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HOMOMORPHISMS ON AN ORTHOGONALLY DECOMPOSABLE HILBERT SPACE

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Dedicated to Professor B.H. Neumann for his eightieth birthday

For a triple (M, H, ξ_0) of a von Neumann algebra M on a Hilbert space H with a cyclic and separating vector ξ_0, every order isomorphism ϕ of H such that $\phi\xi_0 = \xi_0$ is an orthogonal decomposition isomorphism if and only if ξ_0 is a trace vector.

Let M be a von Neumann algebra on a Hilbert space H. We assume that there is a cyclic and separating vector ξ_0 for M. Let J and Δ_{ξ_0} be the conjugation and modular operator respectively associated with (M, H, ξ_0), and let H^+ be the natural cone, that is,

$$H^+ = \overline{\{xj(x)\xi_0 : x \in M\}} = \overline{\{\Delta_{\xi_0}^{1/4} x\xi_0 : x \in M^+\}},$$

where $j(x) = JxJ$ and M^+ is the positive part of M. Let $L(H)$ be the set of all continuous linear operators on H, and $H^J = \{\xi \in H : J\xi = \xi\}$.

An operator $\phi \in L(H)$ is called an *o.d. (orthogonal decomposition) homomorphism* if the following condition is satisfied: if $\xi = \xi^+ - \xi^-$, where $\xi^+ \in H^+$, $\xi^- \in H^+$ and $(\xi^+, \xi^-) = 0$, is the orthogonal decomposition of $\xi \in H^J$, then $\phi\xi \in H^J$ and $\phi\xi = \phi\xi^+ - \phi\xi^-$ is also the orthogonal decomposition of $\phi\xi$. It is easy to see that, for an operator $\phi \in L(H)$, the following conditions are equivalent:

(1) ϕ is an o.d. homomorphism;

(2) $|\phi\xi| = \phi|\xi|$ for every $\xi \in H^+$, where $|\xi| = \xi^+ + \xi^-$;

(3) $\phi(H^+) \subseteq H^+$, and $(\phi\xi, \phi\eta) = 0$ whenever $\xi \in H^+$, $\eta \in H^+$ and $(\xi, \eta) = 0$.

It was proved in [2] that an operator $\phi \in L(H)$ is an o.d. homomorphism if and only if $\phi(H^+) \subseteq H^+$ and $\phi^*\phi \in M \cap M'$.

An *order isomorphism* is a bijective operator $\phi \in L(H)$ such that $\phi(H^+) = H^+$. If ϕ is a bijective o.d. homomorphism and ϕ^{-1} is also an o.d. homomorphism, ϕ is called an *o.d. isomorphism*.

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The aim of this note is to consider the following property of (M, H, ξ_0).

($\star$) Every order isomorphism ϕ such that $\phi\xi_0 = \xi_0$ is an o.d. isomorphism.

When M is commutative, H is lattice-ordered and o.d. homomorphisms are exactly the lattice homomorphisms. Generally, in Banach lattices, all order isomorphisms are lattice homorphisms. However, the corresponding result for H does not hold. It was proved in [6] that every order isomorphism of H is an o.d. isomorphism if and only if M is commutative.

The algebra M itself is also an ordered Banach space with respect to the positive cone M^+. A bijective linear operator $\alpha: M \to M$ such that $\alpha(1) = 1$ is a Jordan isomorphism if and only if $\alpha(|x|) = |\alpha(x)|$ for all self-adjoint elements x of M ([4]). In other words, unital Jordan isomorphisms are exactly unital o.d. isomorphisms of this "orthogonally decomposable" ordered Banach space M. It is a well-known theorem of Kadison [3] that every order isomorphism α of M such that $\alpha(1) = 1$ is a Jordan isomorphism. Thus the property ($\star$) for H corresponds to this theorem of Kadison's for operators on M.

THEOREM. *For (M, H, ξ_0), the following conditions are equivalent:*

(i) (M, H, ξ_0) *satisfies* ($\star$);
(ii) $\Delta_{\xi_0} = 1$;
(iii) ξ_0 *is a trace vector;*
(iv) $\left(\Delta_{\xi_0}^{1/4} x^+ \xi_0, \Delta_{\xi_0}^{1/4} x^- \xi_0\right) = 0$ *for every self-adjoint* $x \in M$;
(v) $|\Delta_{\xi_0}^{1/4} x \xi_0| = \Delta_{\xi_0}^{1/4} |x| \xi_0$ *for every self-adjoint* $x \in M$;
(vi) $x^+ j(x^-) = 0$ *for every self-adjoint* $x \in M$.

PROOF: The equivalence of (ii) and (iii) is known ([5, E.10.5, p.300]). It is obvious that (ii) implies (iv) and (iv) is equivalent to (v). We shall prove that (i) and (ii) are equivalent, (iv) implies (vi), and (vi) implies (ii).

(i) $\Rightarrow$ (ii). Let x be an arbitrary self-adjoint analytic element of M. Since x is analytic, $a = i\Delta^{1/4} x \Delta^{-1/4}$, where $\Delta = \Delta_{\xi_0}$, is an element of M and

$$(a + j(a))\xi_0 = a\xi_0 + Ja\xi_0 = i\Delta^{1/4} x \xi_0 - iJ\Delta^{1/4} x \xi_0 = 0,$$

because, since x is self-adjoint, $J\Delta^{1/4} x \xi_0 = \Delta^{1/4} x \xi_0$. Then, by [1, Theorem 3.4], the operator $e^{t\delta}$, for $\delta = a + j(a)$, is an order isomorphism for each real number t and it satisfies $e^{t\delta}\xi_0 = \xi_0$, because $\xi_0 = 0$. Hence, by assumption, $e^{t\delta}$ is an o.d. isomorphism. Then, by [2, 4.2], we have $\delta + \delta^\star \in M \cap M'$, and hence, $a + a^\star \in M \cap M'$; that is, $a + a^\star = j(a) + j(a^\star)$. Then $a\xi_0 + a^\star\xi_0 = Ja\xi_0 + Ja^\star\xi_0$. Since $a = i\Delta^{1/4} x \Delta^{-1/4}$, we

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have $\Delta^{1/4}x\xi_0 - \Delta^{-1/4}x\xi_0 = -J\Delta^{1/4}x\xi_0 + J\Delta^{-1/4}x\xi_0$, which holds for any self-adjoint element x of M. Then, for an arbitrary $x \in M$,

$$\begin{aligned}\Delta^{1/4}x\xi_0 - \Delta^{-1/4}x\xi_0 &= -J\Delta^{1/4}x^*\xi_0 + J\Delta^{-1/4}x^*\xi_0 \\ &= -\Delta^{1/4}x\xi_0 + \Delta^{3/4}x\xi_0.\end{aligned}$$

Thus we have $\Delta x\xi_0 - 2\Delta^{1/2}x\xi_0 + x\xi_0 = 0$. This implies $\Delta^{1/2}x\xi_0 = x\xi_0$. Therefore $\Delta = 1$.

(ii) $\Rightarrow$ (i). Since ϕ is an order isomorphism such that $\phi\xi_0 = \xi_0$, we can define a unital Jordan isomorphism α on M by

$$\alpha(x)\xi_0 = \phi(x\xi_0).$$

This follows from (b) of Theorem 2.7 in [1]. Since $\Delta_{\xi_0} = 1$, we have $|x\xi_0| = |x|\xi_0$ for every self-adjoint $x \in M$. Therefore

$$|\phi(x\xi_0)| = |\alpha(x)\xi_0| = |\alpha(x)|\xi_0 = \alpha(|x|)\xi_0 = \phi(|x|)\xi_0 = \phi(|x\xi_0|)$$

for every self-adjoint $x \in M$. Hence, by the continuity of ϕ, we have $|\phi\xi| = \phi|\xi|$ for every $\xi \in H^J$. Thus, ϕ is a bijective o.d. homomorphism. Hence, by (3.1) of [2], ϕ is an o.d. isomorphism.

(iv) $\Rightarrow$ (vi). For a self-adjoint $x \in M$,

$$\begin{aligned}\|(x^+)^{1/2}j(x^-)^{1/2}\xi_0\|^2 &= (x^+j(x^-)\xi_0, \xi_0) = (Jx^-\xi_0, x^+\xi_0) \\ &= \left(\Delta_{\xi_0}^{1/2}x^+\xi_0, x^-\xi_0\right) = \left(\Delta_{\xi_0}^{1/4}x^+\xi_0, \Delta_{\xi_0}^{1/4}x^-\xi_0\right).\end{aligned}$$

(vi) $\Rightarrow$ (ii). It follows from the assumption that $(1-p)j(p)\xi_0 = 0$ for any projection p in M. Therefore,

$$\|p\xi_0\| = \|j(p)\xi_0\| = \|pj(p)\xi_0 + (1-p)j(p)\xi_0\| = \|pj(p)\xi_0\|.$$

Hence, $Jp\xi_0 = p\xi_0$ and, by the spectral theory, we have $Jx\xi_0 = x^*\xi_0$ for every $x \in M$. Hence $\Delta_{\xi_0} = 1$. □

M has a trace vector if and only if M is a finite algebra of countable type.

The isomorphism in the condition (i) cannot be replaced by a homomorphism. To see this, let us consider the following property:

($\star\star$) Every order homomorphism ϕ such that $\phi\xi_0 = \xi_0$ is an o.d. homomorphism.

As the following theorem shows, this property is equivalent to the property that every bijective order homomorphism is an order isomorphism. This is in contrast with the fact ([2, 3.1]) that a bijective o.d. homorphism is always an o.d. isomorphism.

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THEOREM. *The following conditions are equivalent:*

(i) (M, H, ξ_0) *satisfies* $(\star\star)$;

(ii) *if* ϕ *is a bijective order homomorphism,* ϕ *is an order isomorphism;*

(iii) *if* ϕ *is a bijective order homomorphism such that* $\phi\xi_0 = \xi_0$, ϕ *is an order isomorphism;*

(iv) H^J *is isomorphic to the one-dimensional ordered space* $\mathbf{R}$ *of all real numbers.*

PROOF: It is obvious that (iv) implies (i) and (ii), and that (ii) implies (iii). Therefore, we need to prove that (i) implies (iii) and (iii) implies (iv).

(i) $\Rightarrow$ (iii). If ϕ satisfies the assumption, it is a bijective o.d. homomorphism by (i). Then, by [**2**, (3.1)], it is an o.d. isomorphism and hence an order isomorphism.

(iii) $\Rightarrow$ (iv). We assume that $\|\xi_0\| = 1$ and set

$$\phi\xi = \frac{1}{2}(\xi + (\xi, \xi_0)\xi_0) \qquad \text{for all } \xi \in H.$$

Then, $\phi(H^+) \subseteq H^+$, $\phi\xi_0 = \xi_0$ and ϕ is bijective. Hence, by the assumption ϕ^{-1} satisfies $\phi^{-1}(H^+) \subseteq H^+$. Since $\phi^{-1}\xi = 2\xi - (\xi, \xi_0)\xi_0$ for all $\xi \in H$, this implies $2\xi \geqslant (\xi, \xi_0)\xi_0$ for all $\xi \in H^+$. For any $\xi \in H^+$, we then have $2(\xi^+, \xi^-) \geqslant (\xi^+, \xi_0)(\xi^-, \xi_0)$. Hence, we have either $\xi^+ = 0$ or $\xi^- = 0$. This means that H^J is totally ordered and is therefore isomorphic to $\mathbf{R}$. ∎

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