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ON THE GROUND STATE OF SPIN-1 BOSE–EINSTEIN CONDENSATES WITH AN EXTERNAL IOFFE–PITCHARD MAGNETIC FIELD

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Abstract

In this paper, we prove the existence of the ground state for the spinor Bose–Einstein condensates with an external Ioffe–Pitchard magnetic field in the one-dimensional case. We also characterise the ground states of spin-1 Bose–Einstein condensates with an external Ioffe–Pitchard magnetic field; that is, for ferromagnetic systems, we show that, under some condition, searching for the ground state of ferromagnetic spin-1 Bose–Einstein condensates with an external Ioffe–Pitchard magnetic field can be reduced to a 'one-component' minimisation problem.

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1. Introduction

The experimental realisation of Bose–Einstein condensates (BEC) in magnetically trapped atomic gases at ultra-low temperatures [1, 4, 8] has spurred great excitement in the atomic physics community and renewed interest in studying the macroscopic quantum behaviour of atoms. In earlier BEC experiments, the atoms were confined in a magnetic trap, in which the spin degree of freedom is frozen. The particles are described by a scalar model and the wave function of the particles is governed by the Gross–Pitaevskii equation within the mean-field approximation [9, 15, 16]. One of the most important recent developments in BEC was the study of spin-1 and spin-2 condensates. In contrast to a single component BEC, a spin-F BEC is described by the coupled Gross–Pitaevskii equations which consist of $2F + 1$ equations, each governing one of the $2F + 1$ hyperfine states ($m_F = -F, -F + 1, \ldots, F + 1, F$) within the mean-field approximation [10, 14]. The spin-1 BEC was realised in experiments recently

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by using both ^{23}Na and ^{87}Rb [13, 17]. In fact, the emergence of spin-1 BEC has created great opportunities for understanding degenerate gases with internal degrees of freedom [2, 3, 5, 7, 10, 11].

In this paper, we consider a spin-1 BEC. For temperatures well below the critical temperature, the dynamics of the spin-1 BEC are well described by the dimensionless Gross–Pitaevskii equations in n dimensions ($n \leqslant 3$) [2, 3, 10]

$$i\partial_t\psi_1(x,t) = (-\tfrac{1}{2}\triangle + V(x) + E_1 + \beta_n\rho + \beta_s(\rho_1 + \rho_0 - \rho_{-1}))\psi_1 + \beta_s\overline{\psi}_{-1}\psi_0^2 + B\psi_0, \quad (1.1)$$

$$\begin{aligned} i\partial_t\psi_0(x,t) &= (-\tfrac{1}{2}\triangle + V(x) + E_0 + \beta_n\rho + \beta_s(\rho_1 + \rho_{-1}))\psi_0 \\ &\quad + 2\beta_s\psi_{-1}\overline{\psi}_0\psi_1 + B(\psi_1 + \psi_{-1}), \end{aligned} \quad (1.2)$$

$$\begin{aligned} i\partial_t\psi_{-1}(x,t) &= (-\tfrac{1}{2}\triangle + V(x) + E_{-1} + \beta_n\rho + \beta_s(\rho_{-1} + \rho_0 - \rho_1))\psi_{-1} \\ &\quad + \beta_s\overline{\psi}_1\psi_0^2 + B\psi_0, \end{aligned} \quad (1.3)$$

where $x \in \mathbb{R}^n$, $t > 0$, and the initial value ($t = 0$)

$$\psi_j(x,0) = \psi_j^0(x), \qquad x \in \mathbb{R}^n,\ j = -1, 0, 1.$$

Here, $\Psi = \Psi(x,t) := (\psi_1(x,t), \psi_0(x,t), \psi_{-1}(x,t))^T$ is the dimensionless wave function of the spin-1 BEC, $V(x)$ is the dimensionless external trapping potential, $\rho_j(x,t) := |\psi_j(x,t)|^2$ is the density of the hyperfine spin component $m_F = j$ ($j = -1, 0, 1$) and $\rho = \rho_1 + \rho_0 + \rho_{-1}$ is the total density. Also, $E_j \in \mathbb{R}$ is the dimensionless Zeeman energy of spin component $m_F = j$ ($j = -1, 0, 1$) in the uniform external magnetic field, $B \in \mathbb{R}$ is the dimensionless external Ioffe–Pitchard magnetic field, and β_n and β_s are the dimensionless mean-field and spin-exchange interaction constants, respectively. Furthermore, $\overline{f}$ denotes the conjugate of the function f. For $\beta_n < 0$ (respectively $\beta_n > 0$) the spin-independent interaction is attractive (respectively repulsive). For $\beta_s < 0$ (respectively $\beta_s > 0$), the spin-exchange interaction is ferromagnetic (respectively anti-ferromagnetic).

For the ferromagnetic system (1.1)–(1.3) with $n = 1$, Cao *et al.* [6] proved the existence of the ground state without the Ioffe–Pitchard magnetic field. Recently, Lin *et al.* [12] characterised the ground states of spin-1 Bose–Einstein condensates under no external magnetic field. Motivated by [6, 12], the aim of this paper is to study the ground state of the ferromagnetic system (1.1)–(1.3) for $n = 1$. We consider the simplest case when $V(x) \equiv 0$ and all ψ_j ($j = -1, 0, 1$) are real. We rename ψ_j by u_j ($j = -1, 0, 1$). From (1.1)–(1.3), the energy functional is

$$\begin{aligned} H(u_{-1}, u_0, u_1) = \int_{\mathbb{R}}\Big(&\sum_{j=-1}^{1}\Big(\frac{1}{2}|u_j'|^2 + E_j\Big)u_j^2 + \frac{\beta_n}{2}u_0^4 + \frac{\beta_n + \beta_s}{2}(u_1^4 + u_{-1}^4 + 2u_0^2(u_1^2 + u_{-1}^2)) \\ &+ (\beta_n - \beta_s)u_1^2u_{-1}^2 + 2\beta_s u_1u_0^2u_{-1} + 2Bu_0(u_1 + u_{-1})\Big)\,dx, \end{aligned}$$

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and the following two integrals are conserved:

$$\int_{\mathbb{R}} \sum_{j=-1}^{1} u_j^2(x)\,dx = N, \tag{1.4}$$

$$\int_{\mathbb{R}} (u_1^2(x) - u_{-1}^2(x))\,dx = M. \tag{1.5}$$

Here we assume that

$$N > 0, \quad |M| < N.$$

For given real numbers (N, M), we define

$$E_{N,M} = \Big\{u = (u_{-1}, u_0, u_1) \mid u_j \in H^1(\mathbb{R}),\ j = 1, 2, 3,$$
$$\int_{\mathbb{R}} \sum_{j=-1}^{1} u_j^2(x)\,dx = N,\ \int_{\mathbb{R}} (u_1^2(x) - u_{-1}^2(x))\,dx = M\Big\}.$$

We consider the minimisation problem

$$H_0 = \inf\{H(u) \mid u \in E_{N,M}\}. \tag{1.6}$$

A solution to (1.6) is called a ground state. A ground state (u_{-1}, u_0, u_1) is nontrivial if $u_j \neq 0$, for $j = -1, 0, 1$.

Our main result in this paper is the following theorem.

THEOREM 1.1. *Let $\beta_n < \beta_s < 0$ and $E_0 = E_{-1} \le E_1 < 0$, $B < 0$, $\min\{|E_j|,\ j = -1, 0, 1\} > 2|B|$. Then a nontrivial ground state exists. Moreover, the ground state (u_{-1}, u_0, u_1) is positive and strictly decreasing.*

Following the method in [6], Theorem 1.1 is proved via approximation. Namely, we consider a related minimisation problem in a bounded interval $I_k := [-k, k]$ and then let $k \to +\infty$. More precisely, let us define an energy functional on I_k:

$$H^k(u_{-1}, u_0, u_1) = \int_{I_k}\Big(\sum_{j=-1}^{1}\Big(\frac{1}{2}|u_j'|^2 + E_j\Big)u_j^2 + \frac{\beta_n}{2}u_0^4 + \frac{\beta_n + \beta_s}{2}(u_1^4 + u_{-1}^4 + 2u_0^2(u_1^2 + u_{-1}^2))$$
$$+ (\beta_n - \beta_s)u_1^2u_{-1}^2 + 2\beta_s u_1 u_0^2 u_{-1} + 2Bu_0(u_1 + u_{-1})\Big)\,dx.$$

For given real numbers (N, M), we define

$$E_{N,M}^k = \Big\{u = (u_{-1}, u_0, u_1) \mid u_j \in H_0^1(I_k),\ j = 1, 2, 3,$$
$$\int_{I_k} \sum_{j=-1}^{1} u_j^2(x)\,dx = N,\ \int_{I_k} (u_1^2(x) - u_{-1}^2(x))\,dx = M\Big\}.$$

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We consider the minimisation problem

$$H_0^k = \inf\{H^k(u) \mid u \in E_{N,M}^k\}. \tag{1.7}$$

It is easy to see that

$$H_0^k \to H_0 \quad \text{as } k \to \infty. \tag{1.8}$$

We will prove the following theorem.

THEOREM 1.2. *Let $\beta_n < \beta_s < 0$ and $E_0 = E_{-1} \le E_1 < 0, B < 0$. Then the minimisation problem (1.8) can be attained by some $u^k = (u_{-1,k}, u_{0,k}, u_{1,k})$ where $u_{j,k} > 0$ and are strictly decreasing.*

Furthermore, in the last section, we will characterise the ground states of spin-1 Bose–Einstein condensates with an external Ioffe–Pitchard magnetic field (1.1)–(1.3).

We organise this paper as follows. In Section 2, we give the proof of Theorem 1.2. In Section 3, we prove Theorem 1.1. In Section 4, following the method in [12] by Lin *et al.*, we characterise the ground state.

2. Proof of Theorem 1.2

We now prove Theorem 1.2.

We rewrite H^k as follows:

$$\begin{aligned} H^k(u_{-1}, u_0, u_1) = &\int_{I_k}\Big(\sum_{j=-1}^{1}\Big(\frac{1}{2}|u_j'|^2 + E_j u_j^2\Big) + \frac{\beta_n}{2}\Big(\sum_{j=-1}^{1} u_j^2\Big)^2\Big)\,dx \\ &+ \frac{\beta_s}{2}\int_{I_k}((u_1^2 - u_{-1}^2)^2 + 2u_0^2(u_1 + u_{-1})^2)\,dx \\ &+ \int_{I_k} 2Bu_0(u_1 + u_{-1})\,dx. \end{aligned}$$

Let $u^l = (u_{-1}^l, u_0^l, u_1^l)$ be a minimising sequence of (1.8). We can always assume that each component u_j^l is nonnegative, since it is easy to see that

$$H^k(|u_{-1}^l|, |u_0^l|, |u_1^l|) \le H^k(u_{-1}^l, u_0^l, u_1^l)$$

and $(|u_{-1}^l|, |u_0^l|, |u_1^l|) \in E_{N,M}^k$. Hence we can replace (u_{-1}^l, u_0^l, u_1^l) by $(|u_{-1}^l|, |u_0^l|, |u_1^l|)$.

For $u \in H^1(\mathbb{R}), u \ge 0$, let us denote its Schwarz symmetrisation by u^*. Then (see [17])

$$\int_{\mathbb{R}} |u_j'|^2\,dx \ge \int_{\mathbb{R}} |(u_j^*)'|^2\,dx, \quad j = -1, 0, 1,$$

$$\int_{\mathbb{R}} u_j^2\,dx = \int_{\mathbb{R}} (u_j^*)^2\,dx, \quad \int_{\mathbb{R}} u_j^4\,dx = \int_{\mathbb{R}} (u_j^*)^4\,dx, \quad j = -1, 0, 1,$$

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$$\int_{\mathbb{R}} u_j u_k\,dx = \int_{\mathbb{R}} (u_j^*)(u_k^*)\,dx, \quad \int_{\mathbb{R}} u_j^2 u_k^2\,dx = \int_{\mathbb{R}} (u_j^*)^2(u_k^*)^2\,dx, \quad j=-1,0,1,$$
$$\int_{\mathbb{R}} u_j u_k\,dx = \int_{\mathbb{R}} (u_j^*)(u_k^*)\,dx, \quad \int_{\mathbb{R}} u_{-1}u_1u_0^2\,dx = \int_{\mathbb{R}} u_{-1}^* u_1^*(u_0^*)^2\,dx, \quad j=-1,0,1,$$

which imply that

$$H^k((u_{-1}^l)^*, (u_0^l)^*, (u_1^l)^*) \le H^k(u_{-1}^l, u_0^l, u_1^l)$$

and $((u_{-1}^l)^*, (u_0^l)^*, (u_1^l)^*) \in E_{N,M}^k$.

Now we can assume that u_j^l are nonnegative, even and nonincreasing in I_k. Next we show that the minimising sequence is uniformly bounded in $H_0^1(I_k)$ for $k \ge 1$.

By the Gagliardo–Nirenberg inequality [2] and Young's inequality, for any $u \in H_0^1(I_k)$, we have that, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\int_{I_k} |u|^4\,dx \le C\varepsilon \int_{I_k} |u'|^2\,dx + C(\varepsilon)\Big(\int_{I_k} |u|^2\,dx\Big)^3. \tag{2.1}$$

Then by the Cauchy inequality and Young's inequality and (2.1),

$$\int_{I_k} u_k u_j\,dx \le \frac{1}{2}\int_{I_k} (u_k^2 + u_j^2)\,dx \le \frac{1}{2}N,$$
$$\int_{I_k} u_k^2 u_j^2\,dx \le C\varepsilon \int_{I_k} (|u_k'|^2 + |u_j'|^2)\,dx + C(\varepsilon)\Big(\Big(\int_{I_k} u_k^2\,dx\Big)^3 + \Big(\int_{I_k} u_j^2\,dx\Big)^3\Big),$$

and

$$\begin{aligned}\int_{I_k} u_l^2 u_k u_j\,dx \le{}& C\varepsilon \int_{I_k} (|u_l'|^2 + |u_k'|^2 + |u_j'|^2)\,dx \\ &+ C(\varepsilon)\Big(\Big(\int_{I_k} u_l^2\,dx\Big)^3 + \Big(\int_{I_k} u_k^2\,dx\Big)^3 + \Big(\int_{I_k} u_j^2\,dx\Big)^3\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2}\int_{I_k} \sum_{j=-1}^{1} |(u_j^l)'|^2\,dx ={}& H_0^k - \int_{I_k} \sum_{j=-1}^{1} E_j(u_j^l)^2\,dx - \frac{\beta_n}{2}\Big(\sum_{j=-1}^{1}(u_j^l)^2\Big)^2\,dx \\ &- \frac{\beta_s}{2}\int_{I_k} (((u_1^l)^2 - (u_{-1}^l)^2)^2 + 2(u_0^l)^2(u_1^l + u_{-1}^l)^2)\,dx \\ &- 2B\int_{I_k} u_0^l(u_1^l + u_{-1}^l)\,dx + o_l(1) \\ \le{}& H_0^k - \frac{C\varepsilon(\beta_n + \beta_s)}{2}\int_{I_k} \sum_{j=-1}^{1} |(u_j^l)'|^2\,dx + C(\varepsilon)N^3 - 2BN.\end{aligned}$$

Choosing ε sufficiently small, we have

$$\int_{I_k}\Big(\sum_{j=-1}^{1}|(u_j^l)'|^2+(u_j^l)^2\Big)\,dx<C,$$

which implies that by Sobolev embedding

$$\|u_j^l\|_{L^\infty}<C,\qquad j=-1,0,1,\,l=1,2,\ldots.$$

Then we can obtain the existence of the minimiser $(u_{-1,k},u_{0,k},u_{1,k})$ by applying compactness of the embedding of the subspace of $H_0^1(I_k)$ that consists of even functions into $L^4(I_k)$. We can also assume that $u_{j,k}\geq 0$, for $j=-1,0,1$, and the least component is not identically 0.

Furthermore, there are Lagrange multipliers λ^k,μ^k such that $(u_{-1,k},u_{0,k},u_{1,k})$ satisfies the Euler–Lagrange equations

$$\begin{cases}\frac{1}{2}u_{-1}''-E_{-1}u_{-1}-(\beta_n+\beta_s)u_{-1}^3-((\beta_n+\beta_s)u_0^2+(\beta_n-\beta_s)u_1^2)u_{-1}-\beta_s u_0^2u_1-Bu_0\\ \qquad=(\lambda^k-\mu^k)u_{-1}\quad\text{in }I_k,\\ \frac{1}{2}u_0''-E_0u_0-\beta_nu_0^3-(\beta_n+\beta_s)(u_1^2+u_{-1}^2)u_0-2\beta_su_{-1}u_0u_1-B(u_1+u_{-1})\\ \qquad=\lambda^ku_0\quad\text{in }I_k,\\ \frac{1}{2}u_1''-E_1u_1-(\beta_n+\beta_s)u_1^3-(\beta_n+\beta_s)u_0^2u_1-(\beta_n-\beta_s)u_{-1}^2u_1-\beta_su_0^2u_{-1}-Bu_0\\ \qquad=(\lambda^k+\mu^k)u_1\quad\text{in }I_k,\\ u_j>0\text{ in }(-k,k),\qquad u_j(\pm k)=0,\ j=-1,0,1.\end{cases}$$

In what follows, we show that $u_j(k)\neq 0$ for $j=-1,0,1$. This will be done by two claims.

Claim 1. $u_{0,k}>0$.

We argue by contradiction. Suppose that $u_{0,k}\geq 0$ and $u_{0,k}(x_0)=0$. Then by the maximum principle, $u_{0,k}\equiv 0$. Hence $(u_{-1,k},u_{1,k})$ is a solution of

$$\begin{cases}\frac{1}{2}u_{-1}''-E_{-1}u_{-1}-(\beta_n+\beta_s)u_{-1}^3-(\beta_n-\beta_s)u_1^2u_{-1}=(\lambda^k-\mu^k)u_{-1}\quad\text{in }I_k,\\ \frac{1}{2}u_1''-E_1u_1-(\beta_n+\beta_s)u_1^3-(\beta_n-\beta_s)u_{-1}^2u_1=(\lambda^k+\mu^k)u_1\quad\text{in }I_k\end{cases}\tag{2.2}$$

satisfying the constraint,

$$\int_{I_k}u_{1,k}^2\,dx=\frac{N+M}{2},\qquad\int_{I_k}u_{-1,k}^2\,dx=\frac{N-M}{2}.$$

Set $(u_{-1},u_0,u_1)=(u_{-1,k}+\varepsilon_{-1}\psi_{-1},\varepsilon_0\psi_0,u_{1,k}+\varepsilon_1\psi_1)$ with $\varepsilon_j>0,\psi_j\in H_0^1(I_k)$ for $j=-1,0,1$ such that

$$\int_{I_k}(|u_{-1,k}+\varepsilon_{-1}\psi_{-1}|^2+|\varepsilon_0\psi_0|^2+|u_{1,k}+\varepsilon_1\psi_1|^2)\,dx=N,$$

$$\int_{I_k}(|u_{1,k}+\varepsilon_1\psi_1|^2-|u_{-1,k}+\varepsilon_{-1}\psi_{-1}|^2)\,dx=M.$$

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This can be done by choosing $\varepsilon_j > 0$, $\psi_j \in H_0^1(I_k)$, $j = -1, 0, 1$ such that $\int_{I_k} u_{-1,k}\psi_{-1}\,dx < 0$, $\int_{I_k} u_{1,k}\psi_1\,dx < 0$ and

$$2\varepsilon_1 \int_{I_k} u_{1,k}\psi_1\,dx + \varepsilon_1^2 \int_{I_k} |\psi_1|^2\,dx = 2\varepsilon_{-1} \int_{I_k} u_{-1,k}\psi_{-1}\,dx + \varepsilon_{-1}^2 \int_{I_k} |\psi_{-1}|^2\,dx,$$
$$\begin{aligned} 2\varepsilon_1 \int_{I_k} u_{1,k}\psi_1\,dx + \varepsilon_1^2 \int_{I_k} |\psi_1|^2\,dx + 2\varepsilon_{-1} \int_{I_k} u_{-1,k}\psi_{-1}\,dx \\ + \varepsilon_{-1}^2 \int_{I_k} |\psi_{-1}|^2\,dx + \varepsilon_0^2 \int_{I_k} |\psi_0|^2\,dx = 0. \end{aligned}$$

Then

$$\varepsilon_1 \int_{I_k} u_{1,k}\psi_1\,dx = \varepsilon_{-1} \int_{I_k} u_{-1,k}\psi_{-1}\,dx + O(\varepsilon_{-1}^2 + \varepsilon_1^2),$$
$$\varepsilon_1 \int_{I_k} u_{1,k}\psi_1\,dx + \varepsilon_{-1} \int_{I_k} u_{-1,k}\psi_{-1}\,dx = -\frac{1}{2}\varepsilon_0^2 \int_{I_k} |\psi_0|^2\,dx + O(\varepsilon_{-1}^2 + \varepsilon_1^2).$$

Applying the above equality to the expression in $E^k(u_{-1}, u_0, u_1)$, we have

$$\begin{aligned} E^k(u_{-1}, u_0, u_1) &= -2\varepsilon_1(\lambda^k + \mu^k) \int_{I_k} u_{1,k}\psi_1\,dx - 2\varepsilon_{-1}(\lambda^k - \mu^k) \int_{I_k} u_{-1,k}\psi_{-1}\,dx \\ &\quad + \varepsilon_0^2 \int_{I_k} \Big(\frac{1}{2}(\psi_0')^2 + E_0\psi_0^2 + (\beta_n + \beta_s)(u_{1,k}^2 + u_{-1,k}^2)\psi_0^2 \\ &\quad + 2\beta_s u_{1,k}u_{-1,k}\psi_0^2\Big)\,dx + E^k(u_{-1,k}, 0, u_{1,k}) + O(\varepsilon_{-1}^4 + \varepsilon_1^4) \\ &= \varepsilon_0^2 \int_{I_k} \Big(\frac{1}{2}|\psi_0'|^2 + (E_0 + \lambda^k)\psi_0^2 + (\beta_n + \beta_s)(u_{1,k}^2 + u_{-1,k}^2)\psi_0^2 \\ &\quad + 2\beta_s u_{1,k}u_{-1,k}\psi_0^2\Big)\,dx + E^k(u_{-1,k}, 0, u_{1,k}) + O(\varepsilon_{-1}^2 + \varepsilon_1^2) \\ &\quad + O(\varepsilon_{-1}^4 + \varepsilon_1^4). \end{aligned} \tag{2.3}$$

By (2.2) satisfied by $u_{-1,k}$, $u_{1,k}$,

$$\begin{aligned} &\int_{I_k} \Big(\frac{1}{2}(|u_{1,k}'|^2 + \eta^2|u_{-1,k}'|^2) + E_1u_{1,k}^2 + \eta^2E_{-1}|u_{-1,k}'|^2 + (\beta_n + \beta_s)(u_{1,k}^4 + \eta^2u_{-1,k}^4) \\ &\quad + (\beta_n - \beta_s)(1 + \eta^2)u_{-1,k}^2u_{1,k}^2\Big)\,dx \\ &\qquad = -(\lambda^k + \mu^k) \int_{I_k} u_{1,k}^2\,dx - (\lambda^k - \mu^k) \int_{I_k} \eta^2u_{-1,k}^2\,dx. \end{aligned} \tag{2.4}$$

By (2.2) satisfied by $u_{-1,k}$, $u_{1,k}$ again,

$$\begin{aligned} &\int_{I_k} (u_{1,k}'u_{-1,k}' + (E_1 + E_{-1})u_{1,k}u_{-1,k} + 2\beta_n(u_{1,k}^3u_{-1,k} + u_{-1,k}^3u_{1,k}))\,dx \\ &\qquad = -2\lambda^k \int_{I_k} u_{1,k}u_{-1,k}\,dx. \end{aligned} \tag{2.5}$$

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Let $\eta = ((N+M)/(N-M))^{1/2}$. Then by $\beta_s < 0$, $E_0 = E_{-1} \le E_1 < 0$ and from (2.4) and (2.5),

$$\int_{I_k} \Big(\frac{1}{2} |(u_{1,k} + \eta u_{-1,k})'|^2 + (E_0 + \lambda^k + (\beta_n + \beta_s)(u_{1,k}^2 + u_{-1,k}^2) + 2\beta_s u_{1,k} u_{-1,k})(u_{1,k} + \eta u_{-1,k})^2 \Big) \, dx < 0. \tag{2.6}$$

Let $\psi_0 = u_{1,k} + \eta u_{-1,k}$. Then by (2.3) and (2.6), when $\varepsilon_{-1}, \varepsilon_0, \varepsilon_1$ are sufficiently large, we have $E^k(u_{-1}, u_0, u_1) < E_0$. This is a contradiction. Hence $u_{0,k} > 0$.

Claim 2. $u_{-1,k} > 0, u_{1,k} > 0$.

Suppose that $u_{1,k} \ge 0$ and $u_{1,k}(x_0) = 0$. By the maximum principle, $u_{1,k}(x) \equiv 0$. By the equation satisfied by $u_{1,k}$,

$$\beta_s u_{0,k}^2 u_{-1,k} + B u_{0,k} = 0.$$

Therefore either $u_{0,k} \equiv 0$ or $\beta_s u_{0,k} u_{-1,k} + B \equiv 0$. But by Claim 1 and $\beta_s < 0, B < 0$, this is impossible. Hence $u_{1,k} > 0$. Similarly, we can prove $u_{-1,k} > 0$. Thus we have completed the proof of Theorem 1.2. □

3. Proof of Theorem 1.1

From Section 2, for each $k \ge 1$, we obtain a minimiser to the minimisation problem (1.7) which satisfies the following Euler–Lagrange equations

$$\begin{cases} \frac{1}{2} u_{-1,k}'' - E_{-1} u_{-1,k} - (\beta_n + \beta_s) u_{-1,k}^3 - [(\beta_n + \beta_s) u_{0,k}^2 + (\beta_n - \beta_s) u_{1,k}^2] u_{-1,k} \\ \quad - \beta_s u_{0,k}^2 u_{1,k} - B u_{0,k} = (\lambda^k - \mu^k) u_{-1,k} \quad \text{in } (-k, k), \\ \frac{1}{2} u_{0,k}'' - E_0 u_{0,k} - \beta_n u_{0,k}^3 - (\beta_n + \beta_s)(u_{1,k}^2 + u_{-1,k}^2) u_{0,k} - 2\beta_s u_{-1,k} u_{0,k} u_{1,k} - B(u_{1,k} + u_{-1,k}) \\ \quad = \lambda^k u_{0,k} \quad \text{in } (-k, k), \\ \frac{1}{2} u_{1,k}'' - E_1 u_{1,k} - (\beta_n + \beta_s) u_{1,k}^3 - (\beta_n + \beta_s) u_{0,k}^2 u_{1,k} - (\beta_n - \beta_s) u_{-1,k}^2 u_{1,k} \\ \quad - \beta_s u_{0,k}^2 u_{-1,k} - B u_{0,k} = (\lambda^k + \mu^k) u_{1,k} \quad \text{in } (-k, k), \\ u_{j,k} > 0 \text{ in } (-k, k), \quad u_{j,k}(\pm k) = 0, \ j = -1, 0, 1. \end{cases}$$

From Section 2, we also know the following results:

(1) $u_{j,k} > 0$ in $(-k, k)$, $u_{j,k}$ is even and decreasing;
(2) for $k \ge k_0$, $H_0^k \le c_0 < 0$.

Indeed, from (1.8), we only need prove $H_0 < 0$. Set $v_j(x) = \rho^{1/2} u_j(\rho x)$ for $j = -1, 0, 1$. Then for any $\rho > 0$, we have that (v_{-1}, v_0, v_1) also satisfies

$$\int_{\mathbb{R}} \sum_{j=-1}^{1} v_j^2(x) \, dx = N,$$

$$\int_{\mathbb{R}} (v_1^2(x) - v_{-1}^2(x)) \, dx = M,$$

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and

$$
\begin{aligned}
H(v_{-1}, v_0, v_1) = \rho^2 \int_{\mathbb{R}} \sum_{j=-1}^{1} \frac{1}{2}|u_j'|^2\,dx &+ \int_{\mathbb{R}} \sum_{j=-1}^{1} E_j u_j^2\,dx + \frac{\rho\beta_n}{2}\int_{\mathbb{R}}\Big(\sum_{j=-1}^{1} u_j^2\Big)^2\,dx \\
&+ \frac{\rho\beta_s}{2}\int_{\mathbb{R}}((u_1^2 - u_{-1}^2)^2 + 2u_0^2(u_1^2 + u_{-1}^2) + 4u_1u_0^2u_{-1} \\
&+ 2u_0^2(u_1 + u_{-1})^2)\,dx + \int_{\mathbb{R}} 2Bu_0(u_1 + u_{-1})\,dx.
\end{aligned}
\tag{3.1}
$$

Then $H_0 < 0$ follows from (3.1) and $\beta_n < \beta_s < 0$, $E_j < 0$, $B < 0$, $\min\{|E_j|, j = -1, 0, 1\} > 2|B|$, by taking ρ small enough.

As well as (1) and (2), we have the following result:

(3) $\|u_{j,k}\|_{H^1(I_k)} \le C$ for some C.

Thus by Morrey's inequality, we can take a subsequence of $k \to \infty$ such that $u_{j,k} \to u_j$ uniformly in $\mathbb{R}$ where $u_j \in H^1(\mathbb{R})$, $u_j \ge 0$ and u_j is decreasing. Then we can conclude that $u_{j,k} \to u_j$ in $L^p(\mathbb{R})$ for $p > 2$. But since $u_{j,k} \to u_j$ uniformly in $\mathbb{R}$, we cannot conclude $u_{j,k} \to u_j$ in $L^2(\mathbb{R})$. Note that if we can prove that $u_{j,k} \to u_j$ in $L^2(\mathbb{R})$, then (u_{-1}, u_0, u_1) satisfies the constraint (1.4)–(1.5) and is a minimiser of the minimisation problem (1.6). By the same arguments as for Claim 1 and Claim 2, we can prove that $u_j > 0$.

Similarly to the proof in [6], we can prove strong convergence in $L^2(\mathbb{R})$. We will give the detailed proof in a few claims.

Claim 3. $\lim_{k\to+\infty}(\lambda^k - \mu^k) \ge 0$, $\lim_{k\to+\infty} \lambda^k \ge 0$, $\lim_{k\to+\infty}(\lambda^k + \mu^k) \ge 0$.

In fact, suppose $\lim_{k\to+\infty}(\lambda^k + \mu^k) < -c_0 < 0$. Then from the equation for $u_{1,k}$, we see that

$$u_{1,k}'' + \frac{c_0}{4}u_{1,k} \le 0, \qquad u_{1,k}(x) > 0 \quad \text{in } (-k, k).$$

But by the Liouville comparison theorem, for k large, $u_{1,k}$ must change signs in $(-\sqrt{c_0}\pi, \sqrt{c_0}\pi)$, which is a contradiction to the fact that $u_{1,k}(x) > 0$ in $(-k, k)$.

The other cases can be proven similarly.

Claim 4. There exists a positive constant $c_0 > 0$ such that

$$\lambda^k N + \mu^k M \ge c_0 > 0.$$

In fact, by integrating by parts,

$$
\begin{aligned}
&- \int_{I_k} \sum_{j=-1}^{1}\Big(\frac{1}{2}(u_{j,k}')^2 + E_j u_{j,k}^2\Big)\,dx - \int_{I_k} \beta_n\Big(\sum_{j=-1}^{1} u_{j,k}^2\Big)^2\,dx \\
&\quad - \int_{I_k} \beta_s((u_{1,k}^2 - u_{-1,k}^2)^2 + 2u_{0,k}^2(u_{1,k} + u_{-1,k})^2)\,dx - \int_{I_k} 2Bu_{0,k}(u_{1,k} + u_{-1,k})\,dx \\
&\quad = \lambda^k N + \mu^k M \ge -H^k(u_{-1,k}, u_{0,k}, u_{1,k}) \ge c_0 > 0
\end{aligned}
$$

for k large.

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Claim 5. There exists $c_0 > 0$ such that $\lambda^k \geq c_0 > 0$ for k large and $\int_{I_k} u_{0,k}^2\,dx \to \int_{\mathbb{R}} u_0^2\,dx$ as $k \to +\infty$.

From Claim 3, we deduce that $\lim_{k\to+\infty}(\lambda^k - |\mu^k|) \geq 0$. By Claim 4,

$$\lambda^k \geq c_0 > 0. \tag{3.2}$$

In fact, if $M = 0$, then (3.2) is obvious. If $M > 0$, then $\lambda^k N + \lambda^k M \geq \lambda^k N + \mu^k M \geq c_0 > 0$ for k large. If $M < 0$, then $\lambda^k N - \lambda^k M \geq \lambda^k N + \mu^k M \geq c_0 > 0$ for k large.

For $\delta = \sqrt{(3/4)c_0/(-3\beta_n - 4\beta_s)}$, we can find $R > 0$ and k_0 such that

$$u_{1,k}(x) \leq \delta, \quad u_{-1,k}(x) \leq \delta \quad \text{for } |x| > R,\ k \geq k_0. \tag{3.3}$$

From the equation for $u_{0,k}$ and (3.3),

$$u_{0,k}'' - \frac{c_0}{4}u_{0,k} \geq 0 \quad \text{for } |x| > R,\ k \geq k_0$$

where R is fixed large number. By the comparison principle,

$$u_{0,k}(x) \leq u_{0,k}(R)e^{-\sqrt{c_0}/4(|x|-R)} \leq Ce^{-\sqrt{c_0}/4|x|}.$$

Note that R depends only on c_0. Thus we conclude that $u_{0,k}$ has exponential decay. So $\int_{I_k} u_{0,k}^2\,dx \to \int_{\mathbb{R}} u_0^2\,dx$ as $k \to \infty$.

Since $\lambda^k \geq c_0 > 0$, we see that either $\lambda^k + \mu^k \geq c_0/2$ or $\lambda^k - \mu^k \geq c_0/2$. Let us assume that $\lambda^k + \mu^k \geq c_0/2$. Then by the same proof as for Claim 5, we have the following.

Claim 6. Assuming that $\lambda^k + \mu^k \geq c_0/2$, we have $\int_{I_k} u_{-1,k}^2\,dx \to \int_{\mathbb{R}} u_{-1}^2\,dx$ as $k \to \infty$.

Now it remains to show that $\int_{I_k} u_{-1,k}^2\,dx \to \int_{\mathbb{R}} u_{-1}^2\,dx$ as $k \to \infty$. Suppose this is not true. By Claim 3, we may assume that $\lim_{k\to+\infty}(\lambda^k - \mu^k) = 0$. In fact, if $\lim_{k\to+\infty}(\lambda^k - \mu^k) \geq C > 0$, then similar arguments to those in Claim 5 show that $u_{-1,k}$ has exponential decay and hence $\int_{I_k} u_{-1,k}^2\,dx \to \int_{\mathbb{R}} u_{-1}^2\,dx$ as $k \to \infty$, which contradicts our assumption.

Claim 7. $u_1u_0^2 \equiv 0$ and $u_{-1} \equiv 0$.

Using $\int_{I_k} u_{-1,k}^2\,dx \to \int_{\mathbb{R}} u_{-1}^2\,dx$ as $k \to \infty$, we see that the limit u_{-1} satisfies

$$\tfrac{1}{2}u_{-1}'' - E_{-1}u_{-1} - (\beta_n + \beta_s)u_{-1}^3 - ((\beta_n + \beta_s)u_0^2 + (\beta_n - \beta_s)u_1^2)u_{-1} - \beta_s u_0^2 u_1 - Bu_0 = 0,$$
in $\mathbb{R}$.

Integrating from 0 to x, we obtain that $|u_{-1}'(x)| \geq C|\int_0^x u_1u_0^2|$. Since $\|u_{-1}\|_{H^1} \leq C$, we derive $u_{-1} = 0$ and $u_1u_0^2 = 0$. If both $u_1 = 0$ and $u_0 = 0$, we then derive $N = -M$ (since $u_{1,k} \to u_1$ and $u_{0,k} \to u_0$ strongly in $L^2(\mathbb{R})$), which is impossible.

There are two cases to be considered.

Case 1. $u_0 > 0$, $u_1 = 0$.

By Claim 5, $\lambda^k \geq C > 0$. Since $u_j \in H^1(\mathbb{R})$ and u_j is decreasing, we see that $u_j(x) \to 0$ as $|x| \to +\infty$. Thus, for any $\delta > 0$, we can find $R_\delta > 0$ such that for $|x| \geq R_\delta$

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we have $u_j(x) \le \delta/2$. As a consequence of the decreasing property of $u_{j,k}$, we can find k_0 such that $u_{j,k}(x) \le u_{j,k}(R_\delta) < \delta$, for $|x| > R_\delta$, $k \ge k_0$. Then by $u_1 = u_{-1} = 0$, we see that $|(\beta_n + \beta_s)u_{1,k}^2 + (\beta_n + \beta_s)u_{-1,k}^2 + 2\beta_s u_{1,k}u_{-1,k} + B(u_{1,k} + u_{-1,k})| \to 0$ uniformly in $\mathbb{R}$. Hence from the equation for $u_{0,k}$ and Claim 7, $u_{0,k}$ satisfies

$$\tfrac{1}{2}u_{0,k}'' - E_0 u_{0,k} - \beta_n u_{0,k}^3 \ge \frac{c_0}{2}u_{0,k},\ u_{0,k} > 0 \quad \text{in } (-k, k), \qquad u_{0,k}(\pm k) = 0. \tag{3.4}$$

Using the equation for $u_{-1,k}$ and $\lim_{k\to+\infty}(\lambda^k - \mu^k) = 0$, we see that $u_{-1,k}$ satisfies

$$\tfrac{1}{2}u_{-1,k}'' - E_{-1}u_{-1,k} - \beta_n u_{0,k}^2 u_{-1,k} \le \frac{c_0}{4}u_{-1,k},\ u_{-1,k} > 0 \quad \text{in } (-k, k), \qquad u_{-1,k}(\pm k) = 0. \tag{3.5}$$

Multiplying (3.4) by $u_{-1,k}$ and (3.5) by $u_{0,k}$ and then integrating over $(-k, k)$, since $E_0 = E_{-1}$, we obtain a contradiction.

Case 2. $u_1 > 0$, $u_2 = 0$.

In this case, we observe that u_1 satisfies

$$\tfrac{1}{2}u_1'' - E_{-1}u_{-1} - (\beta_n + \beta_s)u_1^3 = 2\lambda^0 u_1 \quad \text{in } \mathbb{R},\ u_1 \in H^1(\mathbb{R}) \tag{3.6}$$

where $\lim_{k\to+\infty}\lambda^k = \lim_{k\to+\infty}\mu^k = \lambda^0 > 0$.

On the other hand, $u_{0,k}(x)/u_{0,k}(0) \to \widehat{u}_2(x)$ which satisfies

$$\tfrac{1}{2}\,\widehat{u}_2{}'' - E_0\widehat{u}_2 - (\beta_n + \beta_s)u_1^2\widehat{u}_2 = \lambda^0\widehat{u}_2. \tag{3.7}$$

It is easy to see that $0 < \widehat{u}_2 \le 1$ since $\widehat{u}_2(0) = 1$. Multiplying (3.6) by $\widehat{u}_2$ and (3.7) by u_1 and then integrating over $\mathbb{R}$, since $E_0 = E_{-1}$, we get

$$\lambda^0 \int_{\mathbb{R}} u_1\widehat{u}_2 = 0$$

which is impossible.

In conclusion, we have proved that as $k \to +\infty$ then $\int_{I_k} u_{j,k}^2 \to \int_{\mathbb{R}} u_j^2$ for $j = -1, 0, 1$. This completes the proof of Theorem 1.1. □

4. Characterisation of the ground state

We denote the energy density h by

$$h(\mathbf{u}) = h(u_{-1}, u_0, u_1) = \sum_{j=-1}^{1}\Big(\frac{1}{2}|u_j'|^2 + E_j u_j^2\Big) + \frac{\beta_n}{2}\Big(\sum_{j=-1}^{1} u_j^2\Big)^2$$
$$+ \frac{\beta_s}{2}((u_1^2 - u_{-1}^2)^2 + 2u_0^2(u_1 + u_{-1})^2) + 2Bu_0(u_1 + u_{-1}).$$

Then

$$\int_{\mathbb{R}} h(\mathbf{u})\,dx = H(\mathbf{u}).$$

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We also denote the set of all minimisers of H (over $\mathcal{A}$) by G, where

$$\mathcal{A} = \{\mathbf{u} = (u_{-1}, u_0, u_1) \in E_{N,M} | u_j \geq 0 \text{ for } j = -1, 0, 1\}.$$

The Euler–Lagrange equations for $\mathbf{u} \in G$ are given by the following coupled Gross–Pitaevskii equations:

$$\begin{cases} \frac{1}{2}u_{-1}'' - E_{-1}u_{-1} - (\beta_n + \beta_s)u_{-1}^3 - ((\beta_n + \beta_s)u_0^2 + (\beta_n - \beta_s)u_1^2)u_{-1} - \beta_s u_0^2 u_1 - Bu_0 \\ \qquad = (\lambda - \mu)u_{-1}, \\ \frac{1}{2}u_0'' - E_0 u_0 - \beta_n u_0^3 - (\beta_n + \beta_s)(u_1^2 + u_{-1}^2)u_0 - 2\beta_s u_{-1}u_0u_1 - B(u_1 + u_{-1}) = \lambda u_0, \\ \frac{1}{2}u_1'' - E_1u_1 - (\beta_n + \beta_s)u_1^3 - (\beta_n + \beta_s)u_0^2u_1 - (\beta_n - \beta_s)u_{-1}^2u_1 - \beta_s u_0^2 u_{-1} - Bu_0 \\ \qquad = (\lambda + \mu)u_1, \end{cases}$$

where λ and μ are the Lagrange multipliers.

Similar to the proof of Lemma 2.1 in [12], we give the following lemma.

Lemma 4.1. *If $\mathbf{u} \in G \cap (C^2(\mathbb{R}))^3$, then for each j, either $u_j > 0$ or $u_j \equiv 0$ in $\mathbb{R}$.*

We recall some results on mass-redistribution of n-tuples of real-valued functions.

Definition 4.2 [12]. Let $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in (H^1(\mathbb{R}))^n$ be an n-tuple of real-valued functions and $\mathbf{g} = (g_1, g_2, \ldots, g_m)$ be an m-tuple of nonnegative functions. We say $\mathbf{g}$ is a mass-redistribution of $\mathbf{f}$, if $g_l^2 = \sum_{k=1}^n b_{lk} f_k^2$ for each l, where $b_{lk} \geq 0$ are constants and $\sum_{l=1}^m b_{lk} = 1$ for each k.

We have the following proposition.

Proposition 4.3 [12]. *For any mass-redistribution $\mathbf{g}$ of $\mathbf{f}$ as in Definition 4.2, we have the following results.*

(1) $|\mathbf{g}| = |\mathbf{f}|$.
(2) $|\nabla g|^2 \leq |\nabla \mathbf{f}|^2$. *Moreover, $|\nabla g|^2 = |\nabla \mathbf{f}|^2$ if and only if $f_j \nabla f_k = f_k \nabla f_j$ for each $j \neq k$ with $b_{lj} b_{lk} \neq 0$ for at least one l.*

Our main result in this section is as follows.

Theorem 4.4. *Let $E_{-1} = E_0 = E_1$ in the ferromagnetic system (1.1)–(1.3) for $n = 1$, and let $\mathbf{u} = (u_{-1}, u_0, u_1) \in \mathcal{A}$ be the ground state of (1.1)–(1.3). Then we have the following results.*

(i) *If $\mathbf{u}$ satisfies $u_0(u_{-1} + u_1) \leq \sqrt{\frac{1}{2}(1 - M^2/N^2)}|\mathbf{u}|^2$, then $h(\gamma^*|\mathbf{u}|) \leq h(\mathbf{u})$.*

(ii) *If $\mathbf{u} \in G \cap (C^2(\mathbf{R}))^3$ satisfies $u_0(u_{-1} + u_1) = \sqrt{\frac{1}{2}(1 - M^2/N^2)}|\mathbf{u}|^2$, then $\mathbf{u} = \gamma^*|\mathbf{u}|$.*

Here $\gamma^* = (\gamma_{-1}^*, \gamma_0^*, \gamma_1^*)$ is given by

$$\gamma_{-1}^* = \frac{1}{2}\Big(1 - \frac{M}{N}\Big), \quad \gamma_0^* = \sqrt{\frac{1}{2}\Big(1 - \frac{M^2}{N^2}\Big)}, \quad \gamma_1^* = \frac{1}{2}\Big(1 + \frac{M}{N}\Big).$$

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Proof. By direct calculation,

$$h(\mathbf{u}) - h(\gamma^*|\mathbf{u}|) = (|\nabla \mathbf{u}|^2 - |\nabla|\mathbf{u}||^2) - \beta_s(u_0^2 - 2u_{-1}u_1)^2 + 2B\Big(u_0(u_1 + u_{-1}) - \sqrt{\frac{1}{2}\Big(1 - \frac{M^2}{N^2}\Big)}|\mathbf{u}|^2\Big).$$

Then by Proposition 4.3 and $u_0(u_{-1} + u_1) \le \sqrt{\frac{1}{2}(1 - M^2/N^2)}|\mathbf{u}|^2$, $\beta_s < 0$, we prove (i).

If $\mathbf{u} \in G$, from (i), we have $H(\mathbf{u}) = H(\gamma^*|\mathbf{u}|)$, which in turn implies $h(\mathbf{u}) = h(\gamma^*|\mathbf{u}|)$. Hence from (2) of Proposition 4.3,

$$f_j \nabla f_k = f_k \nabla f_j \quad \text{for } j \neq k; \tag{4.1}$$

$$u_0^2 = 2u_{-1}u_1;$$

$$u_0(u_1 + u_{-1}) = \sqrt{\frac{1}{2}\Big(1 - \frac{M^2}{N^2}\Big)}|\mathbf{u}|^2. \tag{4.2}$$

Now assume $\mathbf{u} \in G \cap (C^2(\mathbf{R}))^3$. From Lemma 4.1, at least one u_j is strictly positive in $\mathbb{R}$. Without loss of generality assume $u_1 > 0$ in $\mathbb{R}$. Then from (4.1)

$$\nabla\Big(\frac{u_0}{u_1}\Big) = \nabla\Big(\frac{u_{-1}}{u_1}\Big) = 0. \tag{4.3}$$

Since $\mathbb{R}$ is connected, by (4.3), it follows that u_{-1} and u_0 are both constant multiples of u_1. Hence (ii) follows by (4.2) and (4.3). □

Remark. Theorem 4.4 implies that, under some condition, searching for the ground state of a ferromagnetic spin-1 BEC with an external Ioffe–Pitchard magnetic field can be reduced to a 'one-component' minimisation problem.

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