (12) with minimal L^2 -deviation on D has the representation*

$$L_{m,n}(x, y) = \sum_{j=0}^{m+n} a_j(m, n) U_{m+n,j}(x, y),$$

where

$$(14) \quad a_j(m, n) = \frac{\sqrt{\pi}}{2^{m+n}} \int_{-1}^1 (\sin \theta_{m+n,j} - is \cos \theta_{m+n,j})^m (\cos \theta_{m+n,j} + is \sin \theta_{m+n,j})^n \, ds,$$

for $j = 0, \dots, m + n$ (and $i^2 = -1$). Moreover, the least deviation $\|L_{m,n}\|_{2,D}$ is given by

$$(15) \quad \|L_{m,n}\|_{2,D} = \frac{\sqrt{m+n+1}}{2^{m+n+3/2}} \left\{ \int_0^{2\pi} \left(\int_{-1}^1 (\sin \theta - is \cos \theta)^m (\cos \theta + is \sin \theta)^n ds \right)^2 d\theta \right\}^{\frac{1}{2}}.$$

Proof We have

$$a_j = (L_{m,n}, U_{m+n,j}) = (x^m y^n, U_{m+n,j}).$$

For the sake of convenience, we shall write θ_j instead of $\theta_{m+n,j}$. After change of variables

$$x = t \sin \theta_j - s \cos \theta_j, \quad y = t \cos \theta_j + s \sin \theta_j,$$

we get

$$\begin{aligned} a_j(m, n) &= \frac{1}{\sqrt{\pi}} \iint_D (t \sin \theta_j - s \cos \theta_j)^m (t \cos \theta_j + s \sin \theta_j)^n U_{m+n}(t) ds dt \\ &= \frac{1}{\sqrt{\pi}} \int_{-1}^1 U_{m+n}(t) \sum_{k=0}^m \sum_{\ell=0}^n c_{k,\ell}(\theta_j) \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} s^{m+n-k-\ell} ds t^{k+\ell} dt, \end{aligned}$$

where

$$c_{k,\ell}(\theta_j) := (-1)^{m-k} \binom{m}{k} \binom{n}{\ell} \sin^{n+k-\ell} \theta_j \cos^{m-k+\ell} \theta_j.$$

But clearly

$$\int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} s^N ds = \begin{cases} 0 & \text{if } N \text{ is odd,} \\ 2 \frac{\sqrt{1-t^2}}{N+1} (1-t^2)^{\frac{N}{2}} & \text{if } N \text{ is even.} \end{cases}$$

Denote, for brevity,

$$\gamma_{k,\ell} := \begin{cases} 0 & \text{if } m+n-k-\ell \text{ is odd,} \\ 2 \frac{(-1)^{(m+n-k-\ell)/2}}{m+n-k-\ell+1} & \text{if } m+n-k-\ell \text{ is even.} \end{cases}$$

Therefore, for even $m+n-k-\ell$,

$$\begin{aligned} t^{k+\ell} \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} s^{m+n-k-\ell} ds &= \frac{2\sqrt{1-t^2}}{m+n-k-\ell+1} t^{k+\ell} (1-t^2)^{(m+n-k-\ell)/2} \\ &= \gamma_{k,\ell} \sqrt{1-t^2} (t^{m+n} - h(t)), \end{aligned}$$

where h is a polynomial of degree less than $m+n$. By the definition of $\gamma_{k,\ell}$ the first and the last expression coincide in the case $m+n-k-\ell$ odd, too. Taking into account that

U_{m+n} is orthogonal on $[-1, 1]$ with respect to the weight $\sqrt{1-t^2}$ to each polynomial of degree not exceeding $m+n-1$ (and, in particular to h), we get

$$a_j(m, n) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^m \sum_{\ell=0}^n c_{k,\ell}(\theta_j) \gamma_{k,\ell} \int_{-1}^1 \sqrt{1-t^2} U_{m+n}(t) t^{m+n} dt.$$

The crucial observation for the proof of (14) is the equality

$$\gamma_{k,\ell} = \int_{-1}^1 (is)^{m+n-k-\ell} ds$$

with $i^2 = -1$. Using this, we can write the double sum above as

$$\sum_{k=0}^m \sum_{\ell=0}^n c_{k,\ell}(\theta_j) \gamma_{k,\ell} = \int_{-1}^1 (\sin \theta_j - is \cos \theta_j)^m (\cos \theta_j + is \sin \theta_j)^n ds.$$

To complete the proof of (14), we only need to note that

$$\int_{-1}^1 \sqrt{1-t^2} U_{m+n}(t) t^{m+n} dt = \frac{1}{2^{m+n}} \int_{-1}^1 \sqrt{1-t^2} U_{m+n}^2(t) dt = \frac{\pi}{2^{m+n+1}}.$$

To prove the second claim of Theorem 4, we introduce the function

$$F(m, n; \theta) := \left(\int_{-1}^1 (\sin \theta - is \cos \theta)^m (\cos \theta + is \sin \theta)^n ds \right)^2.$$

Obviously, $F(m, n; \theta)$ as a function of θ is a trigonometric polynomial of degree at most $2(m+n)$, and, in addition, $F(m, n; \theta + \pi) \equiv F(m, n; \theta)$. We have

$$\begin{aligned} \|L_{m,n}\|_{2,D}^2 &= \sum_{j=0}^{m+n} a_j(m, n)^2 \\ &= \frac{\pi}{2^{2m+2n+2}} \sum_{j=0}^{m+n} F(m, n; \theta_{m+n,j}) \\ &= \frac{m+n+1}{2^{2m+2n+3}} Q_{2m+2n+2}[F(m, n; \cdot)], \end{aligned}$$

where

$$Q_{2m+2n+2}[f] := \frac{2\pi}{2m+2n+2} \sum_{j=0}^{2m+2n+1} f(\theta_{m+n,j})$$

and $\theta_{m+n,j} = 2j\pi/(2m+2n+2)$, $j = 0, \dots, 2m+2n+1$. It is well-known that $Q_{2m+2n+2}[f]$ calculates exactly $\int_0^{2\pi} f(t) dt$ whenever f is a trigonometric polynomial of degree at most $2m+2n+1$ (see, e.g., Braß [5, p. 229, Satz 125]). In particular,

$$Q_{2m+2n+2}[F(m, n; \cdot)] = \int_0^{2\pi} F(m, n; \theta) d\theta,$$

and this completes the proof of Theorem 4. ■

As is well-known, the calculation of polynomials of least L^2 -deviation reduces to solution of a linear system of equations. The first polynomials $L_{m,n}$ are

$$\begin{aligned} L_{1,0}(x, y) &= x, \\ L_{1,1}(x, y) &= xy, \\ L_{2,0}(x, y) &= x^2 - \frac{1}{4}, \\ L_{2,1}(x, y) &= x^2y - \frac{1}{6}y, \\ L_{2,2}(x, y) &= x^2y^2 - \frac{1}{8}x^2 - \frac{1}{8}y^2 + \frac{1}{48}, \\ L_{3,0}(x, y) &= x^3 - \frac{1}{2}x, \\ L_{3,1}(x, y) &= x^3y - \frac{3}{8}xy, \\ L_{3,2}(x, y) &= x^3y^2 - \frac{1}{10}x^3 - \frac{3}{10}xy^2 + \frac{3}{80}, \\ L_{3,3}(x, y) &= x^3y^3 - \frac{1}{4}x^3y - \frac{1}{4}xy^3 + \frac{3}{40}xy, \\ L_{4,0}(x, y) &= x^4 - \frac{3}{4}x^2 + \frac{1}{16}, \\ L_{4,1}(x, y) &= x^4y - \frac{3}{5}x^2y + \frac{3}{80}y, \\ L_{4,2}(x, y) &= x^4y^2 - \frac{1}{12}x^4 - \frac{1}{2}x^2y^2 + \frac{1}{20}x^2 + \frac{1}{40}y^2 - \frac{1}{320}, \\ L_{4,3}(x, y) &= x^4y^3 - \frac{3}{14}x^4y - \frac{3}{7}x^2y^3 + \frac{3}{28}x^2y + \frac{1}{56}y^3 - \frac{3}{560}y, \\ L_{4,4}(x, y) &= x^4y^4 - \frac{3}{8}x^4y^2 - \frac{3}{8}x^2y^4 + \frac{3}{224}x^4 + \frac{9}{56}x^2y^2 \\ &\quad + \frac{3}{224}y^4 - \frac{3}{448}x^2 - \frac{3}{448}y^2 + \frac{3}{8960} \end{aligned}$$

(note that $L_{n,m}(x, y) = L_{m,n}(y, x)$). An interesting observation for the above polynomials is that the monomials $x^k y^\ell$ appearing in $L_{m,n}$ satisfy $k \leq m, \ell \leq n$.

The calculation of the least L^2 -deviation can be performed with the help of (15) without prior knowledge of $L_{m,n}$. In Table 1 on the next page we give the values of $\|L_{m,n}\|_{2,D}$ for $0 \leq n \leq m \leq 8$.

$n \setminus m$	1	2	3	4	5	6	7	8
0	$\frac{\sqrt{\pi}}{2}$	$\frac{\sqrt{\pi}}{4}$	$\frac{\sqrt{\pi}}{8}$	$\frac{\sqrt{\pi}}{16}$	$\frac{\sqrt{\pi}}{32}$	$\frac{\sqrt{\pi}}{64}$	$\frac{\sqrt{\pi}}{128}$	$\frac{\sqrt{\pi}}{256}$
1	$\frac{\sqrt{6\pi}}{12}$	$\frac{\sqrt{5\pi}}{24}$	$\frac{\sqrt{2\pi}}{32}$	$\frac{\sqrt{105\pi}}{480}$	$\frac{\sqrt{\pi}}{96}$	$\frac{\sqrt{21\pi}}{896}$	$\frac{\sqrt{15\pi}}{1536}$	$\frac{\sqrt{33\pi}}{4608}$
2		$\frac{\sqrt{95\pi}}{240}$	$\frac{\sqrt{9\pi}}{160}$	$\frac{\sqrt{73\pi}}{960}$	$\frac{\sqrt{133\pi}}{2688}$	$\frac{\sqrt{14\pi}}{1792}$	$\frac{\sqrt{555\pi}}{23040}$	$\frac{\sqrt{537\pi}}{46080}$
3			$\frac{\sqrt{91\pi}}{1120}$	$\frac{\sqrt{321\pi}}{4480}$	$\frac{\sqrt{47\pi}}{3584}$	$\frac{\sqrt{11\pi}}{3584}$	$\frac{\sqrt{3\pi}}{3840}$	$\frac{\sqrt{5577\pi}}{337920}$
4				$\frac{\sqrt{2509\pi}}{26880}$	$\frac{\sqrt{817\pi}}{32256}$	$\frac{\sqrt{2105\pi}}{107520}$	$\frac{\sqrt{44121\pi}}{1013760}$	$\frac{\sqrt{42097\pi}}{2027520}$
5					$\frac{\sqrt{88594\pi}}{709632}$	$\frac{\sqrt{9061\pi}}{473088}$	$\frac{\sqrt{6238\pi}}{811088}$	$\frac{\sqrt{250081\pi}}{104543104}$
6						$\frac{\sqrt{1400451\pi}}{12300288}$	$\frac{\sqrt{26633\pi}}{3514368}$	$\frac{\sqrt{1232329\pi}}{49201152}$
7							$\frac{\sqrt{1389270\pi}}{52715520}$	$\frac{\sqrt{5225273\pi}}{210862080}$
8								$\frac{\sqrt{1414593953\pi}}{7169310720}$

Table 1: The L^2 -deviation $\|L_{m,n}\|_{2,D}$, $0 \leq n \leq m \leq 8$

4 A Nonlinear L^p -Problem on D

Let P_n^2 be defined as the set of all polynomials f from $\pi_n(\mathbb{R}^2)$ which can be represented in the form

$$f(x, y) = \prod_{k=1}^n \ell_k(x, y),$$

where

$$\ell_k(x, y) = x \cos \alpha_k + y \sin \alpha_k + c_k, \quad \alpha_k \in [0, \pi), c_k \in \mathbb{R}, k = 1, \dots, n.$$

Note that each ℓ_k represents a straight line with a unit normal vector. The lines $\ell_k^*(x, y)$ defined by the equispaced angles

$$\alpha_k^* := \begin{cases} \frac{(2k-1)\pi}{2n} & \text{if } n \text{ even} \\ \frac{(k-1)\pi}{n} & \text{if } n \text{ odd} \end{cases}$$

and by $c_k = 0$ for $k = 1, \dots, n$ play a special role in P_n^2 . Set

$$f^*(x, y) := \prod_{k=1}^n \ell_k^*(x, y).$$

Then we get

Theorem 5 For all $n \in \mathbb{N}$ and $1 \leq p \leq \infty$, the polynomial f^* has a minimal L_p -norm on D among all $f \in P_n^2$, that is

$$\|f^*\|_{D,p} \leq \|f\|_{D,p} \quad \text{for all } f \in P_n^2.$$

In the case $1 \leq p < \infty$, the solution f^* is unique up to rotation. In the case $p = \infty$, f^* and its rotations are not the only solutions.

Proof Let $f \in P_n^2$. We introduce polar coordinates $x = \cos \varphi$, $y = \sin \varphi$ and get

$$(16) \quad f(x, y) = \frac{r^n}{2^{n-1}} \cos(n\varphi - \alpha_1 - \dots - \alpha_n) - g(r, \varphi),$$

where g (as a function of φ) is a trigonometric polynomial of degree at most $n - 1$. After a suitable rotation, we can assume without loss of generality that $\sum_{k=1}^n \alpha_k = 2q\pi$, $q \in \mathbb{Z}$. Hence we have

$$(17) \quad f(x, y) = \frac{r^n}{2^{n-1}} \cos n\varphi + g(r, \varphi).$$

For $1 \leq p < \infty$ we get

$$(18) \quad \begin{aligned} \|f\|_{D,p}^p &= \int_0^1 r \int_0^{2\pi} \left| \frac{r^n}{2^{n-1}} \cos n\varphi + g(r, \varphi) \right|^p d\varphi dr \\ &\geq \int_0^1 \frac{r^{np+1}}{2^{(n-1)p}} dr \cdot (E_{n-1,p}(\cos n\varphi))^p. \end{aligned}$$

Here

$$E_{m,p}(h) := \inf \|h - g\|_{D,p}$$

denotes the distance of the best $L_p[0, 2\pi]$ -approximation from the function h with respect to trigonometric polynomials of degree not exceeding m . It is well known (see Achieser [1, pp. 12–13]) that for $p \geq 1$

$$E_{m,p}(\cos n\varphi) = \|\cos n\varphi\|_{[0,2\pi],p}$$

(that is, the unique best L_p -approximant to $\cos n\varphi$ is zero). Thus, for $1 \leq p < \infty$,

$$(19) \quad \|f\|_{D,p}^p \geq \frac{\|\cos n\varphi\|_{[0,2\pi],p}^p}{2^{(n-1)p(np+2)}}.$$

The lower bound in (19) is attained in the class P_n^2 if we choose $c_k = 0$ and $\alpha_k = \alpha_k^*$ for $k = 1, \dots, n$. Therefore f^* is a polynomial of least deviation from zero. Actually, f^* is the only polynomial with this property. Indeed, if there would be another one, say $h \in P_n^2$, then it must deviate least from zero in L_p on any circle $\partial D_\rho := \{(x, y) : x^2 + y^2 = \rho^2\}$, $0 \leq \rho \leq 1$, and thus the restriction of h to ∂D_ρ must vanish at $2n$ equispaced points on ∂D_ρ . Since this holds for any sufficiently small $\rho < 0$, the lines $\{\ell_k\}$ associated with h must pass through the origin. This property and the normalization condition $\alpha_1 + \dots + \alpha_n = 2q\pi$ determine the extremal polynomial uniquely.

In the case $p = \infty$, we have

$$\begin{aligned} \|f\|_{D,\infty} &= \sup_{0 \leq r \leq 1} \max_{\{(x,y):x^2+y^2=r^2\}} \left| \frac{r^n}{2^{n-1}} \cos n\varphi + g(r, \varphi) \right| \\ &\geq \sup_{0 \leq r \leq 1} \frac{r^n}{2^{n-1}} E_{n-1,\infty}(\cos n\varphi) \geq \sup_{0 \leq r \leq 1} \frac{r^n}{2^{n-1}} \|\cos n\varphi\|_{[0,2\pi],\infty} \\ &= \frac{1}{2^{n-1}}. \end{aligned}$$

For the last inequality we used the fact that zero is the best uniform approximation to $\cos n\varphi$ by trigonometric polynomials of degree not exceeding $n - 1$ on $[0, 2\pi]$. Hence

$$\|f\|_{D,\infty} \geq \frac{1}{2^{n-1}}$$

and equality holds if $f|_{\partial D} = \cos(n\varphi + \vartheta)$ for some $\vartheta \in \mathbb{R}$, and $|f(x, y)| \leq 2^{-(n-1)}$ in the interior of D .

Therefore, up to rotation, we have the same extremizer in the case $p = \infty$, too. However, we have no uniqueness in this case, as can be seen by taking the polynomials

$$\frac{1}{2^{n-1}} T_n(x) \quad \text{and} \quad \frac{1}{2^{n-1}} T_n(y)$$

which belong to P_n^2 and have the same uniform norm on D . ■

5 Remarks

In the univariate case the polynomials of least L^p deviation on $[-1, 1]$ are known explicitly for $p = 1, 2, \infty$. We presented in this paper bivariate polynomials of least deviation on D in the uniform and L^2 -norm. As probably expected, polynomials of least L^1 norm should be found as well. Unfortunately, a careful study of the L^1 case shows that finding the extremal polynomials in L^1 is a fairly hard task. As known from the general theory, it suffices to find a polynomial $g(x, y)$ of the form

$$g(x, y) = x^n y^m + \text{lower degree terms}$$

such that $\text{sign } g(x, y)$ is orthogonal on D to all polynomials of total degree $n + m - 1$. Computer experiments show that the curves of sign change of the extremal polynomials look too complicated to hope to find some simple characterization like in the univariate case. However, there are some particular cases in which error functions in L^1 , and even in L^p for $1 \leq p \leq \infty$, can be easily found using the corresponding result in the univariate case. We shall give such an example. It is based on the following fact mentioned by Oskolkov [14]: *The polynomial of best L^p -approximation of a radial function on D is also radial.* Oskolkov gave a nice proof of this observation using Jensen's inequality. We shall derive it here even simpler: Let f be a radial function, that is, $f(x, y) = F(\sqrt{x^2 + y^2})$ with some univariate F . For any $p, 1 < p < \infty$, the polynomial of best approximation to f on D is unique. Denote it by q . Since f is radial, any rotation in the disk D would transform the function $f - q$ into function of the same form $f - q_1$ which has the same L_p -deviation from zero as the original one. Thus any rotation of q produces a polynomial q_1 of least deviation. Since the extremal polynomial is unique, it follows that q does not change after rotation, therefore q is radial, $q(x, y) = Q(\sqrt{x^2 + y^2})$ with some univariate polynomial Q . Now it is seen that the problem

$$\|f - q\|_p \rightarrow \inf \quad \text{over } q \in \pi_n(\mathbb{R}^2)$$

reduces to the univariate one

$$\|F - Q\|_p \rightarrow \inf \quad \text{over } Q \in \pi_n(\mathbb{R}).$$

The existence of a radial polynomial of least deviation for $p = 1$ and $p = \infty$ follows as the limiting case $p \rightarrow 1$ and $p \rightarrow \infty$. As a consequence, the polynomials of least L^p deviation of the form $(x^2 + y^2)^m - q(x, y)$, $q \in \pi_{4m-1}(\mathbb{R}^2)$ can be characterized. In particular for $p = 1, 2, \infty$ one can write the explicit form of the error function using the corresponding univariate polynomials of least deviation.

Similarly, as shown by Oskolkov [14], the polynomial of best L^p approximation of a ridge function f (that is, a function of the form $f(x, y) = F(ax + by)$ with some univariate F and constants a and b) on D is a ridge polynomial, too. Again, in this case the bivariate least deviation problem is reduced to a univariate one. Theorem 3 of Reimer [17] is dealing with a problem of this kind, since the polynomials he considers are actually ridge functions.

Another particular case can be derived from recent results in [2]. To do this, denote by $V(x, y)$ the polynomial

$$V(x, y) = \prod_{k=1}^n (x^2 + y^2 - \rho_k)$$

with $\{\rho_k\}$ being the zeros of the polynomial $v(t) = t^n + \dots$ of least L^1 -deviation on $[0, 1]$ with a weight $\mu(t) = t$. It was shown in [2] that $\text{sign } V(x, y)$ is orthogonal on D to every polyharmonic function of order $n - 1$ and consequently to every bivariate polynomial of total degree $2n - 3$. This means that V deviates least from zero in L^1 among all polynomials of the form

$$\prod_{k=1}^n (x^2 + y^2 - r_k)$$

with any constants r_1, \dots, r_n .

Finally, let us note that in studying polynomials of least deviation there are various normalization conditions different from the one we imposed in Sections 1–3, namely, a fixed leading monomial $x^m y^n$ with coefficient 1. We only mention a recent paper of Kroó [11] where, instead of our normalization, the homogenous part of the highest total degree is assumed to have a prescribed value at a given point ξ .

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