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MINIMISING QUADRATIC FUNCTIONALS OVER CLOSED CONVEX CONES

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In this article we show that, under suitable conditions a quadratic functional attains its minimum on a closed convex cone (in a finite dimensional real Hilbert space) whenever it is bounded below on the cone. As an application, we solve Generalised Linear Complementarity Problems over closed convex cones.

1. INTRODUCTION

Let H denote a finite-dimensional real Hilbert space. Let T (with adjoint T^*) be a linear operator on H, and let K be a closed convex cone in H. Let $q \in H$ and define $f(x) = \langle Tx + q, x\rangle$.

If T is coercive on K (that is $\exists\, \gamma > 0$ with $\langle Tx, x\rangle \geqslant \gamma \|x\|^2\, \forall x \in K$), then any minimising sequence for the problem, $\min\{f(x)\colon x \in K\}$, is bounded, and hence has a subsequence converging to a limit which minimises $f(x)$ over K.

In this paper, we are interested in a generalisation of the Frank-Wolfe Theorem [1]: If a quadratic function is bounded below on a non-empty polyhedral set then it attains its minimum. Our generalisation of the Frank-Wolfe theorem, to a closed convex cone, is stated in Theorem 1. (See [**3**, **4**] for other generalisations). As an application, we solve a generalised linear complementarity porblem over a cone K.

2. MINIMISING THE FUNCTIONAL $f(x)$

In what follows, $q \otimes q$ denotes the operator on H defined by $(q \otimes q)(x) = \langle q, x\rangle q$.

THEOREM 1. *Suppose that*

(i) $x \in K$, $\langle Tx, x\rangle = 0$ *implies* $(T + T^*)x = 0$, *and*

(ii) $\left[\frac{T+T^*}{2} + q \otimes q\right](K)$ *is closed.*

If $f(x) = \langle Tx + q, x\rangle$ *is bounded below on* K *then there is an* $a_0 \in K$ *such that* $f(a_0) = \min\limits_{x \in K} f(x)$.

PROOF: Without loss of generality we can assume that $T = T^*$. Since the result is trivial if $K = \{0\}$, we assume that $K \neq \{0\}$. Let α be a lower bound for $f(x)$ over

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K. Then, for any $x \in K$ and $\lambda > 0$, $\langle T\lambda x + q, \lambda x\rangle \geqslant \alpha$. Dividing by λ^2 and letting $\lambda \to \infty$, we get

$$(2.1) \qquad \langle Tx, x\rangle \geqslant 0 \qquad (x \in K).$$

Let $M = \operatorname{Ker} T \cap \{q\}^{\perp} = \{x \in H : Tx = 0 \wedge \langle q, x\rangle = 0\}$. ∎

Case 1. : $M \cap K = \{0\}$. Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a dense subset of K. For $n = 1, 2, \ldots$, let K_n be the closed convex cone generated by $\{x_1, x_2, \ldots, x_n\}$. Then, each K_n is a polyhedral cone (by definition), $K_n \subseteq K_{n+1}$, and given any $k \in K$ there are $y_n \in K_n$ such that $y_n \to k$ (as $n \to \infty$). By the Frank-Wolfe theorem (stated in Section 1), applied to T and $K_n (n = 1, 2, \ldots)$ there exists an $a_n \in K_n$ such that $f(x) \geqslant f(a_n)$ $(\forall x \in K_n)$, $n = 1, 2, \ldots$. We claim that $\{a_n\}$ is bounded. Suppose not, and without loss of generality, let $\|a_n\| \to \infty$. Since $a_1 \in K_1 \subseteq K_n$,

$$(2.2) \qquad \langle Ta_1 + q, a_1\rangle \geqslant \langle Ta_n + q, a_n\rangle \geqslant \alpha \quad (n = 1, 2, \ldots).$$

We can assume that $\frac{a_n}{\|a_n\|}$ converges to (say) d. Dividing (2.2) by $\|a_n\|^2$ and letting $n \to \infty$, we get $0 \geqslant \langle Td, d\rangle \geqslant 0$. Therefore $\langle Td, d\rangle = 0$. Since $K_n \subset K$ and K is a closed cone, $\frac{a_n}{\|a_n\|} \in K$ and hence $d \in K$. By condition (i), $Td = 0$. Since $\langle Ta_n, a_n\rangle \geqslant 0$ $(n = 1, 2, \ldots)$ (by 2.1),

$$\langle Ta_1 + q, a_1\rangle \geqslant \langle Ta_n + q, a_n\rangle \geqslant \langle q, a_n\rangle \quad (n = 1, 2, \ldots).$$

Once again, divide throughout by $\|a_n\|$ and let $n \to \infty$. Then

$$(2.3) \qquad 0 \geqslant \langle q, d\rangle.$$

Since $d \in K$, $\langle T\lambda d + q, \lambda d\rangle \geqslant \alpha$ $(\lambda \geqslant 0)$. But $Td = 0$. Hence $\lambda\langle q, d\rangle \geqslant \alpha$ $(\lambda \geqslant 0)$. Thus

$$(2.4) \qquad \langle q, d\rangle \geqslant 0.$$

Since $d \in K$ and $Td = 0$, (2.3) and (2.4) show that $d \in M$. Since $M \cap K = \{0\}$, $d = 0$. This is a contradiction since $\|d\| = 1$. Thus, we have proved that $\{a_n\}$ is bounded. Assume, without loss of generality, that $a_n \to a_0 \in K$. Now for any $x \in K$, there exists $y_n \in K_n$ such that $y_n \to x$. Since $f(y_n) \geqslant f(a_n)$ for all n, we have $f(x) \geqslant f(a_0)$. Thus a_0 minimises $f(x)$ over K.

Case 2. : $M \cap K \neq \{0\}$. Let P denote the orthogonal projection from H onto $M^{\perp} = \operatorname{Ran} T + \operatorname{span}\{q\}$. Since (ii) implies that $K + \operatorname{Ker}(T + q \otimes q) = K + \operatorname{Ker} P$ is closed, we see that $P(K)$ is a closed convex cone in $M^{\perp}$. Let $x \in P(K)$ and $\langle Tx, x\rangle = 0$.

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Upon writing $x = Pk$ for some $k \in K$, we see that $\langle TPk,\, Pk\rangle = 0$. Since P is a projection and $k - Pk \in \ker T$, we have $\langle Tk,\, k\rangle = \langle TPk,\, Pk\rangle = 0$. By (i), $Tk = 0$ which implies that $T(Pk) = 0$. Hence we have proved that $x \in P(K)$ and $\langle Tx,\, x\rangle = 0$ imply $Tx = 0$. We also have for any $x \in K$, $f(x) = \langle Tx + q, x\rangle = \langle TPx + q, Px\rangle = f(Px)$. Clearly $M \cap P(K) = \{0\}$. Thus by Case 1, applied to T and $P(K)$, there exists a $b_0 \in P(K)$ such that $f(z) \geqslant f(b_0)$ $(\forall z \in P(K))$.
Writing $b_0 = Pa_0$ for some $a_0 \in K$, we see that $f(x) = f(Px) \geqslant f(b_0) = f(a_0)$ $(\forall x \in K)$.

The following example shows that if condition (ii) in the above theorem were to be omitted then the result need not be true.

Example. In $\mathbb{R}^3$, let $K = \{(x,y,z)\colon x, z \geqslant 0 \wedge 2xz \geqslant y^2\}$. T is defined by $T(x,y,z) = (x,y,0)$ and $q = (1,1,0)$. Since T is a projection, $\langle Tk,\, k\rangle = 0$ implies $Tk = 0$. For $k = (x,y,z) \in K$, we have $\langle Tk+q, k\rangle = x(x+1) + y(y+1) = \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 - \frac{1}{2} \geqslant -\frac{1}{2}$. Hence the quadratic functional $f(k) = \langle Tk + q, k\rangle$ is bounded below on K. Now

$$\begin{aligned}\inf\{f(x)\colon k \in K\} &= -\frac{1}{2} + \inf\left\{\left\|(x,y) - \left(-\frac{1}{2}, -\frac{1}{2}\right)\right\|^2 : (x,y) = (0,\,0) \text{ or } x > 0\right\},\\ &= -\frac{1}{2} + \frac{1}{4}\\ &= -\frac{1}{4}\end{aligned}$$

but this inf is never attained, since the distance between $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and the open right half plane $\cup\{(0,\,0)\}$ is never attained. We finally observe that $(T + q \otimes q)K$ is not closed. (For example, $(1,\,2,\,0)$ is a limit point of $(T + q \otimes q)K$ that is not in $(T + q \otimes q)K$.)

Remark. Here is a *minimisation result that is valid in a reflexive Banach space.* Let B be a reflexive Banach space (with dual B^*). Let $T : B \to B^*$ be linear and continuous, let K be a closed convex cone in B, and $q \in B^*$. Suppose that

(i) K is separable and $0 \notin$ weak-closure $\{x \in K : \|x\| = 1\}$,

(ii) $x \mapsto \langle Tx,\, x\rangle$ is weak-lsc on K, and

(iii) $\{x \in K\colon \langle Tx,\, x\rangle = 0,\, \langle q,\, x\rangle = 0\} = \{0\}$.

If $f(x) = \langle Tx + q, x\rangle$ *is bounded below on* K *then* f *attains its minimum on* K.
We sketch a proof:

(a) If B is finite dimensional and K is polyhedral, we can identify B (and B^*) with (some) $\mathbb{R}^n$ and use the Frank-Wolfe Theorem.

(b) If K is polyhedral in B then $X = K - K$ is a finite dimensional subspace of B. Let J denote the inclusion map from X into B. Then J^*T maps

X into X^*, and $f(x) = \langle Tx + q, x\rangle = \langle J^*Tx + J^*q, x\rangle$ $(\forall x \in K)$. We can now use (a).

(c) If K is a (general) closed convex cone in B then the result is obtained by appropriately modifying the proof of Theorem 1 (and using the fact that a bounded sequence in B has a weakly convergent subsequence).

3. An application to generalised linear complementarity problems

Given T, K, q, we define a *Generalised Linear Complementarity Problem* as follows:

$$GLCP(T,K,q):\ \text{Find } x_0 \in K \text{ such that } \langle Tx_0 + q, k\rangle \geqslant 0\ (\forall k \in K),$$
$$\text{and } \langle Tx_0 + q, x_0\rangle = 0.$$

If there is an $x \in K$ such that $\langle Tx + q, k\rangle \geqslant 0$ $(\forall k \in K)$ then we say that $GLCP(T,K,q)$ is *feasible*. If

(i) $\langle Tk, k\rangle \geqslant 0$ $(\forall k \in K)$, and

(ii) $k \in K$, $\langle Tk, k\rangle = 0$ implies $(T + T^*)k = 0$, then we say that T is *copositive plus* on K.

Theorem 2. *Suppose that*

(i) $T = T^*$,

(ii) T *is copositive plus on* K, *and*

(iii) $(T + q \otimes q)K$ *is closed.*

Then the feasibility of $GLCP(T,K,q)$ *implies its solvability.*

Lemma. *Suppose that* T *is self-adjoint and copositive plus on* $Cl[K + \operatorname{Ker} T \cap \{q\}^{\perp}]$. *If there is an* $a_0 \in K$ *such that* $\langle Ta_0 + q, k\rangle \geqslant 0$ $(\forall k \in K)$, *then there is an* $\alpha \in \mathbb{R}$ *such that* $\langle Tk + q, k\rangle \geqslant \alpha$ $(\forall k \in K)$.

Proof of the Lemma: Suppose that there exists an $x_n \in K$ such that $\langle Tx_n + q, x_n\rangle \leqslant -n$ $(n = 1, 2, \ldots)$. Clearly $\{x_n\}$ is unbounded. Without loss of generality, we can assume that $\|x_n\| \to \infty$ and $d := \lim \frac{x_n}{\|x_n\|}$ exists in K. We have

$$\langle Tx_n + q, x_n\rangle \leqslant 0\ \ (n = 1, 2, \ldots)\quad \text{and}$$
$$\langle q, x_n\rangle \leqslant -\langle Tx_n, x_n\rangle \leqslant 0\quad (n = 1, 2, \ldots)\quad (\text{since } T \text{ is copositive plus on } K).$$

We see immediately that $\langle Td, d\rangle \leqslant 0$ and $\langle q, d\rangle \leqslant 0$. Since T is self-adjoint and copositive plus on K, $\langle Td, d\rangle \leqslant 0$ implies $Td = 0$. Further, since $\langle Ta_0 + q, k\rangle \geqslant 0$ $(\forall k \in K)$ we have

$$\langle q, d\rangle = \langle Ta_0 + q, d\rangle \geqslant 0.$$

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Thus $\langle q, d\rangle = 0$. Hence $d \in M: = \operatorname{Ker} T \cap \{q\}^{\perp}$. If $M = \{0\}$, we have a contradiction since $\|d\| = 1$. So assume that $M \neq \{0\}$, and let P be the orthogonal projection from H onto $M^{\perp} (= \operatorname{Ran} T + \operatorname{span}\{q\})$. We observe that

(i) T is copositive plus on $P(K)$,

(ii) $\langle T(Pa_0) + q, Pk\rangle = \langle Ta_0 + q, k\rangle \geqslant 0$ $(\forall k \in K)$, and

(iii) $\langle T(Px_n) + q, Px_n\rangle = \langle Tx_n + q, x_n\rangle \leqslant -n$ $(n = 1, 2, \ldots)$.

We can assume that $\overline{d} = \lim \frac{Px_n}{\|Px_n\|}$ exists. Then $\overline{d} \in clP(K)$, $\langle T\overline{d}, \overline{d}\rangle = 0$, and $\langle q, \overline{d}\rangle = 0$. Writing $\overline{d} = Px$ for some $x \in H$, we get $\langle Tx, x\rangle = \langle T\overline{d}, \overline{d}\rangle = 0$ and $\langle q, x\rangle = \langle q, \overline{d}\rangle = 0$. Now x belongs to $P^{-1}(clP(K)) = cl[K + \operatorname{Ker} P]$. Since T is copositive plus on $cl[K + \operatorname{Ker} P]$, $\langle Tx, x\rangle = 0$ gives $Tx = 0$. Thus $Tx = 0$ and $\langle q, x\rangle = 0$. Hence $x \in M$. Since $M = \operatorname{Ker} P$, we have $\overline{d} = Px = 0$ contradicting $\|\overline{d}\| = 1$. Hence $\langle Tk + q, k\rangle$ is bounded below on K. ∎

Proof of Theorem 2: By the feasibility of $GLCP(T, K, q)$, there exists an $a_0 \in K$ such that $\langle Ta_0 + q, k\rangle \geqslant 0$ $(\forall k \in K)$. Thus $2a_0 \in K$ and $\langle T(2a_0) + 2q, k\rangle \geqslant 0$ $(\forall k \in K)$. By (iii), $K + \operatorname{Ker}(T + q \otimes q) = K + \operatorname{Ker} T \cap \{q\}^{\perp}$ is closed. By (ii), T is copositive plus on $K + \operatorname{Ker} T \cap \{q\}^{\perp}$. By the above Lemma, $f(k) = \langle Tk + 2q, k\rangle$ is bounded below on K. Now Theorem 1 shows that there is a $k_0 \in K$ such that $\langle Tk + 2q, k\rangle \geqslant \langle Tk_0 + 2q, k_0\rangle$ $(\forall k \in K)$. Since K is a convex cone, $k_0 + t(k - k_0) \in K$ whenever $k \in K$ and $t \in (0, 1]$. We replace k, in the above inequality, by $k_0 + t(k - k_0)$. Upon expanding the left hand side, we get, after cancellation of suitable terms,

$$t\langle Tk_0 + 2q, k - k_0\rangle + t\langle T(k - k_0), k_0\rangle + t^2\langle T(k - k_0), k - k_0\rangle \geqslant 0.$$

We divide throughout by t and let $t \to 0$. Since T is self-adjoint, we get $\langle 2Tk_0 + 2q, k - k_0\rangle \geqslant 0$ $(k \in K)$. Hence $\langle Tk_0 + q, k - k_0\rangle \geqslant 0$. Finally, putting $k = 0$ and $k = 2k_0$ successively, we get $\langle Tk_0 + q, k_0\rangle = 0$. Also, $\langle Tk_0 + q, k\rangle = \langle Tk_0 + q, k - k_0\rangle \geqslant 0$. Thus k_0 solves the $GLCP(T, K, q)$. ∎

Remark. Theorem 2 also appears in [2] where one finds a direct proof (without using the quadradic functional). [2] contains more results about Generalized Linear Complementarity Problems.

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