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APPROXIMATION OF IRRATIONAL NUMBERS BY PAIRS OF INTEGERS FROM A LARGE SET

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Abstract

We show that there is a set $S \subseteq \mathbb{N}$ with lower density arbitrarily close to 1 such that, for each sufficiently large real number α, the inequality $|m\alpha - n| \geq 1$ holds for every pair $(m, n) \in S^2$. On the other hand, if $S \subseteq \mathbb{N}$ has density 1, then, for each irrational $\alpha > 0$ and any positive ε, there exist $m, n \in S$ for which $|m\alpha - n| < \varepsilon$.

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1. Introduction

By Hurwitz's theorem, for each irrational number $\alpha > 0$, there are infinitely many pairs of positive integers (m, n) such that

$$|m\alpha - n| < \frac{1}{\sqrt{5}m} \tag{1.1}$$

(see, for example, [4, page 189] or [16]). In particular, (1.1) implies that if $\alpha > 0$ is irrational, then, for any $\varepsilon > 0$, there exist $m, n \in \mathbb{N}$ for which

$$|m\alpha - n| < \varepsilon. \tag{1.2}$$

For some infinite subsets S of $\mathbb{N}$, the inequality (1.2) also holds for infinitely many pairs (m, n), where $m \in S$ and $n \in \mathbb{N}$. In [10], such a set S is called a *Heilbronn set*. For example, by Furstenberg's theorem (see [2, 7]), the inequality (1.2) with any $\varepsilon > 0$ holds for some $m \in S$ and $n \in \mathbb{N}$, where $S \subseteq \mathbb{N}$ is a multiplicative semigroup with at least two multiplicatively independent integers, for instance, $S = \{p^u q^v \mid u, v \in \mathbb{N} \cup \{0\}\}$, where $p < q$ are two fixed prime numbers. (See [11, 12, 17, 18] for some generalisations of Furstenberg's theorem.) Also, there are some interesting sets S for which the inequality weaker than (1.1) but stronger than (1.2), namely, $|m\alpha - n| < m^{-\tau}$, has been derived for some τ in the range $0 < \tau < 1$. These are, for example, the set of squares

$S = \{n^2 \mid n \in \mathbb{N}\}$ (see [19]) and the set of prime numbers $S = P = \{p_1 < p_2 < p_3 < \cdots\}$ (see [1, 8, 14]), so they are Heilbronn sets.

In this paper, we are interested in obtaining inequality (1.2) for each irrational $\alpha > 0$ when not only just m but both m and n belong to a subset S of $\mathbb{N}$. For an irrational $\alpha > 0$ it is clear that, for each $\varepsilon > 0$, the inequality (1.2) holds with some $m, n \in S$ if and only if $\liminf_{m,n \in S} |m\alpha - n| = 0$.

For a subset E of the set of real numbers $\mathbb{R}$, we define

$$\Delta(E) := \liminf_{x,y \in E,\, x \neq y} |x - y|. \tag{1.3}$$

It is clear that $\Delta(S) \geq 1$ for $S \subseteq \mathbb{N}$. With the notation as in (1.3), the problem we are interested in can be rephrased as follows: for a given $S \subseteq \mathbb{N}$, determine whether or not, for each irrational $\alpha > 0$,

$$\Delta(S \cup \alpha S) = 0 \tag{1.4}$$

or, alternatively, whether or not there exists an irrational $\alpha > 0$ for which

$$\Delta(S \cup \alpha S) > 0. \tag{1.5}$$

For the set of squares $S = \{n^2 \mid n \in \mathbb{N}\}$, we have option (1.5). Indeed, the distance between any two distinct elements of S is at least 3, while the distance between any two distinct elements of αS is at least 3α. Recall that the number $\beta > 0$ is *badly approximable* if there exists a constant $c = c(\beta) > 0$ such that $|m\beta - n| > c/m$ for all $m, n \in \mathbb{N}$. (A number is badly approximable if and only if the partial quotients of its continued fraction are bounded [4, page 190]. For example, all quadratic algebraic numbers β are badly approximable [4, page 194].) For $\alpha = \beta^2$, where $\beta > 0$ is a badly approximable number, the distance between $\alpha m^2 \in \alpha S$ and $n^2 \in S$ is

$$|m^2\beta^2 - n^2| = |(m\beta - n)(m\beta + n))| \geq \frac{c}{m}|m\beta + n| = \frac{c}{m}(m\beta + n) > c\beta = c\sqrt{\alpha}$$

for some $c > 0$. Hence,

$$\Delta(S \cup \alpha S) \geq \min(3, 3\alpha, c\sqrt{\alpha}) > 0$$

for each such α, which proves (1.5). This example appears in Ruzsa's paper [15] in a slightly different context. (We will also use another idea from the proof of [15, Theorem 1] in the proof of our own Theorem 1.2.)

On the other hand, for the set of primes $S = P$, the problem of determining whether we have option (1.4) or (1.5) seems to be out of reach. Option (1.4) takes place if and only if, for each irrational $\alpha > 0$ and any $\varepsilon > 0$, there are prime numbers p_i, p_j satisfying $|p_i\alpha - p_j| < \varepsilon$. This is true if and only if there is an infinite sequence of primes $q_1 < q_2 < q_3 < \cdots$ such that

$$\|q_j\alpha\| \to 0 \quad \text{as } j \to \infty \tag{1.6}$$

and the nearest integer to αq_j, namely,

$$\lfloor q_j\alpha + 1/2 \rfloor, \tag{1.7}$$

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is a prime number. In particular, condition (1.7) alone, without condition (1.6), is satisfied if and only if there are infinitely many primes p for which $\lfloor p\alpha + 1/2 \rfloor$ is also a prime number. For any $\alpha > 0$, which is not an integer, this problem is completely out of reach (even for rational numbers α). For example, for $\alpha = 1/2$, this problem is equivalent to the following. Are there infinitely many primes p for which $2p - 1$ is also a prime?

As for the problem described in (1.2), in general, it is natural to expect that (1.4) is true when the set S is 'large' whereas (1.5) is true when S is 'small'. However, we show that the answer to the problem does not depend just on the size of S. Recall that the *lower* and the *upper density* of the set $E \subseteq \mathbb{N}$ are defined by

$$\underline{d}(E) = \liminf_{x\to\infty} \frac{\#\{E \cap [1, x]\}}{x} \quad \text{and} \quad \overline{d}(E) = \limsup_{x\to\infty} \frac{\#\{E \cap [1, x]\}}{x},$$

respectively. Clearly, $0 \le \underline{d}(E) \le \overline{d}(E) \le 1$. In the case when $\underline{d}(E) = \overline{d}(E)$, their common value $d(E) = \underline{d}(E) = \overline{d}(E)$ is called the *density* of E.

First, observe that, for any $\delta > 0$, there is a set of positive integers S with density at most δ such that, for each irrational $\alpha > 0$, we have $\Delta(S \cup \alpha S) = 0$. To see this, we can take, for example, an integer $b > 1/\delta$ and $S = \{bk \mid k \in \mathbb{N}\}$. Then the set S has density $d(S) = 1/b < \delta$. Also, by (1.1), for each irrational number $\alpha > 0$ there are infinitely many pairs $(m, n) \in \mathbb{N}^2$ for which

$$|bm\alpha - bn| < \frac{b}{\sqrt{5}m}.$$

For any $\varepsilon > 0$, selecting $m > b/\varepsilon\sqrt{5}$, we see that $0 < |bm\alpha - bn| < \varepsilon$ with $bm, bn \in S$. Hence, $\Delta(S \cup \alpha S) = 0$, as claimed. In this direction, it would be of interest to determine whether or not there is a set $S \subseteq \mathbb{N}$ with density zero such that $\Delta(S \cup \alpha S) = 0$ for each irrational α.

In this paper, we investigate the problem in the opposite direction. First, we show that there is a 'large' set S (much greater than the set of squares $\{n^2 \mid n \in \mathbb{N}\}$ with density zero) for which we have option (1.5).

THEOREM 1.1. *For each $\delta > 0$ and each sufficiently large real number α, there is a set of positive integers S with lower density greater than $1 - \delta$ such that*

$$\Delta(S \cup \alpha S) \ge \Delta\Big(\bigcup_{k=0}^{\infty} \alpha^k S\Big) \ge 1. \tag{1.8}$$

Second, we prove that every set $S \subseteq \mathbb{N}$ with density 1 satisfies option (1.4).

THEOREM 1.2. *If S is a set of positive integers with density 1, then, for each irrational number $\alpha > 0$, we have $\Delta(S \cup \alpha S) = 0$.*

One can also consider approximation weaker than that in (1.2), namely, for a given $S \subseteq \mathbb{N}$, investigate whether or not, for each $\alpha > 0$ and any $\varepsilon > 0$, there are $m, n \in S$ for which

$$\left|\alpha - \frac{n}{m}\right| < \varepsilon. \tag{1.9}$$

For example, for the set of primes $S = P$, this problem has been considered in [9]. It was shown there that the quotients of primes are everywhere dense in $[0, \infty)$, so each $\alpha > 0$ can be approximated as in (1.9) by a quotient of two primes n/m. The density of the sequence of rational numbers of the form b^m/m modulo one, where $b \geq 2$ is a fixed integer and m runs through the set $\mathbb{N}$, and similar sequences, have been considered in [3, 5, 6, 13].

The proofs of Theorems 1.1 and 1.2 will be given in Sections 2 and 3, respectively. In fact, the irrationality of α is not relevant in Theorem 1.2. We show that if $S \subseteq \mathbb{N}$ is a set with density 1, then, for each rational $\alpha > 0$,

$$m\alpha - n = 0 \tag{1.10}$$

for infinitely many pairs $(m, n) \in S^2$ (see the end of Section 3).

2. Proof of Theorem 1.1

By the definition of Δ in (1.3), it is clear that $\Delta(E) \geq \Delta(F)$ whenever $E \subseteq F$. Since $S \cup \alpha S$ is a subset of $\bigcup_{k=0}^{\infty} \alpha^k S$, this immediately implies the first inequality in (1.8).

In order to prove the second inequality in (1.8), we fix δ in the interval $(0, 1)$ and a real number α satisfying

$$\alpha > \frac{3}{\delta} + 1. \tag{2.1}$$

We begin the construction of an infinite set $S = \{s_1 < s_2 < s_3 < \cdots\}$ depending on α by selecting $s_1 = 1$. Assume that, for some $m \in \mathbb{N}$, we have already chosen the first m elements $s_1 < s_2 < \cdots < s_m$ of S. The next element s_{m+1} is always taken as the least positive integer that is greater than s_m and is not equal to any of the numbers

$$\lfloor \alpha^k s_j \rfloor, \quad \lceil \alpha^k s_j \rceil, \quad \text{where } k \in \mathbb{N} \text{ and } j = 1, \ldots, m. \tag{2.2}$$

To see that the integers in (2.2) do not occupy all integers greater than s_m and that such an $s_{m+1} > s_m$ always exists, we choose $t = t(m) \in \mathbb{N}$ so large that $\alpha^t > s_m + 2tm + 1$. (This is possible because $\alpha > 1$.) Then, for $k \geq t$, the numbers in (2.2) are all greater than or equal to

$$\lfloor \alpha^k \rfloor > \alpha^k - 1 \geq \alpha^t - 1 > s_m + 2tm,$$

while for k in the range $1 \leq k \leq t - 1$, there are at most $2m(t-1) < 2mt$ integers of the form (2.2). So, for each $m \in \mathbb{N}$, it is always possible to choose the required integer s_{m+1} in the interval $[s_m + 1, s_m + 2tm]$; therefore, the set S is infinite.

We claim that, for this set S, the distance between any two distinct elements of the set

$$S_\alpha := \bigcup_{k=0}^{\infty} \alpha^k S$$

is at least 1. Indeed, take $x = \alpha^u s_i \in S_\alpha$ and $y = \alpha^v s_j \in S_\alpha$, where $u, v \in \mathbb{N} \cup \{0\}$ and $i, j \in \mathbb{N}$. Assume that $x \neq y$. Then $|x - y| \geq 1$ in the case when $u = v$, since $i \neq j$ and $|x - y| = \alpha^u |s_i - s_j|$. Assume that $u \neq v$. Without restriction of generality, we may assume that $u < v$. Setting $w := v - u \in \mathbb{N}$, we find that

$$|x - y| = |\alpha^u s_i - \alpha^v s_j| = \alpha^u |s_i - \alpha^w s_j| \geq |\alpha^w s_j - s_i|.$$

Now, in the case when $i \leq j$, using (2.1) and $s_j \geq s_i$, we deduce that

$$|\alpha^w s_j - s_i| = \alpha^w s_j - s_i \geq \alpha^w s_j - s_j \geq \alpha^w - 1 \geq \alpha - 1 > \frac{3}{\delta} > 3,$$

so $|x - y| > 3$. In the case when $i > j$, by (2.2), s_i is neither $\lfloor \alpha^w s_j \rfloor$ nor $\lceil \alpha^w s_j \rceil$. Thus, the distance between $\alpha^w s_j$ and $s_i \in \mathbb{N}$ is greater than or equal to 1, that is, $|\alpha^w s_j - s_i| \geq 1$. This yields $|x - y| \geq 1$ and implies that $\Delta(S_\alpha) \geq 1$, which is the second inequality in (1.8).

It remains to show that the lower density of S is greater than $1 - \delta$. Let $x \geq \alpha$ be a real number. Choose the unique $\ell \in \mathbb{N}$ for which $\alpha^\ell \leq x + 1 < \alpha^{\ell+1}$. We derive a lower bound for the number of elements of S in the interval $(x/\alpha, x]$. By (2.2), an integer in this interval belongs to S if and only if it is not of the form $\lfloor \alpha^k s_j \rfloor$ or $\lceil \alpha^k s_j \rceil$ for some $k \in \mathbb{N}$ and some $j \in \mathbb{N}$. Note that it is sufficient to consider k in the range $1 \leq k \leq \ell$, since, otherwise, when $k > \ell$,

$$\lceil \alpha^k s_j \rceil \geq \lfloor \alpha^k s_j \rfloor \geq \lfloor \alpha^k \rfloor \geq \lfloor \alpha^{\ell+1} \rfloor > \alpha^{\ell+1} - 1 > x.$$

Fix $k \in \{1, \ldots, \ell\}$. For this k, at least one of the numbers $\lfloor \alpha^k s_j \rfloor$, $\lceil \alpha^k s_j \rceil$ belongs to the interval $(x/\alpha, x]$ only if j is such that $x/\alpha < \lceil \alpha^k s_j \rceil$ or j is such that $\lfloor \alpha^k s_j \rfloor \leq x$. The first inequality does not hold if

$$x \geq \alpha \lceil \alpha^k s_j \rceil \geq \alpha^{k+1} s_j,$$

while the second inequality does not hold if

$$x < \lfloor \alpha^k s_j \rfloor \leq \alpha^k s_j.$$

Consequently, at least one of the inequalities $x/\alpha < \lceil \alpha^k s_j \rceil$ or $\lfloor \alpha^k s_j \rfloor \leq x$ holds only if j is such that

$$\frac{x}{\alpha^{k+1}} < s_j \leq \frac{x}{\alpha^k}. \tag{2.3}$$

Fix a pair of positive integers (k, j) for which (2.3) is true. Recall that $1 \leq k \leq \ell$. The pair (k, j) prevents at most two integers $\lfloor \alpha^k s_j \rfloor$, $\lceil \alpha^k s_j \rceil$ in the interval $(x/\alpha, x]$ from belonging to the set S. Evidently, for each $k \in \{1, \ldots, \ell\}$, there are at most x/α^k indices

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j satisfying (2.3). So, the collection of all relevant pairs (k, j), where $k = 1, \ldots, \ell$ and j satisfies (2.3), prevents at most

$$2\sum_{k=1}^{\ell} \frac{x}{\alpha^k} < 2\sum_{k=1}^{\infty} \frac{x}{\alpha^k} = \frac{2x}{\alpha - 1}$$

integers of the interval $(x/\alpha, x]$ from being in S. It follows that the intersection $S \cap (x/\alpha, x]$ contains at least

$$\lfloor x \rfloor - \lfloor x/\alpha \rfloor - 1 - \frac{2x}{\alpha - 1} > x - 2 - \frac{x}{\alpha} - \frac{2x}{\alpha - 1} = x\Big(1 - \frac{1}{\alpha} - \frac{2}{\alpha - 1}\Big) - 2$$

elements. Therefore,

$$\underline{d}(S) = \liminf_{x\to\infty} \frac{\#\{S \cap [1, x]\}}{x} \geq \liminf_{x\to\infty} \frac{\#\{S \cap (x/\alpha, x]\}}{x}$$
$$\geq 1 - \frac{1}{\alpha} - \frac{2}{\alpha - 1} > 1 - \frac{3}{\alpha - 1},$$

which is greater than $1 - \delta$ in view of (2.1).

3. Proof of Theorem 1.2

Let S be a set of positive integers with density 1 and let $\alpha > 0$ be an irrational number. It is sufficient to prove that

$$\liminf_{m,n\in S} |m\alpha - n| = 0 \tag{3.1}$$

for each irrational α in the range $0 < \alpha < 1$. Indeed, for irrational $\alpha > 1$, applying (3.1) to the number $\alpha^{-1} \in (0, 1)$, by $|m\alpha^{-1} - n| = \alpha^{-1}|m - n\alpha|$, we deduce that

$$\liminf_{m,n\in S} |m - n\alpha| = 0,$$

and hence $\Delta(S \cup \alpha S) = 0$.

So, from now on, we assume that $0 < \alpha < 1$. Let ε be in the range

$$0 < \varepsilon < \tfrac{1}{9}.$$

Throughout, we consider positive integers n satisfying

$$n > \frac{3}{\varepsilon} \quad \text{and} \quad n > \frac{1}{1 - \alpha}. \tag{3.2}$$

Assume that the nth and the $(n + 1)$st convergents of the continued fraction of α are h_n/k_n and h_{n+1}/k_{n+1} (here $h_n, k_n, h_{n+1}, k_{n+1} \in \mathbb{N}$), which means that

$$\Big|\alpha - \frac{h_n}{k_n}\Big| < \frac{1}{k_n k_{n+1}} \quad \text{and} \quad \Big|\alpha - \frac{h_{n+1}}{k_{n+1}}\Big| < \frac{1}{k_{n+1} k_{n+2}} \tag{3.3}$$

(see [4, page 181]). Let u, v be positive integers satisfying

$$u \leq \varepsilon k_{n+1} \quad \text{and} \quad v \leq \varepsilon k_n. \tag{3.4}$$

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(Such integers exist, because $\varepsilon k_{n+1} \geq \varepsilon(k_n + k_{n-1}) > \varepsilon k_n \geq \varepsilon n > 3$ by the first inequality in (3.2).) Consider the rational number

$$\mu := \frac{uh_n + vh_{n+1}}{uk_n + vk_{n+1}}.$$

It is well known that $h_n/k_n < \alpha < h_{n+1}/k_{n+1}$ for even n and $h_{n+1}/k_{n+1} < \alpha < h_n/k_n$ for odd n (see [4, page 181]). In both cases, the numbers α and μ are between the fractions h_n/k_n and h_{n+1}/k_{n+1}. Therefore, by the identity

$$h_{n+1}k_n - h_n k_{n+1} = (-1)^n \tag{3.5}$$

(see [4, page 180]), we derive

$$\left|\alpha - \frac{uh_n + vh_{n+1}}{uk_n + vk_{n+1}}\right| = |\alpha - \mu| < \left|\frac{h_n}{k_n} - \frac{h_{n+1}}{k_{n+1}}\right| = \frac{1}{k_n k_{n+1}}.$$

This, combined with (3.4), implies that, for

$$s(u, v, n) := uk_n + vk_{n+1} \in \mathbb{N} \quad \text{and} \quad t(u, v, n) := uh_n + vh_{n+1} \in \mathbb{N}, \tag{3.6}$$

we have

$$|s(u, v, n)\alpha - t(u, v, n))| < \frac{uk_n + vk_{n+1}}{k_n k_{n+1}} = \frac{u}{k_{n+1}} + \frac{v}{k_n} \leq 2\varepsilon. \tag{3.7}$$

Now, to complete the proof of the theorem it suffices to show that there are $u, v, n \in \mathbb{N}$ satisfying (3.2) and (3.4) such that the integers $s(u, v, n)$, $t(u, v, n)$ as defined in (3.6) both belong to the set S.

Put

$$L_n := \lfloor 2\varepsilon k_n k_{n+1} \rfloor. \tag{3.8}$$

We first show that, for infinitely many $n \in \mathbb{N}$,

$$\#\{j \notin S,\ 1 \leq j \leq L_n\} \leq \varepsilon^2 L_n. \tag{3.9}$$

Indeed, if the inequality opposite to (3.9) holds for all sufficiently large $n \in \mathbb{N}$, then

$$\frac{\#\{j \in S,\ 1 \leq j \leq L_n\}}{L_n} < \frac{L_n - \varepsilon^2 L_n}{L_n} = 1 - \varepsilon^2,$$

and hence

$$\underline{d}(S) = \liminf_{x\to\infty} \frac{\#\{S \cap [1, x]\}}{x} \leq \liminf_{n\to\infty} \frac{\#\{j \in S,\ 1 \leq j \leq L_n\}}{L_n} \leq 1 - \varepsilon^2,$$

which is contrary to $d(S) = \underline{d}(S) = 1$.

We want to show that there are n satisfying (3.2) and $u, v \in \mathbb{N}$ satisfying (3.4) such that $s(u, v, n)$ and $t(u, v, n)$ both belong to S. Take any $n \in \mathbb{N}$ for which the inequalities (3.2) and (3.9) hold. Note that, by (3.4), (3.6) and (3.8), it follows that $s(u, v, n) \leq L_n$. We claim that

$$t(u, v, n) < s(u, v, n) \leq L_n. \tag{3.10}$$

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Indeed, by the first inequality in (3.3), we find that $|k_n\alpha - h_n| < 1/k_{n+1}$. Hence, $h_n < k_n\alpha + 1/k_{n+1}$. By the second inequality in (3.2), we obtain

$$1 < (1-\alpha)n \le (1-\alpha)k_n \le (1-\alpha)k_n^2.$$

It follows that $k_n\alpha + 1/k_{n+1} < 1/k_n + k_n\alpha < k_n$, and hence $h_n < k_n$. By the same argument, from the second inequality in (3.3), we get $h_{n+1} < k_{n+1}$. Thus,

$$t(u,v,n) = uh_n + vh_{n+1} < uk_n + vh_{n+1} = s(u,v,n),$$

which completes the proof of (3.10) because $s(u,v,n) \le L_n$.

By (3.10), the integers $s(u,v,n)$ and $t(u,v,n)$ are distinct. Assume that, for some two pairs of positive integers $(u,v) \ne (u',v')$ satisfying (3.4), we have $s(u,v,n) = s(u',v',n)$. This implies that $uk_n + vk_{n+1} = u'k_n + v'k_{n+1}$ by (3.6). Hence, $(u-u')k_n = (v'-v)k_{n+1}$. By (3.5), the numbers k_n and k_{n+1} are coprime, which implies that $k_n \mid (v'-v)$. However, by (3.4), $1 \le v, v' \le \varepsilon k_n < k_n$, so this is only possible if $v = v'$. This forces $u = u'$, which is a contradiction. Therefore, $s(u,v,n) \ne s(u',v',n)$. By the same argument, we conclude that $t(u,v,n) \ne t(u',v',n)$.

We call a positive integer *bad* if it does not belong to the set S. Similarly, we call a pair of distinct integers $(s(u,v,n), t(u,v,n))$ *bad* if at least one of those integers is bad. Let us consider all bad integers not exceeding L_n. Because of (3.9), there are at most $\varepsilon^2 L_n$ of them. By what we have just shown above, each of them occurs in at most two pairs $(s(u,v,n), t(u,v,n))$. (It may happen that $s(u,v,n)$ is equal to $t(u',v',n)$ for $(u,v) \ne (u',v')$.) So, by (3.8), at most

$$2\varepsilon^2 L_n \le 4\varepsilon^3 k_n k_{n+1}$$

among the pairs under consideration are bad. Note that, by (3.4), there are exactly $\lfloor \varepsilon k_{n+1} \rfloor \lfloor \varepsilon k_n \rfloor$ pairs $(s(u,v,n), t(u,v,n))$ with u, v satisfying (3.4). Using $\varepsilon k_{n+1} > \varepsilon k_n > 3$ and $0 < \varepsilon < 1/9$, we deduce that

$$\lfloor \varepsilon k_{n+1} \rfloor \lfloor \varepsilon k_n \rfloor > (\varepsilon k_{n+1} - 1)(\varepsilon k_n - 1) > \frac{2\varepsilon k_{n+1}}{3} \cdot \frac{2\varepsilon k_n}{3} > 4\varepsilon^3 k_n k_{n+1}.$$

Consequently, there is a pair $(s(u,v,n), t(u,v,n))$, where u, v satisfy (3.4), which is not bad. This means that these two positive integers $s(u,v,n)$, $t(u,v,n)$ for which (3.7) is true are both in S, which is the desired conclusion. This completes the proof of Theorem 1.2.

Finally, to prove (1.10), we write $\alpha = u/v$, where $u, v \in \mathbb{N}$ are coprime. The result is trivial for $u = v = 1$, so assume that $u \ne v$. Take $N \in \mathbb{N}$ and consider the N pairs $(m,n) = (kv, ku)$ with $k = 1, \dots, N$. As above, a positive integer is called bad if it does not belong to the set S. Since the density of S is 1, for infinitely many $N \in \mathbb{N}$, the set $\{1, 2, \dots, N\max(u,v)\}$ contains at most $N/4$ bad integers. Each of those bad integers can appear in at most two pairs (kv, ku) for $k = 1, 2, \dots, N$. So, for at least $N - 2N/4 = N/2$ indices k in the range $1 \le k \le N$, we must have $m = kv \in S$ and $n = ku \in S$. For each of those k and $(m,n) = (kv, ku) \in S^2$, we get $m\alpha - n = kv(u/v) - ku = 0$, as claimed in (1.10).

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