Canad. Math. Bull. Vol. **62** (4), 2019 pp. 912–922
http://dx.doi.org/10.4153/S0008439518000632

Ricci Solitons on Almost Co-Kähler Manifolds

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Abstract. In this paper, we prove that if an almost co-Kähler manifold of dimension greater than three satisfying η-Einstein condition with constant coefficients is a Ricci soliton with potential vector field being of constant length, then either the manifold is Einstein or the Reeb vector field is parallel. Let M be a non-co-Kähler almost co-Kähler 3-manifold such that the Reeb vector field ξ is an eigenvector field of the Ricci operator. If M is a Ricci soliton with transversal potential vector field, then it is locally isometric to Lie group $E(1,1)$ of rigid motions of the Minkowski 2-space.

1 Introduction

Ricci solitons, known as a natural generalization of Einstein metrics, have attracted a lot of attention in differential geometry of almost contact Riemannian manifolds in the last decade. As one of the most important objects of research in contact geometry, co-Kähler manifolds are really odd-dimensional analogs of Kähler manifolds, because a closed manifold is a co-Kähler manifold if and only if it is a Kähler mapping torus (see [9]). Therefore, the study of existence and classification of Ricci solitons on (almost) co-Kähler manifolds is an interesting problem. As far as we know, there are only a few results related to this topic.

The first attempt in this framework was made by Cho in [5]. Cho proved that if a co-Kähler 3-manifold is a Ricci soliton such that the potential vector field is the Reeb vector field ξ, or is a unit vector field orthogonal to ξ, then the manifold is locally flat (*i.e.,* the Riemannian tensor vanishes). Generalizing this result, the present author proved in [19] that if a co-Kähler 3-manifold is a Ricci soliton, then either the manifold is locally flat or the potential vector field is an infinitesimal contact transformation.

In complex geometry, the celebrated Goldberg conjecture says that any compact Einstein almost Kähler manifold is integrable. Naturally, such conjecture has a counterpart in contact geometry; namely, any compact Einstein almost co-Kähler manifold is a co-Kähler manifold. Although until now Goldberg's conjecture was confirmed only for those manifolds having non-negative scalar curvatures, its counterpart in contact geometry for those manifolds having Killing Reeb vector fields has been proved to be true (see [10]). Extending this result, the present author proved in [18] that if a compact almost co-Kähler manifold is a Ricci soliton such that the

Received by the editors August 29, 2018; revised November 20, 2018.
Published online on Cambridge Core May 3, 2019.
This work was supported by Youth Science Foundation of Henan Normal University (No. 2014QK01).
AMS subject classification: **53D15,** 53C25.
Keywords: almost co-Kähler manifold, Ricci soliton, η-Einstein, Lie group.

potential vector field is pointwise collinear with the Reeb vector field, then the manifold is a Ricci-flat co-Kähler manifold and the soliton is steady. Recently, in [4], some geometric restrictions were given under which a compact Einstein almost co-Kähler manifold is a co-Kähler manifold.

In this paper, we continue the investigation of the existence and classification of Ricci solitons on almost co-Kähler manifolds. For dimensions greater than three, we show that if an almost co-Kähler manifold satisfying η-Einstein condition with constant coefficients is a non-trivial Ricci soliton with potential vector field being of constant length, then the Reeb vector field is parallel. Consequently, the manifold is locally isometric to the product manifold of $\mathbb{R}$ and an almost Kähler Ricci soliton. For dimension three, let M be a non-co-Kähler almost co-Kähler 3-manifold with ξ an eigenvector field of the Ricci operator. If M is a Ricci soliton with transversal potential vector field, then M is locally isometric to the unimodular Lie group $E(1,1)$ of rigid motions of the Minkowski 2-space. Some examples verifying our main results are also given.

2 Almost Co-Kähler Manifolds

Let M^{2n+1} be a smooth differential manifold of dimension $2n+1$. On M^{2n+1}, if there exist a $(1,1)$-type tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

then the triple (ϕ, ξ, η) is called an *almost contact structure*, ξ is called the *Reeb vector field*, and η is called an *almost contact* 1*-form*. If, in addition, there exists a Riemannian metric g on M^{2n+1} such that

$$g(\phi\,\cdot, \phi\,\cdot) = g - \eta \otimes \eta,$$

then g is said to be compatible with the almost contact structure, and M^{2n+1} together with the quadruple (ϕ, ξ, η, g) is called an *almost contact metric manifold*. On the product $M^{2n+1} \times \mathbb{R}$ of an almost contact metric manifold M^{2n+1} and $\mathbb{R}$, there exists an almost complex structure J defined by

$$J\Big(X, f\frac{\mathrm{d}}{\mathrm{d}t}\Big) = \Big(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\Big),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$, and f is a $\mathcal{C}^\infty$-function on $M^{2n+1} \times \mathbb{R}$. We denote by $[\phi, \phi]$ the Nijenhuis tensor of ϕ. If

$$[\phi, \phi] = -2\mathrm{d}\eta \otimes \xi$$

holds, or equivalently, J is integrable, then the almost contact metric structure is said to be *normal*.

In this paper, by an *almost co-Kähler manifold* we mean an almost contact metric manifold M^{2n+1} satisfying $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 0$, where the *fundamental* 2*-form* Φ of the almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y. A normal almost co-Kähler manifold is called a *co-Kähler manifold*. An almost co-Kähler structure is said to be *strictly* almost co-Kähler if it is not a co-Kähler structure.

https://doi.org/10.4153/S0008439518000632 Published online by Cambridge University Press

Note that an (almost) co-Kähler manifold is nothing but an (almost) cosymplectic manifold, defined by Blair in [1] and investigated in some other literature [2, 3, 5, 10–12, 19]. On an almost co-Kähler manifold M^{2n+1} we set $h := \frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ is the Lie differentiation. We consider the Jacobi operator $l = R(\cdot,\xi)\xi$ generated by ξ and define $h' := h\circ\phi$, where R denotes the Riemannian curvature tensor of g. From [11, 12], we know that the three $(1,1)$-type tensor fields l, h' and h are symmetric and satisfy $h\xi = 0$, $l\xi = 0$, $\operatorname{tr} h = 0$, $\operatorname{tr}(h') = 0$, and $h\phi + \phi h = 0$ and

$$\nabla\xi = h'. \tag{2.1}$$

From [1], we see that an almost co-Kähler manifold is co-Kähler if and only if

$$\nabla\phi = 0 (\Longleftrightarrow \nabla\Phi = 0).$$

In particular, an almost co-Kähler 3-manifold is co-Kähler if and only if h vanishes (see [11]). In this paper, all manifolds are assumed to be smooth and connected.

3 Ricci Solitons on η-Einstein Almost Co-Kähler Manifolds

On a Riemannian manifold (M,g) if there exist a vector field V and a constant λ such that

$$\frac{1}{2}\mathcal{L}_V g + \operatorname{Ric} = \lambda g, \tag{3.1}$$

where Ric denotes the Ricci tensor and $\mathcal{L}$ is the Lie derivative, then we say that the triple (M,V,λ), or simply, M is a *Ricci soliton*. V and λ are called the *potential vector field* and the *soliton constant*, respectively. A Ricci soliton is said to be shrinking, steady, or expanding according to whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. If the potential vector field is either Killing or vanishes, then the Ricci soliton (3.1) reduces to an Einstein metric and in this case the soliton is said to be trivial. Hamilton [6] introduced the notion of Ricci flow in order to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined by

$$\frac{\partial}{\partial_t} g_{ij}(t) = -2\operatorname{Ric}_{ij}(t).$$

Ricci solitons are self-similar solutions to the Ricci flow. If the potential vector field V is the gradient of some function f on M, then the Ricci soliton equation becomes

$$\operatorname{Hess} f + \operatorname{Ric} = \lambda g$$

and characterizes what is called a *gradient Ricci soliton*. It is known that a Ricci soliton on any compact manifold is always a gradient Ricci soliton (see [14]).

An almost contact metric manifold M^{2n+1} is said to be *η-Einstein* if the Ricci operator satisfies

$$Q = \alpha\mathrm{id} + \beta\eta\otimes\xi, \tag{3.2}$$

where both α and β are smooth functions, and the Ricci operator Q is defined by $\operatorname{Ric}(X,Y) = g(X,QY)$. If, in particular, both α and β are constants, then we say that M satisfies *η-Einstein condition with constant coefficients*.

As seen in [19], any co-Kähler 3-manifold is always η-Einstein. More precisely, we have

$$Q = \frac{r}{2}(\mathrm{id} - \eta \otimes \xi),$$

where r denotes the scalar curvature that is not necessarily a constant. Thus, according to [16, Corollary 4.3], we have the following proposition.

Proposition 3.1 *A co-Kähler 3-manifold is η-Einstein with constant coefficients if and only if it is locally isometric to the product of $\mathbb{R}$ and a 2-manifold of constant Gauss curvature.*

For dimension greater than three, we have the following proposition.

Proposition 3.2 *If, on an almost co-Kähler manifold of dimension greater than three, the Reeb vector field is Killing, then the coefficients of η-Einstein condition are constants.*

Proof Suppose that an almost co-Kähler manifold M has a Killing Reeb vector field. From (2.1), we have $\nabla\xi = 0$ and hence $Q\xi = 0$. If M is η-Einstein, from (3.2), we have $\alpha + \beta = 0$. In this paper, we denote by ∇f the gradient of a smooth function f. Applying the formula $\operatorname{div} Q = \frac{1}{2}\nabla r$ on (3.2), we obtain

$$\frac{1}{2}Y(r) = Y(\alpha) + \xi(\beta)\eta(Y) \tag{3.3}$$

for any vector field Y. It follows directly from (3.2) that $r = (2n+1)\alpha + \beta$. Applying this and $\alpha + \beta = 0$ in (3.3) gives $(n-1)Y(\alpha) + \xi(\alpha)\eta(Y) = 0$ for any vector field Y. In this relation, replacing Y by ξ gives $\xi(\alpha) = 0$, and hence we obtain $(n-1)Y(\alpha) = 0$ for any vector field Y. This completes the proof, because of the assumption $\dim M > 3$. ∎

The coefficients of the η-Einstein condition on a strictly almost co-Kähler manifold with non-Killing Reeb vector field are not necessarily constants for dimension greater than three. For dimension three, note that the η-Einstein condition on an almost co-Kähler manifold implies that the Reeb vector field is harmonic; *i.e.,* it is a critical point for the energy function (see [15, Proposition 4.2]). Therefore, from [17, Theorem 4.1], we have the following proposition.

Proposition 3.3 *A strictly almost co-Kähler 3-manifold is η-Einstein if and only if it is locally isometric to Lie group $E(1,1)$ of rigid motions of the Minkowski 2-space.*

The construction of almost co-Kähler structure on $E(1,1)$ can be seen in [17]. Proposition 3.3 is a nice complement to Proposition 3.1 for the non-co-Kähler case. In view of Propositions 3.1 and 3.2, we next consider η-Einstein condition with constant coefficients on strictly almost co-Kähler manifolds with higher dimensions and prove the following theorem.

Theorem 3.1 *If an almost co-Kähler manifold M of dimension greater than three satisfying η-Einstein condition with constant coefficients is a Ricci soliton with potential vector field being of constant length, then either M is Einstein or the Reeb vector field is parallel.*

Proof Suppose that an almost co-Kähler manifold M of dimension greater than three is η-Einstein with constant coefficients. From [12, Corollary 4.1], we have $\operatorname{Ric}(\xi,\xi) = -\|\nabla\xi\|^2$. Because $h' = h\circ\phi$ has the same norm with h, with the aid of (2.1), it follows that $\operatorname{Ric}(\xi,\xi) = -\|h\|^2$, and this implies

$$\alpha+\beta = -\|h\|^2, \tag{3.4}$$

because of (3.2). According to (3.2) we also have $r = (2n+1)\alpha+\beta$.

If, in addition, M is a Ricci soliton, inserting (3.2) into (3.1) gives

$$(\mathcal{L}_V g)(X,Y) = 2(\lambda-\alpha)g(X,Y) - 2\beta\eta(X)\eta(Y) \tag{3.5}$$

for any vector fields X, Y. Taking the covariant derivative of (3.5) and making use of (2.1), we acquire

$$(\nabla_X\mathcal{L}_V g)(Y,Z) = -2\beta\eta(Z)g(h'X,Y) - 2\beta\eta(Y)g(h'X,Z) \tag{3.6}$$

for any vector fields X, Y, Z. According to Yano [20, p. 23], we write

$$\begin{aligned}(\mathcal{L}_V\nabla_X g - \nabla_X\mathcal{L}_V g - \nabla_{[V,X]}g)(Y,Z) = \\ - g\big((\mathcal{L}_V\nabla)(X,Y),Z\big) - g\big((\mathcal{L}_V\nabla)(X,Z),Y\big)\end{aligned}$$

for any vector fields X, Y, Z. Because of the parallelism of the Riemannian metric g, it follows directly that

$$(\nabla_X\mathcal{L}_V g)(Y,Z) = g\big((\mathcal{L}_V\nabla)(X,Y),Z\big) + g\big((\mathcal{L}_V\nabla)(X,Z),Y\big)$$

for any vector fields X, Y, Z. Interchanging cyclicly the roles of X, Y, and Z in the above relation and by a direct calculation, we obtain

$$2g\big((\mathcal{L}_V\nabla)(X,Y),Z\big) = (\nabla_X\mathcal{L}_V g)(Y,Z) + (\nabla_Y\mathcal{L}_V g)(Z,X) - (\nabla_Z\mathcal{L}_V g)(X,Y)$$

for any vector fields X, Y, Z, where we have used the fact that $\mathcal{L}_X\nabla$ is symmetric. Applying (3.6) in the above relation gives

$$(\mathcal{L}_V\nabla)(X,Y) = -2\beta g(h'X,Y)\xi \tag{3.7}$$

for any vector fields X and Y, where we have used the fact that h' is a symmetric operator. Taking the covariant derivative of (3.7), we acquire

$$(\nabla_X\mathcal{L}_V\nabla)(Y,Z) = -2\beta g\big((\nabla_X h')Y,Z\big)\xi - 2\beta g(h'Y,Z)h'X \tag{3.8}$$

for any vector fields X, Y, Z, where we have used (2.1). Also, according to Yano [20, p. 23], we write

$$(\mathcal{L}_V R)(X,Y,Z) = (\nabla_X\mathcal{L}_V\nabla)(Y,Z) - (\nabla_Y\mathcal{L}_V\nabla)(X,Z)$$

for any vector fields X, Y, Z. Inserting (3.8) into the above relation yields

$$\begin{aligned}(\mathcal{L}_V R)(X,Y,Z) = -2\beta\big(g((\nabla_X h')Y,Z\big)\,\xi + g(h'Y,Z)h'X \\ - g\big((\nabla_Y h')X,Z)\xi - g(h'X,Z)h'Y\big)\end{aligned}$$

for any vector fields X, Y, Z. Contracting the above equation over X gives

$$(\mathcal{L}_V \operatorname{Ric})(Y,Z) = -2\beta g\big((\nabla_\xi h')Y,Z\big) \tag{3.9}$$

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for any vector fields Y, Z, where we have used $\operatorname{tr} h' = 0$, (2.1), and the symmetry property of h'. On the other hand, from (3.2), we have

$$(\mathcal{L}_V \operatorname{Ric})(Y,Z) = \alpha\big(g(\nabla_Y V, Z) + g(\nabla_Z V, Y)\big) \\ + \beta\eta(Z)\big(g(h'Y, V) + g(\nabla_Y V, \xi)\big) + \beta\eta(Y)\big(g(h'Z, V) + g(\nabla_Z V, \xi)\big)$$

for any vector fields Y, Z, where we have used (2.1). Obviously, comparing the previous equation with (3.9) gives

$$2\beta g\big((\nabla_\xi h')Y, Z\big) + \alpha\big(g(\nabla_Y V, Z) + g(\nabla_Z V, Y)\big) \\ + \beta\eta(Z)\big(g(h'Y, V) + g(\nabla_Y V, \xi)\big) + \beta\eta(Y)\big(g(h'Z, V) + g(\nabla_Z V, \xi)\big) = 0$$

for any vector fields Y, Z, which is simplified by (3.5), giving

$$2\beta g\big((\nabla_\xi h')Y, Z\big) + 2\alpha(\lambda - \alpha)g(Y,Z) - 2\alpha\beta\eta(Y)\eta(Z) \\ + \beta\eta(Z)\big(g(h'Y, V) + g(\nabla_Y V, \xi)\big) \\ + \beta\eta(Y)\big(g(h'Z, V) + g(\nabla_Z V, \xi)\big) = 0 \tag{3.10}$$

for any vector fields Y, Z.

In view of $\operatorname{tr} h' = 0$, by a direct calculation, we have $\operatorname{tr} \nabla_\xi h' = 0$. Let $\{e_i : i = 1, \dots, 2n+1\}$ be a local orthonormal frame of the tangent space for each point of M. Thus, putting $Y = Z = e_i$ in (3.10) and summing i over $\{1, 2, \dots, 2n+1\}$, we obtain

$$(2n+1)\alpha(\lambda - \alpha) - \alpha\beta + \beta\eta(\nabla_\xi V) = 0, \tag{3.11}$$

where we have used the fact $h'\xi = 0$. On the other hand, replacing both Y and Z by ξ in (3.10) yields

$$\alpha(\lambda - \alpha - \beta) + \beta\eta(\nabla_\xi V) = 0. \tag{3.12}$$

Obviously, subtracting equation (3.12) from (3.11) gives that

$$\alpha(\lambda - \alpha) = 0. \tag{3.13}$$

If $\beta = 0$, M is Einstein. In what follows, let us consider the other case; *i.e.*, β is a non-zero constant, and this immediately reduces to $\eta(\nabla_\xi V) = \alpha$, because of (3.11) (or (3.12)) and (3.13). Then replacing both X and Y by ξ in (3.5) gives

$$\lambda - 2\alpha - \beta = 0. \tag{3.14}$$

If $\alpha \neq 0$, from (3.13) we obtain $\lambda = \alpha$, and putting this into (3.14), we acquire $\alpha + \beta = 0$. Applying this in (3.4) we obtain $\|h\|^2 = 0$, and hence we have $\nabla\xi = 0$ because of (2.1). This clearly implies that ξ is parallel and hence a Killing vector field. Finally, we consider the last case, that is, $\alpha = 0$ and $\beta \neq 0$, and prove that it is impossible.

In this context, according to (3.11) or (3.12), we have $\eta(\nabla_\xi V) = 0$. Applying this and replacing Y by ξ in (3.10), with the aid of $\alpha = 0$, $\beta \neq 0$, and (2.1), we acquire $Z(g(V, \xi)) = 0$ for any vector field Z, and this means that the component of V along ξ is a constant. Because the length of the potential vector field is a constant, with the aid of $\alpha = 0$ and $\lambda = \beta \neq 0$, putting $X = Y = V$ in (3.5) we observe that V is collinear with the Reeb vector field ξ. Therefore, we can assume that $V = \theta\xi$

with $\theta \in \mathbb{R}$. Substituting this into (3.5), with the aid of (2.1), $\alpha = 0$, $\lambda = \beta \neq 0$, and (3.14), we obtain

$$\theta g(h'X, Y) = \beta\big(g(X, Y) - \eta(X)\eta(Y)\big) \tag{3.15}$$

for any vector fields X, Y. In view of $\operatorname{tr} h' = 0$, putting $X = Y = e_i$ into (3.15) and summing i over $\{1, 2, \dots, 2n+1\}$, we obtain $\beta = 0$, a contradiction. This completes the proof. ∎

Remark 3.1 According to the last part of proof of Theorem 3.1, we observe that conclusions of Theorem 3.1 are still true even if the assumption "potential vector field is of constant length" was weakened to "the length of potential vector field V is constant along V".

Remark 3.2 When studying Ricci solitons on co-Kähler manifolds, "the length of the potential vector field is a constant" was employed by Cho [5].

It was proved in [8, Theorem 2.1] that the product manifold $\mathbb{R} \times F$ is a Ricci soliton if and only if F is a Ricci soliton. Moreover, if the Reeb vector field of an almost co-Kähler manifold is Killing, then the manifold is locally isometric to the product of $\mathbb{R}$ and an almost Kähler manifold (see [11]). Therefore, from Theorem 3.1, we have the following corollary.

Corollary 3.1 *Let M be an almost co-Kähler manifold M of dimension greater than three satisfying η-Einstein condition with constant coefficients. Then M is a Ricci soliton with potential vector field being of constant length if and only if either M is Einstein or M is locally isometric to the product of $\mathbb{R}$ and an almost Kähler Ricci soliton.*

Next we construct two examples to verify our main results.

Example 3.1 Let G be a connected, simply connected Lie group with Lie algebra $\mathfrak{g} = \{e_1, e_2, e_3, e_4, e_5\}$ whose structure equations are given by

$$\begin{aligned}
&de^1 = \frac{\sqrt{3}}{2} e^2 \wedge e^5 + \frac{1}{2} e^1 \wedge e^4, \quad de^2 = \frac{\sqrt{3}}{2} e^1 \wedge e^5 + \frac{1}{2} e^2 \wedge e^4,\\
&de^3 = e^1 \wedge e^2 + e^3 \wedge e^4, \quad de^4 = 0, \quad de^5 = 0,
\end{aligned}$$

where $\{e^1, e^2, e^3, e^4, e^5\}$ is the dual basis for $\mathfrak{g}^*$. Let g be the left invariant metric on G given as

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2.$$

Conti and Fernández [4] proved that G admits an Einstein non-co-Kähler almost co-Kähler structure $(\eta := e^5, \Phi := -e^1 \wedge e^2 - e^3 \wedge e^4, g)$.

Let M be a strictly almost Kähler manifold. Then the product $\mathbb{R} \times M$ admits a strictly almost co-Kähler structure (see [11]) for which the Reeb vector field is Killing. It was proved in [13] that on the product of $\mathbb{R}$ and a hyperbolic 3-space $\mathbb{H}^3(-1)$ there exists a strictly almost Kähler structure. Therefore, we have the following example.

Example 3.2 The product $\mathbb{R}^2 \times \mathbb{H}^3(-1)$ admits a strictly almost co-Kähler structure satisfying η-Einstein condition with constant coefficients; namely, the Ricci operator is given by $Q = -\frac{3}{2}(\mathrm{id} - \eta \otimes \xi)$.

4 Ricci Solitons on Strictly Almost Co-Kähler 3-manifolds

In this section, we denote by M a non-co-Kähler almost co-Kähler 3-manifold. Because of $h \neq 0$, there exists a local orthonormal basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of h for the tangent space at each point $p \in M$. We set $he = \mu e$ and hence $h\phi e = -\mu\phi e$, where μ is a positive function.

Lemma 4.1 ([15, Lemma 2.1]) *On M, the Levi-Civita connection is given by*

$$\nabla_\xi \xi = 0, \quad \nabla_\xi e = a\phi e, \quad \nabla_\xi \phi e = -ae, \quad \nabla_e \xi = -\mu\phi e, \quad \nabla_{\phi e}\xi = -\mu e,$$
$$\nabla_e e = \frac{1}{2\mu}\big(\phi e(\mu) + \sigma(e)\big)\phi e, \quad \nabla_{\phi e}\phi e = \frac{1}{2\mu}\big(e(\mu) + \sigma(\phi e)\big)e,$$
$$\nabla_{\phi e} e = \mu\xi - \frac{1}{2\mu}\big(e(\mu) + \sigma(\phi e)\big)\phi e, \quad \nabla_e \phi e = \mu\xi - \frac{1}{2\mu}\big(\phi e(\mu) + \sigma(e)\big)e,$$

where a is a smooth function and σ is the 1*-form defined by* $\sigma(\cdot) = \mathrm{Ric}(\cdot, \xi)$.

Applying Lemma 4.1, the Ricci operator Q is expressed (see [15]) by

$$\begin{aligned} Q\xi &= -2\mu^2\xi + \sigma(e)e + \sigma(\phi e)\phi e, \\ Qe &= \sigma(e)\xi + \frac{1}{2}(r + 2\mu^2 - 4\mu a)e + \xi(\mu)\phi e, \\ Q\phi e &= \sigma(\phi e)\xi + \xi(\mu)e + \frac{1}{2}(r + 2\mu^2 + 4\mu a)\phi e, \end{aligned} \tag{4.1}$$

with respect to the local basis $\{\xi, e, \phi e\}$, where r denotes the scalar curvature.

On an almost contact metric manifold, if the potential vector field of a Ricci soliton is orthogonal to the Reeb vector field ξ, then we say that the potential vector field is *transversal* (see also [5]).

Theorem 4.1 *Let M be a strictly almost co-Kähler* 3*-manifold such that ξ is an eigenvector field of the Ricci operator. If M is a Ricci soliton with transversal potential vector field, then M is locally isometric to Lie group* $E(1,1)$ *of rigid motions of the Minkowski 2-space. Moreover, M is η-Einstein.*

Proof If the Reeb vector field ξ is an eigenvector field of the Ricci operator, we have $\sigma(e) = \sigma(\phi e) = 0$. Since M is a Ricci soliton, it follows from (3.1) that

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + 2g(X, QY) = 2\lambda g(X, Y) \tag{4.2}$$

for any vector fields X, Y. In view of $Q\xi = -2\mu^2\xi$, replacing both X and Y by ξ in (4.2) and using the fact $\nabla_\xi \xi = 0$ (see (2.1)), we obtain

$$\lambda = -2\mu^2,$$

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where we have used that the potential vector field is orthogonal to ξ. Now from (4.1) we have

$$Qe = \frac{1}{2}(r + 2\mu^2 - 4\mu a)e, \quad Q\phi e = \frac{1}{2}(r + 2\mu^2 + 4\mu a)\phi e. \tag{4.3}$$

Because the potential vector field V is assumed to be orthogonal to the Reeb vector field ξ, we set $V = f_1 e + f_2 \phi e$ for certain smooth functions f_1 and f_2. Inserting $X = Y = e$ in (4.2) gives

$$e(f_1) = \frac{1}{2}\left(4\mu a - r - 6\mu^2\right), \tag{4.4}$$

where we have used Lemma 4.1 and the first term of (4.3). Similarly, inserting $X = Y = \phi e$ in (4.2) gives

$$\phi e(f_2) = -\frac{1}{2}(4\mu a + r + 6\mu^2), \tag{4.5}$$

where we have used Lemma 4.1 and the second term of (4.3). Inserting $X = \xi$ and $Y = e$ in (4.2) gives

$$\xi(f_1) + (\mu - a)f_2 = 0, \tag{4.6}$$

where we have used Lemma 4.1 and $Q\xi = -2\mu^2\xi$. Inserting $X = \xi$ and $Y = \phi e$ in (4.2) gives

$$\xi(f_2) + (\mu + a)f_1 = 0, \tag{4.7}$$

where we have used Lemma 4.1 and $Q\xi = -2\mu^2\xi$. Finally, inserting $X = e$ and $Y = \phi e$ in (4.2) gives

$$e(f_2) + \phi e(f_1) = 0, \tag{4.8}$$

where we have used Lemma 4.1 and (4.3).

Because $Q\xi = -2\mu^2\xi$ and μ is a constant, from Lemma 4.1 we obtain $[\xi, e] = (\mu + a)\phi e$, $[e, \phi e] = 0$, and $[\phi e, \xi] = (a - \mu)e$. The application of the three relations in the Jacobi identity $[[\xi, e], \phi e] + [[e, \phi e], \xi] + [[\phi e, \xi], e] = 0$ implies

$$e(a) = \phi e(a) = 0. \tag{4.9}$$

The application of Lemma 4.1 together with $Q\xi = -2\mu^2\xi$ and (4.3) gives

$$(\nabla_\xi Q)\xi = 0, \quad (\nabla_e Q)e = \frac{1}{2}e(r)e, \quad (\nabla_{\phi e} Q)\phi e = \frac{1}{2}\phi e(r)e,$$

where we have used (4.9) and the fact that μ is a constant. Thus, the application of the above relation on formula $\frac{1}{2}\nabla r = \operatorname{div} Q$ gives $\xi(r) = 0$.

Applying $[\xi, e] = (\mu + a)\phi e$ on f_2 yields $\xi(e(f_2)) - e(\xi(f_2)) = (\mu + a)\phi e(f_2)$. Putting (4.7) into this relation gives

$$\xi\big(e(f_2)\big) = (\mu + a)\big(\phi e(f_2) - e(f_1)\big). \tag{4.10}$$

Similarly, applying $[\xi, \phi e] = (\mu - a)e$ on f_1 yields $\xi(\phi e(f_1)) - \phi e(\xi(f_1)) = (\mu - a)e(f_1)$. Putting (4.6) into this relation gives

$$\xi\big(\phi e(f_1)\big) = (a - \mu)\big(\phi e(f_2) - e(f_1)\big). \tag{4.11}$$

In view of (4.8), the addition of (4.10) to (4.11) yields $a(\phi e(f_2) - e(f_1)) = 0$. Here we conclude that this implies only one case, that is, $a = 0$. Otherwise, if $a \neq 0$,

https://doi.org/10.4153/S0008439518000632 Published online by Cambridge University Press

we have $\phi e(f_2) = e(f_1)$. By applying this, comparing (4.4) with (4.5), we have $a = 0$, because M is a strictly almost co-Kähler manifold, a contradiction. Finally, we have

$$[\xi, e] = \mu\phi e, \quad [e, \phi e] = 0, \quad [\phi e, \xi] = -\mu e \tag{4.12}$$

for a certain positive constant μ. Following (4.12), we state that the manifold is locally isometric to the unimodular Lie group $E(1,1)$ of rigid motions of the Minkowski 2-space.

Applying Lemma 4.1, by a direct calculation we have $R(e,\xi)\xi = -\mu^2 e$ and $R(\phi e,\xi)\xi = \mu^2 e$. It follows that $Qe = 0$, and hence by (4.3) we obtain $Q\phi e = 0$ and $r = -2\mu^2$. Thus, M is η-Einstein, namely $Q = -2\mu^2\eta \otimes \xi$. ■

The present author proved in [19, p. 983] that if a co-Kähler 3-manifold is a Ricci soliton, then either the manifold is locally flat or $\nabla_\xi V = \lambda\xi$. For the later case, if, in addition, the potential vector field is orthogonal to ξ, the soliton constant is given by $\lambda = \eta(\nabla_\xi V) = 0$. Therefore, the following corollary follows directly from [19, Theorem 3.1].

Corollary 4.1 *Let M be a co-Kähler 3-manifold. If M is a Ricci soliton with transversal potential vector field, then either it is locally flat or the soliton is steady.*

Acknowledgment The author would like to thank the reviewer for his or her careful reading and useful suggestions.

References

[1] D. E. Blair, *The theory of quasi-Sasakian structures*. J. Differential Geometry 1(1967), 331–345.
[2] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*. Progress in Mathematics, 203, Birkhäuser, Boston, 2010. https://doi.org/10.1007/978-0-8176-4959-3.
[3] B. Cappelletti-Montano, A. D. Nicola, and I. Yudin, *A survey on cosymplectic geometry*. Rev. Math. Phys. 25(2013), 1343002, 55 pp. https://doi.org/10.1142/S0129055X13430022.
[4] D. Conti and M. Fernández, *Einstein almost cokähler manifolds*. Math. Nachr. 289(2016), 1396–1407. https://doi.org/10.1002/mana.201400412.
[5] J. T. Cho, *Almost contact 3-manifolds and Ricci solitons*. Int. J. Geom. Methods Mod. Phys. 10(2013), 1220022, 7 pp. https://doi.org/10.1142/S0219887812200228.
[6] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*. J. Differ. Geom. 17(1982), 255–306. https://doi.org/10.4310/jdg/1214436922.
[7] R. S. Hamilton, *The Ricci flow on surfaces*. Contemp. Math., 71, American Mathematicl Society, Providence, RI, 1988. https://doi.org/10.1090/conm/071/954419.
[8] S. D. Lee, H. K. Byung, and J. H. Choi, *On a classification of warped product spaces with gradient Ricci solitons*. Korean J. Math. 24(2016), 627–636. https://doi.org/10.11568/kjm.2016.24.4.627.
[9] H. Li, *Topology of co-symplectic/co-Kähler manifolds*. Asian J. Math. 12(2008), 527–544. https://doi.org/10.4310/AJM.2008.v12.n4.a7.
[10] B. C. Montano and A. M. Pastore, *Einstein-like conditions and cosymplectic geometry*. J. Adv. Math. Stud. 3(2010), 27–40.
[11] Z. Olszak, *On almost cosymplectic manifolds*. Kodai Math. J. 4(1981), 239–250.
[12] Z. Olszak, *Locally conformal almost cosymplectic manifolds*. Colloq. Math. 57(1989), 73–87. https://doi.org/10.4064/cm-57-1-73-87.
[13] T. Oguro and K. Sekigawa, *Almost Kähler structures on the Riemannian product of a 3-dimensional hyperbolic space and a real line*. Tsukuba J. Math. 20(1996), 151–161. https://doi.org/10.21099/tkbjm/1496162985.
[14] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*. 2012. arxiv:math/0211159.

[15] D. Perrone, *Minimal Reeb vector fields on almost cosymplectic manifolds*. Kodai Math. J. 36(2013), 258–274. https://doi.org/10.2996/kmj/1372337517.

[16] W. Wang, *A class of three dimensional almost co-Kähler manifolds*. Palest. J. Math. 6(2017), 111–118.

[17] W. Wang and X. Liu, *Three-dimensional almost co-Kähler manifolds with harmonic Reeb vector field*. Rev. Un. Mat. Argentina 58(2017), 307–317.

[18] Y. Wang, *A generalization of the Goldberg conjecture for co-Kähler manifolds*. Mediterr. J. Math. 13(2016), 2679–2690. https://doi.org/10.1007/s00009-015-0646-8.

[19] Y. Wang, *Ricci solitons on* 3*-dimensional cosymplectic manifolds*. Math. Slovaca 67(2017), 979–984. https://doi.org/10.1515/ms-2017-0026.

[20] K. Yano, *Integral formulas in Riemannian geometry*. Pure and Applied Mathematics, No. 1, Marcel Dekker, New York, 1970.

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