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SOME INTEGER-VALUED TRIGONOMETRIC SUMS

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It is shown that for $m = 1, 2, 3, \ldots,$ the trigonometric sums $\sum_{k=1}^{n}(-1)^{k-1}\cot^{2m-1}((2k-1)\pi/4n)$ and $\sum_{k=1}^{n}\cot^{2m}((2k-1)\pi/4n)$ can be represented as integer-valued polynomials in n of degrees $2m-1$ and $2m$, respectively. Properties of these polynomials are discussed, and recurrence relations for the coefficients are obtained. The proofs of the results depend on the representations of particular polynomials of degree $n-1$ or less as their own Lagrange interpolation polynomials based on the zeros of the nth Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$.

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1. Introduction

An identity of M. Riesz [**4**] (see also Zygmund [**6**, Volume II, p. 10]) states that if S_n is a trigonometric polynomial of degree at most n, then for arbitrary θ,

$$S_n'(\theta) = \frac{1}{4n}\sum_{k=1}^{2n}(-1)^{k-1}\frac{S_n(\theta+\theta_k)}{\sin^2(\theta_k/2)},$$

where $\theta_k = (2k-1)\pi/2n$. Setting $S_n(\theta) = \sin\theta$ and $\theta = 0$ establishes that

$$\sum_{k=1}^{n}(-1)^{k-1}\cot\left(\frac{(2k-1)\pi}{4n}\right) = n, \tag{1.1}$$

while putting $S_n(\theta) = \sin n\theta$ and $\theta = 0$ yields

$$\sum_{k=1}^{n}\cot^2\left(\frac{(2k-1)\pi}{4n}\right) = 2n^2 - n. \tag{1.2}$$

The purpose of this note is to point out that the identities (1.1) and (1.2) can be generalized to sums of arbitrary odd and even powers of $\cot((2k-1)\pi/4n)$, respectively, and that somewhat surprisingly these sums are integers for each value of n.

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Our main results are presented in Theorems 1 and 2. Note that a polynomial $p(x)$ is said to be *integral-valued* if $p(x)$ is an integer whenever x is an integer.

Theorem 1. *For* $m = 1, 2, 3, \ldots,$ *the sum* $\sum_{k=1}^{n}(-1)^{k-1}\cot^{2m-1}((2k-1)\pi/4n)$ *is an odd, integral-valued polynomial* $p_m(n)$ *in* n *of degree* $2m-1$*, of the form*

$$\sum_{k=1}^{n}(-1)^{k-1}\cot^{2m-1}\left(\frac{(2k-1)\pi}{4n}\right) = p_m(n) = \sum_{j=1}^{m} a_{m,j}n^{2j-1}, \tag{1.3}$$

where $a_{m,1} = (-1)^{m-1}$. *The remaining* $a_{m,j}$ *can be determined recursively from the relations*

$$a_{m,j} = \frac{1}{2^{2(m-j)}-1}\sum_{r=1}^{m-j}(-1)^r\binom{2m-1}{r}a_{m-r,j} \qquad (j < m), \tag{1.4}$$

and

$$\sum_{j=1}^{m} a_{m,j} = 1.$$

Hence the leading coefficients of $p_m(n)$ *are given explicitly by*

$$\begin{aligned} a_{m,m} &= \frac{2^{2m-2}}{(2m-2)!}E_{2m-2}, \qquad a_{m,m-1} = -\frac{(2m-1)2^{2m-4}}{3(2m-4)!}E_{2m-4},\\ a_{m,m-2} &= \frac{(2m-1)(5m-6)2^{2m-6}}{45(2m-6)!}E_{2m-6},\\ a_{m,m-3} &= -\frac{(2m-1)(70m^2-217m+153)2^{2m-8}}{2835(2m-8)!}E_{2m-8}, \end{aligned} \tag{1.5}$$

where the E_{2j} *are the even-numbered Euler numbers, defined by*

$$\sec x = \sum_{j=0}^{\infty} E_{2j}\frac{x^{2j}}{(2j)!} \qquad (|x| < \pi/2).$$

Theorem 2. *For* $m = 1, 2, 3, \ldots,$ *the sum* $\sum_{k=1}^{n}\cot^{2m}((2k-1)\pi/4n)$ *is an integral-valued polynomial* $q_m(n)$ *in* n *of degree* $2m$*, of the form*

$$\sum_{k=1}^{n}\cot^{2m}\left(\frac{(2k-1)\pi}{4n}\right) = q_m(n) = (-1)^m n + \sum_{j=1}^{m} b_{m,j}n^{2j}. \tag{1.6}$$

The $b_{m,j}$ can be determined recursively from the relations

$$b_{m,j} = \frac{1}{2^{2(m-j)} - 1} \sum_{r=1}^{m-j} (-1)^r \binom{2m}{r} b_{m-r,j} \qquad (j < m), \tag{1.7}$$

and

$$\sum_{j=1}^{m} b_{m,j} = 1 + (-1)^{m-1}. \tag{1.8}$$

Thus the leading coefficients of $q_m(n)$ are given explicitly by

$$\begin{aligned} b_{m,m} &= \frac{2^{2m-1}}{(2m-1)!} E_{2m-1}, \qquad b_{m,m-1} = -\frac{m2^{2m-2}}{3(2m-3)!} E_{2m-3}, \\ b_{m,m-2} &= \frac{m(10m-7)2^{2m-5}}{45(2m-5)!} E_{2m-5}, \\ b_{m,m-3} &= -\frac{m(70m^2 - 147m + 62)2^{2m-6}}{2835(2m-7)!} E_{2m-7}, \end{aligned} \tag{1.9}$$

where the E_{2j-1} are the odd-numbered Euler numbers (also known as the tangent numbers), defined by

$$\tan x = \sum_{j=1}^{\infty} E_{2j-1} \frac{x^{2j-1}}{(2j-1)!} \qquad (|x| < \pi/2).$$

The proofs of the theorems will be presented in Section 2, and depend on the Lagrange formula for polynomial interpolation based on the zeros of the nth Chebyshev polynomial

$$T_n(x) = \cos(n \arccos x) \qquad (-1 \le x \le 1). \tag{1.10}$$

Calogero and Perelomov [**1**] have obtained trigonometric summation formulas that are similar in type to those reported here. (These results are also described in Gradshteyn and Ryzhik [**2**, pp. 1122–3].) For example, Calogero and Perelomov obtain formulas for $\sum_{k=1}^{n-1} \csc^{2m}(k\pi/n)$, $m = 1, 2, 3, 4$, as polynomials in n of degree $2m$. However, their method of deriving the formulas (by eigenvalue calculations for particular off-diagonal Hermitian matrices) is very different to our approach, and the trigonometric sums they consider are not necessarily integer-valued.

2. Proofs of the Theorems

Our starting point is the Lagrange formula for polynomial interpolation of a given function $f(x)$ at the nodes

$$x_k = x_{k,n} = \cos\left(\frac{(2k-1)\pi}{2n}\right) \qquad (k = 1, 2, \ldots, n). \tag{2.1}$$

(These nodes are the zeros of the nth Chebyshev polynomial $T_n(x)$, defined by (1.10).) The unique polynomial $L_{n-1}(x) = L_{n-1}(f, x)$ of degree $n-1$ or less which agrees with $f(x)$ at all the nodes x_k $(k = 1, 2, \ldots, n)$ is given by

$$L_{n-1}(x) = \frac{T_n(x)}{n}\sum_{k=1}^{n}(-1)^{k-1}\frac{(1-x_k^2)^{1/2}}{x-x_k}f(x_k).$$

(See, for example, Rivlin [5, Section 1.3].) Now, if $f(x)$ is itself a polynomial $p_{n-1}(x)$ of degree $n-1$ or less, then $L_{n-1}(x) \equiv p_{n-1}(x)$, and so

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{(1-x_k^2)^{1/2}}{x-x_k}p_{n-1}(x_k) = n\frac{p_{n-1}(x)}{T_n(x)}.$$

Differentiating this formula $r(\geq 0)$ times, then putting $x = 1$, gives

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{(1-x_k^2)^{1/2}}{(1-x_k)^{r+1}}p_{n-1}(x_k) = (-1)^r\frac{n}{r!}\left[\frac{d^r}{dx^r}\left(\frac{p_{n-1}(x)}{T_n(x)}\right)\right]_{x=1},$$

or (on employing (2.1)),

$$\sum_{k=1}^{n}(-1)^{k-1}\csc^{2r}\frac{\theta_k}{2}\cot\frac{\theta_k}{2}p_{n-1}(x_k) = (-1)^r\frac{2^r n}{r!}\left[\frac{d^r}{dx^r}\left(\frac{p_{n-1}(x)}{T_n(x)}\right)\right]_{x=1}, \tag{2.2}$$

where $\theta_k = (2k-1)\pi/2n$. We will shortly exploit this formula by choosing specific $p_{n-1}(x)$, but firstly we establish a lemma that will be used to interpret the right-hand side of (2.2).

Lemma 1. *For $k = 0, 1, 2, \ldots,$ put*

$$A_k = A_k(n) = \frac{1}{k!}\left[\frac{d^k}{dx^k}(T_n(x))^{-1}\right]_{x=1}.$$

Then A_k is an even integral-valued polynomial in n of degree no greater than $2k$, with constant term of 1 if $k = 0$ and 0 if $k \geq 1$.

Proof. Since $T_n(1) = 1$, the result is true for $k = 0$. Suppose, by induction, that the lemma holds true for $k = 0, 1, 2, \ldots, \ell - 1$ ($\ell \geq 1$). Then, on differentiating ℓ times the identity $T_n(x)(T_n(x))^{-1} = 1$, then putting $x = 1$, we obtain

$$\sum_{k=0}^{\ell} \binom{\ell}{k} T_n^{(\ell-k)}(1) k! A_k = 0,$$

or

$$A_\ell = -\sum_{k=0}^{\ell-1} \binom{\ell}{k} \frac{k!}{\ell!} T_n^{(\ell-k)}(1) A_k. \tag{2.3}$$

By Rivlin [5, p. 38],

$$T_n^{(r)}(1) = 2^{r-1}(r-1)!n\binom{n+r-1}{2r-1} \qquad (r = 1, 2, 3, \ldots). \tag{2.4}$$

(This result is obtained by differentiating $r-1$ times the equation $(1-x^2)T_n'' - xT_n' + n^2T_n = 0$, then putting $x = 1$ to yield the recurrence relation $T_n^{(r)}(1) = (n^2 - (r-1)^2)(2r-1)^{-1}T_n^{(r-1)}(1)$.) Hence (2.3) can be written as

$$A_\ell = -\sum_{k=0}^{\ell-1} 2^{\ell-k-1} \frac{n}{\ell-k} \binom{n+\ell-k-1}{2\ell-2k-1} A_k. \tag{2.5}$$

Now, an even polynomial $P(x)$ of degree $2m$ is integral-valued if and only if it can be written as

$$P(x) = d_0 + d_1 \frac{x}{1}\binom{x}{1} + d_2 \frac{x}{2}\binom{x+1}{3} + \ldots + d_m \frac{x}{m}\binom{x+m-1}{2m-1},$$

where $d_0, d_1, d_2, \ldots, d_m$ are integers. (See, for example, Pólya and Szegö [3, pp. 129–130].) Thus for $0 \leq k \leq \ell - 1$, the quantity

$$2^{\ell-k-1} \frac{n}{\ell-k} \binom{n+\ell-k-1}{2\ell-2k-1} \tag{2.6}$$

is an even integral-valued polynomial in n of degree $2\ell - 2k$, and so by the induction assumption, the right-hand side of (2.5) is a sum of even integral-valued polynomials in n of degree 2ℓ or less. Hence A_ℓ is an even integral-valued polynomial in n of degree no greater than 2ℓ. Further, n^2 is a factor of each term of the form (2.6) for $k = 0, 1, \ldots, \ell - 1$, and so n^2 is a factor of A_ℓ. Thus A_ℓ has zero constant term, and the lemma is established. □

Proof of Theorem 1. Put $p_{n-1}(x) \equiv 1$ in (2.2), so that

$$\sum_{k=1}^{n}(-1)^{k-1}\csc^{2r}\frac{\theta_k}{2}\cot\frac{\theta_k}{2} = (-1)^r 2^r n A_r.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k-1}\cot^{2m-1}\frac{\theta_k}{2} &= (-1)^{m-1}\sum_{k=1}^{n}(-1)^{k-1}\left(1-\csc^2\frac{\theta_k}{2}\right)^{m-1}\cot\frac{\theta_k}{2}\\ &= (-1)^{m-1}\sum_{r=0}^{m-1}(-1)^r\binom{m-1}{r}\sum_{k=1}^{n}(-1)^{k-1}\csc^{2r}\frac{\theta_k}{2}\cot\frac{\theta_k}{2}\\ &= (-1)^{m-1}\sum_{r=0}^{m-1}\binom{m-1}{r}2^r n A_r.\end{aligned}$$

By Lemma 1 it follows that $\sum_{k=1}^{n}(-1)^{k-1}\cot^{2m-1}\frac{\theta_k}{2}$ is an odd integral-valued polynomial in n of degree no greater than $2m-1$, whose coefficient of n is $(-1)^{m-1}$.

To obtain the recurrence relation (1.4), use the identity $\cot 2\theta = (\cot\theta - \tan\theta)/2$, so that

$$\begin{aligned}p_m(n) &= \sum_{k=1}^{n}(-1)^{k-1}\cot^{2m-1}\left(\frac{(2k-1)\pi}{4n}\right)\\ &= \frac{1}{2^{2m-1}}\sum_{k=1}^{n}(-1)^{k-1}\sum_{r=0}^{2m-1}(-1)^r\binom{2m-1}{r}\cot^{2m-2r-1}\left(\frac{(2k-1)\pi}{8n}\right)\\ &= \frac{1}{2^{2m-1}}\sum_{k=1}^{n}(-1)^{k-1}\sum_{r=0}^{m-1}(-1)^r\binom{2m-1}{r}\left[\cot^{2m-2r-1}\left(\frac{(2k-1)\pi}{8n}\right)\right.\\ &\qquad\left.-\tan^{2m-2r-1}\left(\frac{(2k-1)\pi}{8n}\right)\right]\\ &= \frac{1}{2^{2m-1}}\sum_{r=0}^{m-1}(-1)^r\binom{2m-1}{r}\sum_{k=1}^{n}(-1)^{k-1}\left[\cot^{2m-2r-1}\left(\frac{(2k-1)\pi}{8n}\right)\right.\\ &\qquad\left.-\cot^{2m-2r-1}\left(\frac{\pi}{2}-\frac{(2k-1)\pi}{8n}\right)\right]\\ &= \frac{1}{2^{2m-1}}\sum_{r=0}^{m-1}(-1)^r\binom{2m-1}{r}\sum_{k=1}^{2n}(-1)^{k-1}\cot^{2m-2r-1}\left(\frac{(2k-1)\pi}{8n}\right)\\ &= \frac{1}{2^{2m-1}}\sum_{r=0}^{m-1}(-1)^r\binom{2m-1}{r}p_{m-r}(2n).\end{aligned}$$

Thus

$$\sum_{j=1}^{m} a_{m,j} n^{2j-1} = \frac{1}{2^{2m-1}} \sum_{r=0}^{m-1} (-1)^r \binom{2m-1}{r} \sum_{j=1}^{m-r} a_{m-r,j} (2n)^{2j-1}$$
$$= \frac{1}{2^{2m-1}} \sum_{j=1}^{m} 2^{2j-1} \left[\sum_{r=0}^{m-j} (-1)^r \binom{2m-1}{r} a_{m-r,j} \right] n^{2j-1}.$$

Equating coefficients of like powers of n gives

$$a_{m,j} = \frac{1}{2^{2(m-j)}} \sum_{r=0}^{m-j} (-1)^r \binom{2m-1}{r} a_{m-r,j} \qquad (1 \le j \le m),$$

and so

$$a_{m,j} = \frac{1}{2^{2(m-j)} - 1} \sum_{r=1}^{m-j} (-1)^r \binom{2m-1}{r} a_{m-r,j} \qquad (1 \le j < m),$$

which is (1.4).

The recurrence relation that has just been derived enables all the coefficients in the polynomial representation of $\sum_{k=1}^{n} (-1)^{k-1} \cot^{2m-1}\left(\frac{(2k-1)\pi}{4n}\right)$, except for the leading coefficient $a_{m,m}$, to be determined from the coefficients in the representations of $\sum_{k=1}^{n} (-1)^{k-1} \cot^{2j-1}\left(\frac{(2k-1)\pi}{4n}\right)$, where $j < m$. Further, by putting $n = 1$ in (1.3), we obtain $\sum_{j=1}^{m} a_{m,j} = 1$, and so $a_{m,m}$ can be determined from the $a_{m,j}$ $(1 \le j < m)$. An alternate approach to finding the coefficients of $p_m(n)$ is as follows.

For $0 < \theta < \pi/2$, we can write $\cot\theta = \theta^{-1} + O(\theta)$, so $\cot^{2m-1}\theta = \theta^{-(2m-1)} + O(\theta^{-(2m-3)})$. Thus

$$\sum_{k=1}^{n} (-1)^{k-1} \cot^{2m-1}\left(\frac{(2k-1)\pi}{4n}\right) = \left(\frac{4}{\pi}\right)^{2m-1} \left(\sum_{k=1}^{n} \frac{(-1)^{k-1}}{(2k-1)^{2m-1}} \right) n^{2m-1} + O(n^{2m-2}). \qquad (2.7)$$

Now,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{(2k-1)^{2m-1}} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^{2m-1}} - \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^{2m-1}}$$
$$= \frac{\pi^{2m-1}}{2^{2m}(2m-2)!} E_{2m-2} + O(n^{-(2m-1)}). \qquad (2.8)$$

(The summation formula for $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^{2m-1}}$ can be found in, for example, Gradshteyn and Ryzhik [2, p. 7].) From (2.7) and (2.8) it follows that

$$\sum_{k=1}^{n} (-1)^{k-1} \cot^{2m-1}\left(\frac{(2k-1)\pi}{4n}\right) = \frac{2^{2m-2}}{(2m-2)!} E_{2m-2} n^{2m-1} + O(n^{2m-2}). \tag{2.9}$$

Comparing (1.3) and (2.9) gives

$$a_{m,m} = \frac{2^{2m-2}}{(2m-2)!} E_{2m-2}.$$

The remaining formulas in (1.5) (and analogous, though progressively more complicated, formulas for $a_{m,m-j}$, where $j \geq 4$) then follow recursively from (1.4). □

Proof of Theorem 2. Put $p_{n-1}(x) \equiv T_n'(x)$ in (2.2). Since $T_n'(x_k) = (-1)^{k-1} n \csc\theta_k$ (by (1.10)), we obtain

$$\sum_{k=1}^{n} \csc^{2r+2}\frac{\theta_k}{2} = (-1)^r \frac{2^{r+1}}{r!} \sum_{k=0}^{r} \binom{r}{k} k! A_k T_n^{(r-k+1)}(1),$$

and on replacing r with $r-1$ and employing (2.4), it follows that

$$\begin{aligned}\sum_{k=1}^{n} \csc^{2r}\frac{\theta_k}{2} &= (-1)^{r-1} \frac{2^r}{(r-1)!} \sum_{k=0}^{r-1} \binom{r-1}{k} 2^{r-k-1}(r-k-1)!\, n \binom{n+r-k-1}{2r-2k-1} k! A_k \\ &= (-1)^{r-1} 2^r \sum_{k=0}^{r-1} (r-k) 2^{r-k-1} \frac{n}{r-k} \binom{n+r-k-1}{2r-2k-1} A_k.\end{aligned}$$

Now, $(r-k)2^{r-k-1}\frac{n}{r-k}\binom{n+r-k-1}{2r-2k-1}$ is an even integral-valued polynomial in n (with zero constant term) for $k = 0, 1, \ldots, r-1$, and so by Lemma 1, $\sum_{k=1}^{n} \csc^{2r}\frac{\theta_k}{2}$ is an even integral-valued polynomial in n (with zero constant term) for $r = 1, 2, 3, \ldots$ Since $\cot^{2m}\frac{\theta_k}{2} = \left(\csc^2\frac{\theta_k}{2} - 1\right)^m$, we conclude that $\sum_{k=1}^{n} \cot^{2m}\frac{\theta_k}{2}$ is an integral-valued polynomial in n of the form (1.6). Finally, we remark that the verification of the recurrence relation (1.7) and the identities (1.8) and (1.9) can be made in a very similar manner to that employed to prove the corresponding results of Theorem 1. □

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