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Differential forms on universal $K3$ surfaces

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Abstract

We give a vanishing and classification result for holomorphic differential forms on smooth projective models of the moduli spaces of pointed $K3$ surfaces. We prove that there is no nonzero holomorphic k -form for $0 < k < 10$ and for even $k > 19$. In the remaining cases, we give an isomorphism between the space of holomorphic k -forms with that of vector-valued modular forms ($10 \leq k \leq 18$) or scalar-valued cusp forms (odd $k \geq 19$) for the modular group. These results are in fact proved in the generality of lattice-polarisation.

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1. Introduction

Let $\mathcal{F}_{g,n}$ be the moduli space of n -pointed $K3$ surfaces of genus $g > 2$, i.e., primitively polarised of degree $2g - 2$. It is a quasi-projective variety of dimension $19 + 2n$ with a natural morphism $\mathcal{F}_{g,n} \rightarrow \mathcal{F}_g$ to the moduli space \mathcal{F}_g of $K3$ surfaces of genus g , which is generically a $K3^n$ -fibration. In this paper we study holomorphic differential k -forms on a smooth projective model of $\mathcal{F}_{g,n}$. They do not depend on the choice of a smooth projective model, and thus are fundamental birational invariants of $\mathcal{F}_{g,n}$. We prove a vanishing result for about half of the values of the degree k , and for the remaining degrees give a correspondence with modular forms on the period domain.

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Our main result is stated as follows.

THEOREM 1.1. *Let $\bar{\mathcal{F}}_{g,n}$ be a smooth projective model of $\mathcal{F}_{g,n}$ with $g > 2$. Then we have a natural isomorphism:*

$$H^0(\bar{\mathcal{F}}_{g,n}, \Omega^k) \simeq \begin{cases} 0 & 0 < k \leq 9 \\ M_{\wedge^k, k}(\Gamma_g) & 10 \leq k \leq 18 \\ 0 & k > 19, k \in 2\mathbb{Z} \\ S_{19+m}(\Gamma_g, \det) \otimes \mathbb{C}\mathcal{S}_{n,m} & k = 19 + 2m, 0 \leq m \leq n \end{cases}. \quad (1.1)$$

Here Γ_g is the modular group for $K3$ surfaces of genus g , which is defined as the kernel of $O^+(L_g) \rightarrow O(L_g^\vee/L_g)$ where $L_g = 2U \oplus 2E_8 \oplus \langle 2 - 2g \rangle$ is the period lattice of $K3$ surfaces of genus g . In the second case, $M_{\wedge^k, k}(\Gamma_g)$ stands for the space of vector-valued modular forms of weight (\wedge^k, k) for Γ_g (see [4]). In the last case, $S_{19+m}(\Gamma_g, \det)$ stands for the space of scalar-valued cusp forms of weight $19 + m$ and determinant character for Γ_g , and $\mathcal{S}_{n,m}$ stands for the right quotient $\mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$, which is a left \mathfrak{S}_n -set. Theorem 1.1 is actually formulated and proved in the generality of lattice-polarisation (Theorem 2.6).

In the case of the top degree $k = 19 + 2n$, namely for canonical forms, the isomorphism (1.1) is proved in [2]. Theorem 1.1 is the extension of this result to all degrees $k < 19 + 2n$. The spaces in the right-hand side of (1.1) can also be geometrically explained as follows. In the case $k \leq 18$, $M_{\wedge^k, k}(\Gamma_g)$ is identified with the space of holomorphic k -forms on a smooth projective model of \mathcal{F}_g , pulled back by $\mathcal{F}_{g,n} \rightarrow \mathcal{F}_g$. In the case $k = 19 + 2m$, $S_{19+m}(\Gamma_g, \det)$ is identified with the space of canonical forms on $\bar{\mathcal{F}}_{g,m}$, and the tensor product $S_{19+m}(\Gamma_g, \det) \otimes \mathbb{C}\mathcal{S}_{n,m}$ is the direct sum of pullback of such canonical forms by various projections $\mathcal{F}_{g,n} \rightarrow \mathcal{F}_{g,m}$. Therefore Theorem 1.1 can be understood as a kind of classification result which says that except for canonical forms, there are essentially no new differential forms on the tower $(\mathcal{F}_{g,n})_n$ of moduli spaces. In fact, this is how the proof proceeds.

The space $S_l(\Gamma_g, \det)$ is nonzero for every sufficiently large l , so the space $H^0(\bar{\mathcal{F}}_{g,n}, \Omega^k)$ for odd $k \geq 19$ is typically nonzero (at least when k is large). On the other hand, it is not clear at present whether $M_{\wedge^k, k}(\Gamma_g) \neq 0$ or not in the range $10 \leq k \leq 18$. This is a subject of study in the theory of vector-valued orthogonal modular forms.

The isomorphism (1.1) in the case $k = 19 + 2m$ is an \mathfrak{S}_n -equivariant isomorphism, where \mathfrak{S}_n acts on $H^0(\bar{\mathcal{F}}_{g,n}, \Omega^k)$ by its permutation action on $\mathcal{F}_{g,n}$, while it acts on $S_{19+m}(\Gamma_g, \det) \otimes \mathbb{C}\mathcal{S}_{n,m}$ by its natural left action on $\mathcal{S}_{n,m}$. Therefore, taking the \mathfrak{S}_n -invariant part, we obtain the following simpler result for the unordered pointed moduli space $\mathcal{F}_{g,n}/\mathfrak{S}_n$, which is birationally a $K3^{[n]}$ -fibration over \mathcal{F}_g .

COROLLARY 1.2. *Let $\overline{\mathcal{F}_{g,n}/\mathfrak{S}_n}$ be a smooth projective model of $\mathcal{F}_{g,n}/\mathfrak{S}_n$. Then we have a natural isomorphism:*

$$H^0(\overline{\mathcal{F}_{g,n}/\mathfrak{S}_n}, \Omega^k) \simeq \begin{cases} 0 & 0 < k \leq 9 \\ M_{\wedge^k, k}(\Gamma_g) & 10 \leq k \leq 18 \\ 0 & k > 19, k \in 2\mathbb{Z} \\ S_{19+m}(\Gamma_g, \det) & k = 19 + 2m, 0 \leq m \leq n \end{cases}.$$

The universal $K3$ surface $\mathcal{F}_{g,1}$ is an analogue of elliptic modular surfaces ([6]), and the moduli spaces $\mathcal{F}_{g,n}$ for general n are analogues of the so-called Kuga varieties over modular curves ([7]). Starting with the case of elliptic modular surfaces [6], holomorphic differential forms on the Kuga varieties have been described in terms of elliptic modular forms: [7] for canonical forms, and [1] for the case of lower degrees (somewhat implicitly). Theorem 1.1 can be regarded as a $K3$ version of these results.

As a final remark, in view of the analogy between universal $K3$ surfaces and elliptic modular surfaces, invoking the classical fact that elliptic modular surfaces have maximal Picard number ([6]) now raises the question if $H^{k,0}(\bar{\mathcal{F}}_{g,n}) \oplus H^{0,k}(\bar{\mathcal{F}}_{g,n})$ is a sub \mathbb{Q} -Hodge structure of $H^k(\bar{\mathcal{F}}_{g,n}, \mathbb{C})$. This is independent of the choice of a smooth projective model $\bar{\mathcal{F}}_{g,n}$.

The rest of this paper is devoted to the proof of Theorem 1.1. In Section 2.1 we compute a part of the holomorphic Leray spectral sequence associated to a certain type of $K3^n$ -fibration. This is the main step of the proof. In Section 2.2 we study differential forms on a compactification of such a fibration. In Section 2.3 we deduce (a generalised version of) Theorem 1.1 by combining the result of Section 2.2 with some results from [2–5]. Sometimes we drop the subscript X from the notation Ω_X^k when the variety X is clear from the context.

2. Proof

2.1. Holomorphic Leray spectral sequence

Let $\pi : X \rightarrow B$ be a smooth family of $K3$ surfaces over a smooth connected base B . In this subsection X and B may be analytic. We put the following assumption:

Condition 2.1. In a neighbourhood of every point of B , the period map is an embedding.

This is equivalent to the condition that the differential of the period map

$$T_b B \rightarrow \text{Hom}(H^{2,0}(X_b), H^{1,1}(X_b))$$

is injective for every $b \in B$, where X_b is the fiber of π over b .

For a natural number $n > 0$ we denote by $X_n = X \times_B \cdots \times_B X$ the n -fold fiber product of X over B , and let $\pi_n : X_n \rightarrow B$ be the projection. We denote by Ω_{π_n} the relative cotangent bundle of π_n , and $\Omega_{\pi_n}^p = \wedge^p \Omega_{\pi_n}$ for $p \geq 0$ as usual.

PROPOSITION 2.2. *Let $\pi : X \rightarrow B$ be a $K3$ fibration satisfying Condition 2.1. Then we have a natural isomorphism:*

$$(\pi_n)_* \Omega_{X_n}^k \simeq \begin{cases} \Omega_B^k & k \leq \dim B \\ 0 & k > \dim B, \quad k \not\equiv \dim B \pmod{2} \\ K_B \otimes (\pi_n)_* \Omega_{\pi_n}^{2m} & k = \dim B + 2m, \quad 0 \leq m \leq n \end{cases}$$

This assertion amounts to a partial degeneration of the holomorphic Leray spectral sequence. Recall ([8, section 5.2]) that $\Omega_{X_n}^k$ has the holomorphic Leray filtration $L^\bullet \Omega_{X_n}^k$ defined by

$$L^l \Omega_{X_n}^k = \pi_n^* \Omega_B^l \wedge \Omega_{X_n}^{k-l},$$

whose graded quotients are naturally isomorphic to

$$\mathrm{Gr}_L^l \Omega_{X_n}^k = \pi_n^* \Omega_B^l \otimes \Omega_{\pi_n}^{k-l}.$$

This filtration induces the holomorphic Leray spectral sequence

$$(E_r^{l,q}, d_r) \Rightarrow E_\infty^{l+q} = R^{l+q}(\pi_n)_* \Omega_{X_n}^k$$

which converges to the filtration

$$L^l R^{l+q}(\pi_n)_* \Omega_{X_n}^k = \mathrm{Im}(R^{l+q}(\pi_n)_* L^l \Omega_{X_n}^k \rightarrow R^{l+q}(\pi_n)_* \Omega_{X_n}^k).$$

By [8, proposition 5.9], the E_1 page coincides with the collection of the Koszul complexes associated to the variation of Hodge structures for π_n :

$$(E_1^{l,q}, d_1) = (\mathcal{H}^{k-l, l+q} \otimes \Omega_B^l, \bar{\nabla}). \quad (2.1)$$

Here $\mathcal{H}^{*,*}$ are the Hodge bundles associated to the fibration $\pi_n: X_n \rightarrow B$, and

$$\bar{\nabla}: \mathcal{H}^{*,*} \otimes \Omega_B^* \rightarrow \mathcal{H}^{*-1, *+1} \otimes \Omega_B^{*+1}$$

are the differentials in the Koszul complexes (see [8, section 5.1.3]). For degree reasons, the range of (l, q) in the E_1 page satisfies the inequalities

$$0 \leq l \leq \dim B, \quad 0 \leq k-l \leq 2n, \quad 0 \leq l+q \leq 2n.$$

The first two can be unified:

$$\max(0, k-2n) \leq l \leq \min(\dim B, k), \quad 0 \leq l+q \leq 2n. \quad (2.2)$$

We calculate the E_1 to E_2 pages on the edge line $l+q=0$.

LEMMA 2.3. *The following holds:*

- (1) $E_1^{l,-l} = 0$ when $l \leq \min(\dim B, k)$ with $l \not\equiv k \pmod{2}$;
- (2) $E_2^{l,-l} = 0$ when $l < \min(\dim B, k)$;
- (3) For $l_0 = \min(\dim B, k)$ we have $E_1^{l_0, -l_0} = E_2^{l_0, -l_0} = \dots = E_\infty^{l_0, -l_0}$.

Proof. By (2.1), we have $E_1^{l,-l} = \mathcal{H}^{k-l, 0} \otimes \Omega_B^l$. By the Künneth formula, the fiber of $\mathcal{H}^{k-l, 0}$ over a point $b \in B$ is identified with

$$H^{k-l, 0}(X_b^n) = \bigoplus_{(p_1, \dots, p_n)} H^{p_1, 0}(X_b) \otimes \dots \otimes H^{p_n, 0}(X_b), \quad (2.3)$$

where (p_1, \dots, p_n) ranges over all indices with $\sum_i p_i = k-l$ and $0 \leq p_i \leq 2$.

(1) When $k-l$ is odd, every index (p_1, \dots, p_n) in (2.3) must contain a component $p_i = 1$. Since $H^{1, 0}(X_b) = 0$, we see that $H^{k-l, 0}(X_b^n) = 0$. Therefore $\mathcal{H}^{k-l, 0} = 0$ when $k-l$ is odd.

(3) Let $l_0 = \min(\dim B, k)$. By the range (2.2) of (l, q) , we see that for every $r \geq 1$ the source of d_r that hits $E_r^{l_0, -l_0}$ is zero, and the target of d_r that starts from $E_r^{l_0, -l_0}$ is also zero. This proves our assertion.

(2) Let $l < \min(\dim B, k)$. In view of (1), we may assume that $l = k-2m$ for some $m > 0$. By (2.2), the source of d_1 that hits $E_1^{l,-l}$ is zero. We shall show that $d_1: E_1^{l,-l} \rightarrow E_1^{l+1, -l}$ is

injective. By (2.1), this morphism is identified with

$$\bar{\nabla}: \mathcal{H}^{2m,0} \otimes \Omega_B^l \rightarrow \mathcal{H}^{2m-1,1} \otimes \Omega_B^{l+1}. \quad (2.4)$$

By the Künneth formula as in (2.3), the fibers of the Hodge bundles $\mathcal{H}^{2m,0}$, $\mathcal{H}^{2m-1,1}$ over $b \in B$ are respectively identified with

$$H^{2m,0}(X_b^n) = \bigoplus_{|\sigma|=m} H^{2,0}(X_b)^{\otimes \sigma}, \quad (2.5)$$

$$\begin{aligned} H^{2m-1,1}(X_b^n) &= \bigoplus_{|\sigma'|=m-1} \bigoplus_{i \notin \sigma'} H^{2,0}(X_b)^{\otimes \sigma'} \otimes H^{1,1}(X_b) \\ &= \bigoplus_{|\sigma|=m} \bigoplus_{i \in \sigma} H^{2,0}(X_b)^{\otimes \sigma - \{i\}} \otimes H^{1,1}(X_b). \end{aligned} \quad (2.6)$$

In (2.5), σ ranges over all subsets of $\{1, \dots, n\}$ consisting of m elements, and $H^{2,0}(X_b)^{\otimes \sigma}$ stands for the tensor product of $H^{2,0}(X_b)$ for the j th factors X_b of X_b^n over all $j \in \sigma$. The notations σ' , σ in (2.6) are similar, and $H^{1,1}(X_b)$ in (2.6) is the $H^{1,1}$ of the i th factor X_b of X_b^n .

Let us write $V = H^{2,0}(X_b)$ and $W = (T_b B)^\vee$ for simplicity. The homomorphism (2.4) over $b \in B$ is written as

$$\bigoplus_{|\sigma|=m} \left(V^{\otimes \sigma} \otimes \wedge^l W \rightarrow \bigoplus_{i \in \sigma} V^{\otimes \sigma - \{i\}} \otimes H^{1,1}(X_b) \otimes \wedge^{l+1} W \right). \quad (2.7)$$

By [8, lemma 5.8], the (σ, i) -component

$$V^{\otimes \sigma} \otimes \wedge^l W \rightarrow V^{\otimes \sigma - \{i\}} \otimes H^{1,1}(X_b) \otimes \wedge^{l+1} W \quad (2.8)$$

factorises as

$$\begin{aligned} V^{\otimes \sigma} \otimes \wedge^l W &\rightarrow V^{\otimes \sigma - \{i\}} \otimes H^{1,1}(X_b) \otimes W \otimes \wedge^l W \\ &\rightarrow V^{\otimes \sigma - \{i\}} \otimes H^{1,1}(X_b) \otimes \wedge^{l+1} W, \end{aligned}$$

where the first map is induced by the adjunction $V \rightarrow H^{1,1}(X_b) \otimes W$ of the differential of the period map for the i th factor X_b , and the second map is induced by the wedge product $W \otimes \wedge^l W \rightarrow \wedge^{l+1} W$. By linear algebra, this composition can also be decomposed as

$$\begin{aligned} V^{\otimes \sigma} \otimes \wedge^l W &\rightarrow V^{\otimes \sigma - \{i\}} \otimes V \otimes W^\vee \otimes \wedge^{l+1} W \\ &\rightarrow V^{\otimes \sigma - \{i\}} \otimes H^{1,1}(X_b) \otimes \wedge^{l+1} W, \end{aligned} \quad (2.9)$$

where the first map is induced by the adjunction $\wedge^l W \rightarrow W^\vee \otimes \wedge^{l+1} W$ of the wedge product, and the second map is induced by the adjunction $V \otimes W^\vee \rightarrow H^{1,1}(X_b)$ of the differential of the period map. By our initial Condition 2.1, the second map of (2.9) is injective. Moreover, since $l+1 \leq \dim W$ by our assumption, the wedge product $\wedge^l W \times W \rightarrow \wedge^{l+1} W$ is nondegenerate, so its adjunction $\wedge^l W \rightarrow W^\vee \otimes \wedge^{l+1} W$ is injective. Thus the first map of (2.9) is also injective. It follows that (2.8) is injective. Since the map (2.7) is the direct sum of its (σ, i) -components, it is injective. This finishes the proof of Lemma 2.3.

We can now complete the proof of Proposition 2.2.

Proof of Proposition 2.2. By Lemma 2.3 (2), we have $E_\infty^{l,-l} = 0$ when $l < l_0 = \min(\dim B, k)$. Together with Lemma 2.3 (3), we obtain

$$(\pi_n)_* \Omega_{X_n}^k = E_\infty^0 = E_\infty^{l_0, -l_0} = E_1^{l_0, -l_0}.$$

When $k \leq \dim B$, we have $l_0 = k$, and $E_1^{l_0, -l_0} = \Omega_B^k$ by (2.1). When $k > \dim B$, we have $l_0 = \dim B$, and $E_1^{l_0, -l_0} = \mathcal{H}^{k-\dim B, 0} \otimes K_B$ by (2.1). When $k - \dim B$ is odd, this vanishes by Lemma 2.3 (1).

In the case $k = \dim B + 2m$, the vector bundle $\mathcal{H}^{2m, 0} \otimes K_B = (\pi_n)_* \Omega_{\pi_n}^{2m} \otimes K_B$ can be written more specifically as follows. For a subset σ of $\{1, \dots, n\}$ with cardinality $|\sigma| = m$, we denote by $X_\sigma \simeq X_m$ the fiber product of the i th factors $X \rightarrow B$ of $X_n \rightarrow B$ over all $i \in \sigma$. We denote by

$$X_n \xrightarrow{\pi_\sigma} X_\sigma \xrightarrow{\pi^\sigma} B$$

the natural projections. The Künneth formula (2.5) says that

$$(\pi_n)_* \Omega_{\pi_n}^{2m} \simeq \bigoplus_{|\sigma|=m} \pi_*^\sigma K_{\pi^\sigma}.$$

Combining this with the isomorphism

$$\pi_*^\sigma K_{X_\sigma} \simeq K_B \otimes \pi_*^\sigma K_{\pi^\sigma} \quad (2.10)$$

for each X_σ , we can rewrite the isomorphism in the last case of Proposition 2.2 as

$$(\pi_n)_* \Omega_{X_n}^{\dim B + 2m} \simeq \bigoplus_{|\sigma|=m} \pi_*^\sigma K_{X_\sigma}. \quad (2.11)$$

2.2. Extension over compactification

Let $\pi: X \rightarrow B$ be a $K3$ fibration as in Section 2.1. We now assume that X, B are quasi-projective and π is a morphism of algebraic varieties. We take smooth projective compactifications of X_n, X_σ, B and denote them by $\bar{X}_n, \bar{X}_\sigma, \bar{B}$ respectively.

PROPOSITION 2.4. *We have*

$$H^0(\bar{X}_n, \Omega^k) \simeq \begin{cases} H^0(\bar{B}, \Omega^k) & k \leq \dim B \\ 0 & k > \dim B, \ k \not\equiv \dim B \pmod{2} \\ \bigoplus_\sigma H^0(\bar{X}_\sigma, K_{\bar{X}_\sigma}) & k = \dim B + 2m, \ 0 \leq m \leq n \end{cases}$$

In the last case, σ ranges over all subsets of $\{1, \dots, n\}$ with $|\sigma| = m$. The isomorphism in the first case is given by the pullback by $\pi_n: X_n \rightarrow B$, and the isomorphism in the last case is given by the direct sum of the pullbacks by $\pi_\sigma: X_n \rightarrow X_\sigma$ for all σ .

Proof. The assertion in the case $k > \dim B$ with $k \not\equiv \dim B \pmod{2}$ follows directly from the second case of Proposition 2.2. Next we consider the case $k \leq \dim B$. We may assume that $\pi_n: X_n \rightarrow B$ extends to a surjective morphism $\bar{X}_n \rightarrow \bar{B}$. Let ω be a holomorphic k -form on \bar{X}_n . By the first case of Proposition 2.2, we have $\omega|_{X_n} = \pi_n^* \omega_B$ for a holomorphic

k -form ω_B on B . Since ω is holomorphic over \bar{X}_n , ω_B is holomorphic over \bar{B} as well by a standard property of holomorphic differential forms. (Otherwise ω must have pole at the divisors of \bar{X}_n dominating the divisors of \bar{B} where ω_B has pole.) Therefore the pullback $H^0(\bar{B}, \Omega^k) \rightarrow H^0(\bar{X}_n, \Omega^k)$ is surjective.

Finally, we consider the case $k = \dim B + 2m$, $0 \leq m \leq n$. Let ω be a holomorphic k -form on \bar{X}_n . By (2.11), we can uniquely write $\omega|_{X_n} = \sum_{\sigma} \pi_{\sigma}^* \omega_{\sigma}$ for some canonical forms ω_{σ} on X_{σ} .

Claim 2.5. For each σ , ω_{σ} is holomorphic over \bar{X}_{σ} .

Proof. We identify X_n with the fiber product $X_{\sigma} \times_B X_{\tau}$ where $\tau = \{1, \dots, n\} - \sigma$ is the complement of σ . We may assume that this fiber product diagram extends to a commutative diagram of surjective morphisms

$$\begin{array}{ccc} \bar{X}_n & \xrightarrow{\pi_{\tau}} & \bar{X}_{\tau} \\ \pi_{\sigma} \downarrow & & \downarrow \pi^{\tau} \\ \bar{X}_{\sigma} & \xrightarrow{\pi^{\sigma}} & \bar{B} \end{array}$$

between smooth projective models. We take an irreducible subvariety $\tilde{B} \subset \bar{X}_{\tau}$ such that $\tilde{B} \rightarrow \bar{B}$ is surjective and generically finite. Then $\pi_{\tau}^{-1}(\tilde{B}) \subset \bar{X}_n$ has a unique irreducible component dominating \tilde{B} . We take its desingularisation and denote it by Y . By construction $\pi_{\sigma}|_Y: Y \rightarrow \bar{X}_{\sigma}$ is dominant (and so surjective) and generically finite. On the other hand, for any $\sigma' \neq \sigma$ with $|\sigma'| = m$, the projection $\pi_{\sigma'}|_Y: Y \dashrightarrow X_{\sigma'}$ is not dominant. Indeed, such σ' contains at least one component $i \in \tau$, so if $Y \dashrightarrow X_{\sigma'}$ was dominant, then the i th projection $Y \dashrightarrow X$ would be also dominant, which is absurd because it factorises as $Y \rightarrow \tilde{B} \subset \bar{X}_{\tau} \dashrightarrow X$.

We pullback the differential form $\omega = \pi_{\sigma}^* \omega_{\sigma} + \sum_{\sigma' \neq \sigma} \pi_{\sigma'}^* \omega_{\sigma'}$ to Y and denote it by $\omega|_Y$. Since ω is holomorphic over \bar{X}_n , $\omega|_Y$ is holomorphic over Y . Since $\pi_{\sigma'}^* \omega_{\sigma'}|_Y$ is the pullback of the canonical form $\omega_{\sigma'}$ on $X_{\sigma'}$ by the non-dominant map $Y \dashrightarrow X_{\sigma'}$, it vanishes identically. Hence $\pi_{\sigma}^* \omega_{\sigma}|_Y = \omega|_Y$ is holomorphic over Y . Since $\pi_{\sigma}|_Y: Y \rightarrow \bar{X}_{\sigma}$ is surjective, this implies that ω_{σ} is holomorphic over \bar{X}_{σ} as before.

The above argument will be clear if we consider over the generic point η of B : we restrict ω to the fiber of $(X_{\eta})^n \rightarrow (X_{\eta})^{\tau}$ over the geometric point \tilde{B} of $(X_{\eta})^{\tau}$ over η .

By Claim 2.5, the pullback

$$(\pi_{\sigma}^*)_{\sigma}: \bigoplus_{|\sigma|=m} H^0(\bar{X}_{\sigma}, K_{\bar{X}_{\sigma}}) \rightarrow H^0(\bar{X}_n, \Omega^{\dim B + 2m})$$

is surjective. It is also injective as implied by (2.11). This proves Proposition 2.4.

2.3. Universal K3 surface.

Now we prove Theorem 1.1, in the generality of lattice-polarisation. Let L be an even lattice of signature $(2, d)$ which can be embedded as a primitive sublattice of the K3 lattice $3U \oplus 2E_8$. We denote by

$$\mathcal{D} = \{ \mathbb{C}\omega \in \mathbb{P}L_{\mathbb{C}} \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \}^+$$

the Hermitian symmetric domain associated to L , where $+$ means a connected component.

Let $\pi: X \rightarrow B$ be a smooth projective family of $K3$ surfaces over a smooth quasi-projective connected base B . We say ([3]) that the family $\pi: X \rightarrow B$ is *lattice-polarised with period lattice L* if there exists a sub local system Λ of $R^2\pi_*\mathbb{Z}$ such that each fiber Λ_b is a hyperbolic sublattice of the Néron-Severi lattice $NS(X_b)$ and the fibers of the orthogonal complement Λ^\perp are isometric to L . Then we have a period map

$$\mathcal{P}: B \rightarrow \Gamma \backslash \mathcal{D}$$

for some finite-index subgroup Γ of $O^+(L)$. By Borel's extension theorem, \mathcal{P} is a morphism of algebraic varieties.

Let us put the assumption

$$\mathcal{P} \text{ is birational and } -\text{id} \notin \Gamma. \quad (2.12)$$

For such a family $\pi: X \rightarrow B$, if we shrink B as necessary, then \mathcal{P} is an open immersion and Condition 2.1 is satisfied. For example, the universal $K3$ surface $\mathcal{F}_{g,1} \rightarrow \mathcal{F}_g$ for $g > 2$ restricted over a Zariski open set of \mathcal{F}_g satisfies this assumption with $L = L_g$ and $\Gamma = \Gamma_g$ (see Section 1 for these notations).

As in Section 1, we denote by $M_{\wedge^k, k}(\Gamma)$ the space of vector-valued modular forms of weight (\wedge^k, k) for Γ , $S_l(\Gamma, \det)$ the space of scalar-valued cusp forms of weight l and character \det for Γ , and $\mathcal{S}_{n,m} = \mathcal{S}_n / (\mathcal{S}_m \times \mathcal{S}_{n-m})$.

THEOREM 2.6. *Let $\pi: X \rightarrow B$ be a lattice-polarised $K3$ family with period lattice L of signature $(2, d)$ with $d \geq 3$ and monodromy group Γ satisfying (2.12). Then we have an \mathcal{S}_n -equivariant isomorphism*

$$H^0(\bar{X}_n, \Omega^k) \simeq \begin{cases} 0 & 0 < k < d/2 \\ M_{\wedge^k, k}(\Gamma) & d/2 \leq k < d \\ 0 & k > d, k - d \notin 2\mathbb{Z} \\ S_{d+m}(\Gamma, \det) \otimes \mathbb{C}\mathcal{S}_{n,m} & k = d + 2m, 0 \leq m \leq n \end{cases}.$$

Proof. When $k \leq d$, we have $H^0(\bar{X}_n, \Omega^k) \simeq H^0(\bar{B}, \Omega^k)$ by Proposition 2.4. Then \bar{B} is a smooth projective model of the modular variety $\Gamma \backslash \mathcal{D}$. By a theorem of Pommerening [5], the space $H^0(\bar{B}, \Omega^k)$ for $k < d$ is isomorphic to the space of Γ -invariant holomorphic k -forms on \mathcal{D} , which in turn is identified with the space $M_{\wedge^k, k}(\Gamma)$ of vector-valued modular forms of weight (\wedge^k, k) for Γ (see [4]). The vanishing of this space in $0 < k < d/2$ is proved in [4, theorem 1.2] in the case when L has Witt index 2, and in [4, theorem 1.5 (1)] in the case when L has Witt index ≤ 1 .

The vanishing in the case $k > d$ with $k \not\equiv d \pmod{2}$ follows from Proposition 2.4. Finally, we consider the case $k = d + 2m$, $0 \leq m \leq n$. By Proposition 2.4, we have a natural \mathcal{S}_n -equivariant isomorphism

$$H^0(\bar{X}_n, \Omega^{d+2m}) \simeq \bigoplus_{|\sigma|=m} H^0(\bar{X}_\sigma, K_{\bar{X}_\sigma}),$$

where \mathcal{S}_n permutes the subsets σ of $\{1, \dots, n\}$. Here note that the stabiliser of each σ acts on $H^0(\bar{X}_\sigma, K_{\bar{X}_\sigma})$ trivially by (2.10). Therefore, as an \mathcal{S}_n -representation, the right-hand side can be written as

$$H^0(\bar{X}_m, K_{\bar{X}_m}) \otimes \left(\bigoplus_{|\sigma|=m} \mathbb{C}\sigma \right) \simeq H^0(\bar{X}_m, K_{\bar{X}_m}) \otimes \mathbb{C}\mathcal{S}_{n,m}.$$

Finally, we have $H^0(\bar{X}_m, K_{\bar{X}_m}) \simeq S_{d+m}(\Gamma, \det)$ by [3, theorem 3.1].

Remark 2.7. The case $k \geq d$ of Theorem 2.6 holds also when $d = 1, 2$. We put the assumption $d \geq 3$ for the requirement of the Koecher principle from [5]. Therefore, in fact, only the case $(d, k) = (2, 1)$ with Witt index 2 is not covered.

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