J. Appl. Probab. **61,** 802–809 (2024)
*doi:*10.1017/jpr.2023.81

CHARACTERISTICS OF THE SWITCH PROCESS AND GEOMETRIC DIVISIBILITY

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Abstract

The switch process alternates independently between 1 and -1, with the first switch to 1 occurring at the origin. The expected value function of this process is defined uniquely by the distribution of switching times. The relation between the two is implicitly described through the Laplace transform, which is difficult to use for determining if a given function is the expected value function of some switch process. We derive an explicit relation under the assumption of monotonicity of the expected value function. It is shown that geometric divisible switching time distributions correspond to a non-negative decreasing expected value function. Moreover, an explicit relation between the expected value of a switch process and the autocovariance function of the switch process stationary counterpart is obtained, leading to a new interpretation of the classical Pólya criterion for positive-definiteness.

Keywords: Renewal theory; geometric divisibility; binary processes

2020 Mathematics Subject Classification: Primary 60K05
Secondary 60G55; 60E07; 60G10

1. Introduction

The study of binary stochastic processes has a long-standing tradition in probability theory. There exist many versions of such processes, for example the telegraph process in continuous time or the simple discrete-time Markov chain. These processes found applications in many fields, for example in renewal theory, signal processing [8], and statistical physics [2].

The focus of this paper is the switch process with independent switching times. More specifically, we consider a continuous-time stochastic process taking values in $\{-1, 1\}$, starting at 1 at the origin, and then switching according to an independent and identically distributed (i.i.d.) sequence of non-negative random variables. The switch process always starts from one and hence is not stationary; however, a convenient stationary counterpart can be defined. This counterpart will be referred to as the stationary switch process.

The expected value of the switch process is intrinsically connected with the switching time distribution. This is also the case for the covariance of the stationary switch process. Formalizing this connection is the main contribution of the paper, among other contributions such as formulating and deriving the underlying properties of the switch process. The connection also leads to a class of distributions that constitutes a proper sub-class of geometric infinite divisible distributions introduced in [6].

Received 30 September 2022; accepted 9 October 2023.

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The main results of this paper answer interesting questions related to renewal theory and signal processing. In the context of renewal theory, finding the original distribution given that we observe the thinned process was considered in [11]. From that perspective, Theorem 1 provides additional criteria for when such an inverse problem can be solved. In the signal processing context, Theorem 2 provides a partial solution to the classical difficult problem of obtaining the covariance function from the statistical properties of the point process used to construct binary random signals; see [8] for further discussion of such problems.

The structure of the paper is as follows. In Section 2, the basic concepts are defined. Section 3 contains the first main theorem relating expected value functions to the class of geometric divisible distributions. The second main theorem connecting the covariance of a stationary version of the switch process with the expected value of the switch process is presented in Section 4. In Section 5, a possible application for deriving results that can be used to approximate level-crossing distributions is elaborated on.

2. Preliminaries

2.1. The switch process and its expected value

Let T_k, $k = 1, 2, 3, \ldots$, be a sequence of i.i.d. non-negative random variables with the distribution function F, which is assumed to be absolutely continues with respect to the Lebesgue measure. Additionally, let the corresponding density f associated with F be bounded on any closed interval of the positive half-line. Define a renewal count process for $t \in [0, \infty)$ by

$$N(t) = \begin{cases} \sup\left\{n \in \mathbb{N}; \sum_{k=1}^{n} T_k \leqslant t\right\}, & t \geqslant T_1, \\ 0, & 0 \leqslant t < T_1. \end{cases}$$

In other words, $N(t)$ is the number of renewal events up to a time point t.

Definition 1. Let N(t), $t \geqslant 0$, be a count process. Then the switch process is defined by $X(t) = (-1)^{N(t)}$, $t \geqslant 0$.

The process $X(t)$ switches between the values 1 and -1 at each renewal event, hence the name.

One of the main objects of interest is the expected value function of the switch process $E(t) = \mathbb{E}X(t)$. The relation between E and the switching time distribution, F, is implicit in the time domain. There exist some elementary properties of $E(t)$, which are important but straightforward to derive; see, for example, [3]. First, we have the limiting results for $t \geqslant 0$, which follow from the key renewal theorem:

$$\lim_{t\to 0^+} E(t) = 1, \qquad \lim_{t\to\infty} E(t) = 0. \tag{1}$$

The existence of $E'(t)$ is of importance for the main results of the paper. Under the assumptions stated in this section and assuming $\sup_{t>0} f(t) < \infty$ on $F(t)$, $E'(t)$ exists and is well-defined. The last assumption can be relaxed. Specifically, there needs to exist $l \in \mathbb{N}\colon\ \sup_{u>0} f^{*l}(u) < \infty$, which allows for more general switching time distributions, e.g. those with unbounded densities at zero.

Let $\mathcal{L}(\cdot)$ denote the Laplace transform and, in particular, let $\Psi_F(s) = \mathcal{L}(f)(s)$, where f is the derivative of F when it exists. The Laplace transform of this probability-generating function is

https://doi.org/10.1017/jpr.2023.81 Published online by Cambridge University Press

well-known and has the following form for $s > 0$:

$$\mathcal{L}(E)(s) = \frac{1}{s}\frac{1-\Psi_F(s)}{1+\Psi_F(s)}. \tag{2}$$

This expression is easily solved for the switching time distribution,

$$\Psi_F(s) = \frac{1-s\mathcal{L}(E)(s)}{1+s\mathcal{L}(E)(s)}. \tag{3}$$

Although the above explicit relations tie the distribution of the switch process and its expected value, they do not provide an explicit condition when a function E is the expected value of a switch process. Naturally, by Bernstein's theorem, see [5, Theorem 1, p. 415], we could say that this is the case whenever the right-hand side of the above equation is a complete monotone function, but this condition is not easy to check in a concrete case. In Section 3, easy-to-check conditions for E are presented.

2.2. Geometric divisibility

The concept of geometric infinite divisibility was introduced in [6] and further treated in [1, 7]. It describes distributions that can be represented as a sum of i.i.d. random variables where the number of terms in the sum follows a geometric distribution with an arbitrary parameter $p \in (0, 1)$. The main focus here is on a weaker concept, defined next.

Definition 2. Let ν_p be a geometric random variable with $\mathbb{P}(\nu_p = k) = (1-p)^{k-1}p$ for $k = 1, 2, \ldots$, and $\{\tilde{W}_k\}_{k\geqslant 1}$ a sequence of i.i.d. non-negative random variables independent of ν_p. If the random variable W, with the distribution function F, has the stochastic representation $W = \sum_{k=1}^{\nu_p} \tilde{W}_k$, then W follows an r-geometric divisible distribution with $r = \mathbb{E}\nu_p$ and is said to belong to the class GD(r); we write $F \in GD(r)$.

The distribution of $\tilde{W}$ is then called the r-geometric divisor of the distribution of W. There are two important properties of a $GD(r)$ distribution, which are presented in the following propositions.

Proposition 1.

(i) *The Laplace transform of $F \in GD(r)$ is*

$$\Psi_F(s) = \frac{(1/r)\Psi_{\tilde{F}}(s)}{1-(1-(1/r))\Psi_{\tilde{F}}(s)}.$$

(ii) *The function*

$$\frac{r\Psi_F(s)}{1+(r-1)\Psi_F(s)}$$

is completely monotone if and only if $F \in GD(r)$.

The second important property is key to generalizing the main result in Section 3.

Proposition 2. *Let $u \in \mathbb{R}\colon 1 < u \leqslant r < \infty$; then $GD(r) \subseteq GD(u)$.*

The proofs of both propositions follow using standard methods. However, it should be noted that the first result follows from Bernstein's theorem and the second result follows from the first by using r/u instead of $1/r$ in (ii).

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3. Switch processes with monotonic expected value function

In this section we fully characterize switch processes with monotonic expected value functions. For the main result, we recall the assumptions on the switching time distribution: $F(t)$ has support on $(0, \infty)$ with a density $f(t)$ for which there exists $l \in \mathbb{N}$: $\sup_{t>0} f^{*l}(t) < \infty$ (a technical requirement for the existence of $E'(t)$).

Theorem 1. *Let X(t) be a switch process with E(t) its expected value function and the switching time distribution F(t). Then the following conditions are equivalent:*

(i) $F(t) \in GD(2)$.

(ii) *E(t) is non-negative and decreasing.*

Proof. (i) ⇒ (ii): Since $F(t) \in GD(2)$ it has the following Laplace transform, as described in Section 2:

$$\Psi_F(s) = \frac{\frac{1}{2}\Psi_{\tilde{F}(s)}}{1 - \frac{1}{2}\Psi_{\tilde{F}(s)}}.$$

Substituting this into (2), we have

$$\mathcal{L}(E)(s) = \frac{1}{s}\frac{1 - \dfrac{\frac{1}{2}\Psi_{\tilde{F}}(s)}{1 - \frac{1}{2}\Psi_{\tilde{F}}(s)}}{1 + \dfrac{\frac{1}{2}\Psi_{\tilde{F}}(s)}{1 - \frac{1}{2}\Psi_{\tilde{F}}(s)}} = \frac{1}{s}(1 - \Psi_{\tilde{F}}(s)),$$

which is equivalent to $s\mathcal{L}(E)(s) - 1 = -\Psi_{\tilde{F}}(s)$. The existence of $E'(t)$ is needed in order to use the Laplace transform $\mathcal{L}(E')(s) = s\mathcal{L}(E)(s) - E(0)$. It follows from the stated assumptions by a rather standard although technical argument, see [4, Exercise 4.4.3]. Using the above-stated property of the Laplace transform and the limits of $E(t)$, $\mathcal{L}(-E')(s) = \Psi_{\tilde{F}}(s)$. By taking the inverse Laplace transform, this implies that $-E'(t)$ is a probability density function. Therefore, to satisfy the limiting results of (1), $E(t)$ must satisfy the conditions of (ii).

(ii) ⇒ (i): Under the assumptions of (ii) and the limits of (1) we have

$$\int_0^\infty E'(t)\,\mathrm{d}t = \lim_{t\to\infty} E(t) - \lim_{t\to 0} E(t) = -1;$$

$-E'(t)$ is thus a probability density function. Combining this with the derivative property of the Laplace transform and the limits in (1), (3) becomes

$$\Psi_F(s) = \frac{1 - s\mathcal{L}(E)(s)}{1 + s\mathcal{L}(E)(s)} = \frac{\mathcal{L}(-E')(s)}{2 - \mathcal{L}(-E')(s)} = \frac{\frac{1}{2}\mathcal{L}(-E')(s)}{1 - \frac{1}{2}\mathcal{L}(-E')(s)}.$$

This is the Laplace transform of a $GD(2)$ distribution, as described in Section 2. Therefore, $F(t) \in GD(2)$ with the divisor $-E'(t)$, which yields (i). □

Remark 1. The switch process is a special case of the process $\alpha^{N(t)}$, where $\alpha = -1$. For any α not equal to minus one, the process will either diverge or converge to zero. For $\alpha \in [-1, 0)$, the expected value function is positive and decreasing if and only if the switching time distribution belongs to $GD(1 - \alpha)$. This is shown using an argument similar to the proof of Theorem 1.

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Theorem 1 directly relates functional properties of the expected value of the switch process with the switching time distribution for the class of $GD(2)$ distributions. By combining Theorem 1 and properties of $E(t)$ derived in Section 2, a partial solution can be obtained for the case when the switching time distribution belongs to $GD(2)$. To highlight this partial characterization we have the following corollary, which follows from the second part of the proof of Theorem 1.

Corollary 1. *Let E(t) be a function for* $t \geqslant 0$ *such that the following conditions are satisfied:* $\lim_{t\to 0^+} E(t) = 1$, $\lim_{t\to\infty} E(t) = 0$, *E(t) is at least once differentiable on* $(0, \infty)$, *and* $E'(t) \leqslant 0$ *for all* $t \geqslant 0$; *then it is an expected value function of a switch process with a GD(2) switching time distribution.*

Corollary 2 gives an explicit representation of the distribution function and density for the 2-geometric divisor of the switching time distribution in terms of $E(t)$.

Corollary 2. *Let the switching time distribution, F(t), belong to GD(2), with the divisor* $\tilde{F}(t)$; *then, for* $t \geqslant 0$,

$$E(t) = 1 - \tilde{F}(t), \qquad E'(t) = -\tilde{f}(t).$$

Proposition 2 can be used to extend the results of Theorem 1.

Corollary 3. *Let the switching time distribution be GD(r), for some* $r \geqslant 2$; *then the corresponding expected value function of the switch process, E(t), is non-negative and decreasing for* $t \geqslant 0$.

However, the opposite is not necessarily true, i.e. a non-negative and decreasing expected value function does not necessarily imply an r-geometric divisible switching time for $r > 2$.

Let us consider a switch process constructed from a count process $N(t)$ and satisfying the conditions of Theorem 1. Further, let $\tilde{N}(t)$ be a count process with the arrival times distributed according to the divisor of this switch process. The two count processes are related through thinning. More specifically, $N(t)$ is a thinning of $\tilde{N}(t)$, with the probability of thinning equal to $\frac{1}{2}$. Thus we have the following result.

Corollary 4. *A switch process X(t) is* $\frac{1}{2}$*-thinned if and only if its expected value is non-negative and decreasing.*

From a given trajectory of $N(t)$, the trajectory of process $\tilde{N}(t)$ cannot be recovered, in general. However, it follows from Corollary 1 that the distribution of arrival times of $\tilde{N}(t)$ can be recovered. For further relations between geometric divisibility of the switching time distribution and the thinned renewal processes, see [11, 12].

4. The autocovariance of the stationary switch process

A stationary version of the switch process can be constructed by addressing the behavior around zero. Let $\mu < \infty$ be the expected value of the switching time distribution, and $((A, B), \delta)$ be non-negative random variables, mutually independent and independent of $X(t)$, such that δ takes values $\{-1, 1\}$ with equal probability and such that $f_{A,B}(a, b) = (1/\mu) f_T(a + b)$ so that the marginals of $f_{A,B}$ are $f_A(t) = f_B(t) = (1 - F_T(t))/\mu$.

Definition 3. Let $X_+(t)$ and $X_-(t)$ be two independent switch processes, and $((A, B), \delta)$ be as described above. Define

$$Y(t) = \begin{cases} -\delta, & -B < t < A, \\ \delta X_+(t-A), & t \geqslant A, \\ -\delta X_-(-(t+B)), & t \leqslant -B. \end{cases}$$

Then Y(t) is called a stationary switch process.

The stationarity of $Y(t)$ follows from standard results in renewal theory. In the same way as the switch process is characterized by its expected value function $E(t)$, the stationary switch process is characterized by its covariance function $C(t)$. There exists a relation between $E(t)$ and $C(t)$ presented in the next theorem.

Theorem 2. *Let C(t) be the covariance of the stationary switch process, E(t) be the expected value function of the switch process, and μ be the expected value of the switching time distribution. Then, for $t \geqslant 0$,*

$$C'(t) = -(2/\mu)E(t).$$

Proof. Starting with the covariance of Y(t), and utilizing symmetry, we have

$$(-Y(t) \mid \delta = 1) \overset{\mathrm{d}}{=} (Y(t) \mid \delta = -1)$$

and, for $t > 0$,

$$\begin{aligned} C(t) = \mathbb{E}(\mathbb{E}(Y(t)Y(0) \mid \delta)) = -\mathbb{E}(Y(t) \mid \delta = 1) \\ &= -\int_0^\infty \mathbb{E}(Y(t) \mid \delta = 1, A = x) f_{A \mid \delta=1}(x)\,\mathrm{d}x \\ &= -\bigg(\int_0^t \mathbb{E}(\delta X(t-x) \mid \delta = 1, A = x) f_A(x)\,\mathrm{d}x + \int_t^\infty (-1) f_A(x)\,\mathrm{d}x\bigg). \end{aligned}$$

Since $E(t-x) = 0$, for $x > t$ we obtain $C(t) = 1 - F_A(t) - (E * f_A)(t)$. Using the above expression and (2),

$$\begin{aligned} \mathcal{L}(C)(s) = \frac{1}{s} - \mathcal{L}(f_A)(s)\bigg(\frac{1}{s} + \frac{1}{s}\frac{1-\Psi_F(s)}{1+\Psi_F(s)}\bigg) &= \frac{1}{s} - \frac{1-\Psi_F(s)}{\mu s}\bigg(\frac{2}{s}\frac{1}{1+\Psi_F(s)}\bigg) \\ &= \frac{1}{s}\bigg(1 - \frac{2}{\mu}\mathcal{L}(E)(s)\bigg). \end{aligned}$$

Using $\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$ and $C(0) = 1$,

$$s\mathcal{L}(C)(s) - 1 = -\frac{2}{\mu}\mathcal{L}(E)(s), \qquad C'(t) = -\frac{2}{\mu}E(t). \qquad \square$$

Theorem 2 allows us to use functional properties of the expected value of the switch process to investigate the covariance of the stationary switch process. In particular, combining

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Theorem 2 with Theorem 1 and Proposition 1 yields a partial characterization of the covariance functions.

Corollary 5. *Let C(t) be a symmetric function around zero, $t \in \mathbb{R}$, such that the following conditions are satisfied for all $t \in [0, \infty)$: $C(t) \geqslant 0$, $C'(t) \leqslant 0$, $C''(t) \geqslant 0$, and $C(0) = 1$. Then C(t) is the covariance function of a stationary switch process with a GD(2) switching time distribution.*

Remark 2. Interestingly, the conditions of Corollary 5 are essentially equivalent to those in [9, Theorem 1]. Thus, the above corollary can be viewed as an alternative proof of the Pólya criterion of positive definiteness and consequently implying that characteristic functions satisfying the conditions in [9] can be characterized as covariance functions of stationary switch processes with $GD(2)$ switching time distribution.

By combining Theorem 1 and Theorem 2, the divisor's density and distribution can be derived from the covariance function.

Corollary 6. *Let C(t) be the covariance of the stationary switch process and the switching time distribution belong to GD(2), with the divisor distribution $\tilde{F}$; then, for $t \geqslant 0$,*

$$1 + \frac{\mu}{2} C'(t) = \tilde{F}(t), \qquad \frac{\mu}{2} C''(t) = \tilde{f}(t),$$

where $\mu = -2C'(0^+)$.

The identification of μ does not require geometric divisibility, since it follows from the limits of $E(t)$ and Theorem 2.

Even if the switching time distribution does not belong to $GD(2)$ Theorem 2 is still applicable, as illustrated in the next example.

Example 1. Consider a switch process with $\Gamma(2, 2)$ switching time distribution. The expected value of this switch process, $E(t) = \sqrt{2} \sin((2t + \pi)/4) e^{-t/2}$ is oscillating so that the switching time distribution does not belong to GD(2). By Theorem 2, $C(t) = \cos(t/2) e^{-t/2}$ is the covariance of the stationary switch process.

5. Conclusions

To characterize which functions correspond to the expected value of the switch process is a difficult problem. By exploring the relationship between the functional properties of the expected value and the class of 2-geometric divisible distributions, a partial answer to the problem is given.

An explicit relation between the expected value function of the switch process and the covariance function of the stationary switch process is presented. It leads to corresponding relations between the 2-geometric divisible switching time distributions and the covariance of the stationary switch process. It enables the recovery of the switching time distribution from the covariance function under conditions that are easy to verify. This constitutes a partial solution to the well-known open problem of obtaining the switching time distribution from the covariance function of a continuous-time binary process. Complete answers to both the above-mentioned problems are still unknown.

Finally, it is apposite to mention the connection the presented results have to the persistence studies that are a long-standing and heavily investigated problem of statistical physics, see [2, 10]. There, the independent interval approximation (IIA) framework has been used to approximate the tail distribution and its tail index (persistency exponent). The results of this paper allow us to obtain an explicit IIA representation for many stochastic processes commonly used in statistical physics. This not only provides information about the tail behavior but also yields the explicit approximated distribution of excursions above or below zero. Explicit applications of the obtained results for the independent interval approximations of the level-crossing distributions is planned in future work.

Acknowledgements

The author is grateful for the fruitful discussions and input from Krzysztof Podgórski, Tomasz Kozubowski, Yvette Baurne, Joel Danielsson, and Georg Lindgren. Additionally, the author is very grateful for the precise and constructive comments from the anonymous reviewer and editors.

Funding information

Financial support from the Swedish Research Council (VR) Grant DNR: 2020-05168 is acknowledged.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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