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LOGARITHMIC COEFFICIENTS OF SOME CLOSE-TO-CONVEX FUNCTIONS

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Abstract

The logarithmic coefficients γ_n of an analytic and univalent function f in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalisation $f(0) = 0 = f'(0) - 1$ are defined by $\log(f(z)/z) = 2\sum_{n=1}^{\infty} \gamma_n z^n$. In the present paper, we consider close-to-convex functions (with argument 0) with respect to odd starlike functions and determine the sharp upper bound of $|\gamma_n|$, $n = 1, 2, 3$, for such functions f.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalised by $f(0) = 0 = f'(0) - 1$. Any function f in $\mathcal{A}$ has the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The class of univalent (that is, one-to-one) functions in $\mathcal{A}$ is denoted by $\mathcal{S}$. A function $f \in \mathcal{A}$ is called starlike (respectively, convex) if $f(\mathbb{D})$ is starlike (respectively, convex) with respect to the origin. Let $\mathcal{S}^*$ and C denote the classes of starlike and convex functions in $\mathcal{S}$, respectively. It is well known that a function $f \in \mathcal{A}$ is in $\mathcal{S}^*$ if and only if $\mathrm{Re}(zf'(z)/f(z)) > 0$ for $z \in \mathbb{D}$. Similarly, a function $f \in \mathcal{A}$ is in C if and only if $\mathrm{Re}(1 + zf''(z)/f'(z)) > 0$ for $z \in \mathbb{D}$. From the above it is easy to see that $f \in C$ if and only if $zf' \in \mathcal{S}^*$. Given $\alpha \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, a function $f \in \mathcal{A}$ is said to be close-to-convex with argument α and with respect to g if

$$\mathrm{Re}\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

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Let $\mathcal{K}_\alpha(g)$ denote the class of all such functions. Let

$$\mathcal{K}(g) := \bigcup_{\alpha\in(-\pi/2,\pi/2)} \mathcal{K}_\alpha(g) \quad \text{and} \quad \mathcal{K}_\alpha := \bigcup_{g\in\mathcal{S}^*} \mathcal{K}_\alpha(g)$$

be the classes of close-to-convex functions with respect to g and close-to-convex functions with argument α, respectively. The class

$$\mathcal{K} := \bigcup_{\alpha\in(-\pi/2,\pi/2)} \mathcal{K}_\alpha = \bigcup_{g\in\mathcal{S}^*} \mathcal{K}(g)$$

is the class of all close-to-convex functions. It is well known that every close-to-convex function is univalent in $\mathbb{D}$ (see [5]). Geometrically, $f \in \mathcal{K}$ means that the complement of the image domain $f(\mathbb{D})$ is the union of nonintersecting half-lines.

For a function $f \in \mathcal{S}$, the logarithmic coefficients γ_n $(n = 1, 2, \ldots)$ are defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}. \tag{1.2}$$

Bazilevich first noticed that the logarithmic coefficients are essential in the coefficient problem of univalent functions. In [2, 3], he gave estimates in terms of the positive Hayman constant (see [10]) for how close the coefficients γ_n $(n = 1, 2, \ldots)$ of the functions of class $\mathcal{S}$ are to the relative logarithmic coefficients of the Koebe function $k(z) = z/(1 - z)^2$. He also estimated $\sum_{n=1}^{\infty} n|\gamma_n|^2 r^{2n}$, which after multiplication by π is equal to the area of the image of the disc $|z| < r < 1$ under the function $\frac{1}{2}\log(f(z)/z)$ for $f \in \mathcal{S}$. The celebrated de Branges' inequalities (the former Milin conjecture) for univalent functions f state that

$$\sum_{k=1}^{n}(n-k+1)|\gamma_k|^2 \le \sum_{k=1}^{n} \frac{n+1-k}{k}, \quad n = 1, 2, \ldots,$$

with equality if and only if $f(z) = e^{-i\theta}k(e^{i\theta}z), \theta \in \mathbb{R}$ (see [4]). De Branges [4] used this inequality to prove the celebrated Bieberbach conjecture. Moreover, the de Branges' inequalities have also been the source of many other interesting inequalities involving logarithmic coefficients of $f \in \mathcal{S}$ such as (see [6])

$$\sum_{k=1}^{\infty} |\gamma_k|^2 \le \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

More attention has been given to the results in an average sense (see [5, 6, 14]) than the exact upper bounds for $|\gamma_n|$ for functions in the class $\mathcal{S}$ and few exact upper bounds for $|\gamma_n|$ have been established. For the Koebe function $k(z) = z/(1 - z)^2$, the logarithmic coefficients are $\gamma_n = 1/n$. Since the Koebe function $k(z)$ plays the role of an extremal function for most of the extremal problems in the class $\mathcal{S}$, it is expected that $|\gamma_n| \le 1/n$ for functions in $\mathcal{S}$. But this is not true in general, even in order of magnitude. Indeed, there exists a bounded function f in the class $\mathcal{S}$ with logarithmic coefficients $\gamma_n \ne O(n^{-0.83})$ (see [5, Theorem 8.4]).

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By differentiating (1.2) and equating coefficients,

$$\gamma_1 = \tfrac{1}{2}a_2, \tag{1.3}$$
$$\gamma_2 = \tfrac{1}{2}(a_3 - \tfrac{1}{2}a_2^2), \tag{1.4}$$
$$\gamma_3 = \tfrac{1}{2}(a_4 - a_2a_3 + \tfrac{1}{3}a_2^3). \tag{1.5}$$

If $f \in \mathcal{S}$, then $|\gamma_1| \le 1$ follows from (1.3). Using the Fekete–Szegö inequality [5, Theorem 3.8] in (1.4), it is easy to obtain the sharp estimate

$$|\gamma_2| \le \tfrac{1}{2}(1 + 2e^{-2}) = 0.635\ldots.$$

For $n \ge 3$, the problem seems much harder and no significant upper bounds for $|\gamma_n|$ when $f \in \mathcal{S}$ appear to be known.

For functions in the class $\mathcal{S}^*$, by the analytic characterisation $\mathrm{Re}(zf'(z)/f(z)) > 0$ for $z \in \mathbb{D}$, it is easy to prove that $|\gamma_n| \le 1/n$ for $n \ge 1$ and equality holds for the Koebe function $k(z) = z/(1-z)^2$. The inequality $|\gamma_n| \le 1/n$ for $n \ge 2$ for functions in the class $\mathcal{K}$ was claimed in a paper of Elhosh [7]. However, Girela [8] pointed out an error in the proof of Elhosh [7] and, hence, the result is not substantiated. Indeed, Girela proved that for each $n \ge 2$, there exists a function $f \in \mathcal{K}$ such that $|\gamma_n| > 1/n$. Recently, it has been proved [15] that $|\gamma_3| \le \frac{7}{12}$ for functions in $\mathcal{K}_0$ (close-to-convex functions with argument 0) with the additional assumption that the second coefficient of the corresponding starlike function g is real. But this estimate is not sharp, as pointed out in [1], where the authors proved that $|\gamma_3| \le \frac{1}{18}(3 + 4\sqrt{2}) = 0.4809$ for functions in $\mathcal{K}_0$ without the additional assumption that the second coefficient of the corresponding starlike function g is real. In the same paper, the authors also determined the sharp upper bound $|\gamma_3| \le \frac{1}{243}(28 + 19\sqrt{19}) = 0.4560$ for close-to-convex functions with argument 0 and with respect to the Koebe function and conjectured that this upper bound is also true for the whole class $\mathcal{K}_0$ (see also [13]).

Let $\mathcal{S}_2^*$ denote the class of odd starlike functions and $\mathcal{F}$ the class of close-to-convex functions with argument 0 and with respect to odd starlike functions. That is,

$$\mathcal{F} = \Big\{ f \in \mathcal{A} : \mathrm{Re}\, \frac{zf'(z)}{g(z)} > 0, z \in \mathbb{D}, \text{for some } g \in \mathcal{S}_2^* \Big\}.$$

It is important to note that the class $\mathcal{F}$ is rotationally invariant. In the present article, we determine the sharp upper bound of $|\gamma_n|$, $n = 1, 2, 3$, for functions in $\mathcal{F}$.

2. Main results

Let $\mathcal{P}$ denote the class of analytic functions P of the form

$$P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2.1}$$

such that $\mathrm{Re}\, P(z) > 0$ in $\mathbb{D}$. Functions in $\mathcal{P}$ are sometimes called Carathéodory functions. To prove our main results, we need some preliminary lemmas.

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LEMMA 2.1 [5, page 41]. *For a function $P \in \mathcal{P}$ of the form (2.1), the sharp inequality $|c_n| \le 2$ holds for each $n \ge 1$. Equality holds for the function $P(z) = (1+z)/(1-z)$.*

LEMMA 2.2 [12]. *Let $P \in \mathcal{P}$ be of the form (2.1) and μ be a complex number. Then*

$$|c_2 - \mu c_1^2| \le 2\max\{1, |2\mu - 1|\}.$$

The result is sharp for the functions $P(z) = (1+z^2)/(1-z^2)$ and $P(z) = (1+z)/(1-z)$.

LEMMA 2.3 [11]. *Let $P \in \mathcal{P}$ be of the form (2.1). Then there exist $x, t \in \mathbb{C}$ with $|x| \le 1$ and $|t| \le 1$ such that*

$$2c_2 = c_1^2 + x(4 - c_1^2) \quad \textit{and}$$
$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)t.$$

THEOREM 2.4. *Let $f \in \mathcal{F}$ be of the form (1.1). Then we have $|\gamma_1| \le \frac{1}{2}$, $|\gamma_2| \le \frac{1}{2}$ and $|\gamma_3| \le \frac{1}{972}(95 + 23\sqrt{46})$. The inequalities are sharp.*

PROOF. Let $f \in \mathcal{F}$ be of the form (1.1). Then there exist an odd starlike function $g(z) = z + \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1}$ and a Carathéodory function $P \in \mathcal{P}$ of the form (2.1) with

$$zf'(z) = g(z)P(z). \tag{2.2}$$

Comparing the coefficients on both sides of (2.2),

$$a_2 = \tfrac{1}{2}c_1, \quad a_3 = \tfrac{1}{3}(b_3 + c_2) \quad \text{and} \quad a_4 = \tfrac{1}{4}(b_3 c_1 + c_3). \tag{2.3}$$

Substituting a_2, a_3 and a_4 given by (2.3) in (1.3), (1.4) and (1.5) and simplifying,

$$\gamma_1 = \tfrac{1}{2}a_2 = \tfrac{1}{4}c_1, \tag{2.4}$$
$$\gamma_2 = \tfrac{1}{2}(a_3 - \tfrac{1}{2}a_2^2) = \tfrac{1}{6}b_3 + \tfrac{1}{6}(c_2 - \tfrac{3}{8}c_1^2), \tag{2.5}$$
$$2\gamma_3 = a_4 - a_2 a_3 + \tfrac{1}{3}a_2^3 = \tfrac{1}{24}(2c_1 b_3 + c_1^3 - 4c_1 c_2 + 6c_3). \tag{2.6}$$

By Lemma 2.1, it follows from (2.4) that $|\gamma_1| \le \frac{1}{2}$ and equality holds for a function f defined by $zf'(z) = g(z)P(z)$, where $g(z) = z/(1-z^2)$ and $P(z) = (1+z)/(1-z)$. Since g is an odd starlike function, $|b_3| \le 1$ (see [9, Ch. 4, Theorem 3, page 35]). Using Lemma 2.2, it follows from (2.5) that

$$|\gamma_2| \le \tfrac{1}{6}|b_3| + \tfrac{1}{6}|c_2 - \tfrac{3}{8}c_1^2| \le \tfrac{1}{6} + \tfrac{1}{3} = \tfrac{1}{2}$$

and equality holds for a function f defined by $zf'(z) = g(z)P(z)$, where $g(z) = z/(1-z^2)$ and $P(z) = (1+z^2)/(1-z^2)$.

From (2.6), after writing c_2 and c_3 in terms of c_1 with the help of Lemma 2.3,

$$48\gamma_3 = 2c_1 b_3 + \tfrac{1}{2}c_1^3 + c_1 x(4 - c_1^2) - \tfrac{3}{2}c_1 x^2(4 - c_1^2) + 3(4 - c_1^2)(1 - |x|^2)t, \tag{2.7}$$

where $|x| \le 1$ and $|t| \le 1$. Since the class $\mathcal{F}$ is invariant under rotation, without loss of generality we can assume that $c_1 = c$, where $0 \le c \le 2$. Taking the modulus on both the sides of (2.7) and then applying the triangle inequality and $|b_3| \le 1$,

$$48|\gamma_3| \le 2c + \left|\tfrac{1}{2}c^3 + cx(4 - c^2) - \tfrac{3}{2}cx^2(4 - c^2)\right| + 3(4 - c^2)(1 - |x|^2),$$

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where we have also used the fact that $|t| \le 1$. Let $x = re^{i\theta}$, where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. For simplicity, write $\cos\theta = p$. Then

$$48|\gamma_3| \le \psi(c,r) + |\phi(c,r,p)| =: F(c,r,p), \tag{2.8}$$

where $\psi(c,r) = 2c + 3(4-c^2)(1-r^2)$ and

$$\begin{aligned}\phi(c,r,p) = (\tfrac{1}{4}c^6 + c^2r^2(4-c^2)^2 + \tfrac{9}{4}c^2r^4(4-c^2)^2 + c^4(4-c^2)rp \\ - \tfrac{3}{2}c^4r^2(4-c^2)(2p^2-1) - 3c^2(4-c^2)r^3p)^{1/2}.\end{aligned}$$

Thus, we need to find the maximum value of $F(c,r,p)$ over the rectangular cube $R := [0,2] \times [0,1] \times [-1,1]$. By elementary calculus,

$$\max_{0\le r\le 1} \psi(0,r) = \psi(0,0) = 12, \quad \max_{0\le r\le 1} \psi(2,r) = 4, \quad \max_{0\le c\le 2} \psi(c,0) = \psi(\tfrac{1}{3},0) = \tfrac{37}{3},$$
$$\max_{0\le c\le 2} \psi(c,1) = \psi(2,1) = 4 \quad \text{and} \quad \max_{(c,r)\in[0,2]\times[0,1]} \psi(c,r) = \psi(\tfrac{1}{3},0) = \tfrac{37}{3}.$$

We first find the maximum value of $F(c,r,p)$ on the boundary of R, that is, on the six faces of the rectangular cube R. On the face $c = 0$, we have $F(0,r,p) = \psi(0,r)$ for $(r,p) \in R_1 := [0,1] \times [-1,1]$. Thus,

$$\max_{(r,p)\in R_1} F(0,r,p) = \max_{0\le r\le 1} \psi(0,r) = \psi(0,0) = 12.$$

On the face $c = 2$, we have $F(2,r,p) = 8$ for $(r,p) \in R_1$. On the face $r = 0$, we have $F(c,0,p) = 2c + 3(4-c^2) + \frac{1}{2}c^3$ for $(c,p) \in R_2 := [0,2] \times [-1,1]$. Note that $F(c,0,p)$ is independent of p. Thus, by using elementary calculus it is easy to see that

$$\max_{(c,p)\in R_2} F(c,0,p) = F(\tfrac{2}{3}(3-\sqrt{6}),0,p) = \tfrac{8}{9}(9+\sqrt{6}) = 12.3546.$$

On the face $r = 1$, we have $F(c,1,p) = \psi(c,1) + |\phi(c,1,p)|$ for $(c,p) \in R_2$. We first prove that $\phi(c,1,p) \ne 0$ in the interior of R_2. On the contrary, if $\phi(c,1,p) = 0$ in the interior of R_2, then

$$|\phi(c,1,p)|^2 = \left|\tfrac{1}{2}c^3 + ce^{i\theta}(4-c^2) - \tfrac{3}{2}ce^{2i\theta}(4-c^2)\right|^2 = 0,$$

giving the simultaneous equations

$$\tfrac{1}{2}c^3 + cp(4-c^2) - \tfrac{3}{2}c(4-c^2)(2p^2-1) = 0 \quad \text{and}$$
$$c(4-c^2)\sin\theta - \tfrac{3}{2}c(4-c^2)\sin 2\theta = 0.$$

On further simplification, this reduces to

$$\tfrac{1}{2}c^2 + p(4-c^2) - \tfrac{3}{2}(4-c^2)(2p^2-1) = 0 \quad \text{and} \quad 1 - 3p = 0,$$

which is equivalent to $p = 1/3$ and $c^2 = 6$. This contradicts the range of $c \in (0,2)$. Thus, $\phi(c,1,p) \ne 0$ in the interior of R_2.

Next, we find the maximum value $F(c, 1, p)$ in the interior of R_2. Suppose that $F(c, 1, p)$ has a maximum at an interior point of R_2. At such a point $\partial F(c, 1, p)/\partial c = 0$ and $\partial F(c, 1, p)/\partial p = 0$. From $\partial F(c, 1, p)/\partial p = 0$ (for points in the interior of R_2), a straightforward calculation gives

$$p = \frac{2(c^2 - 3)}{3c^2}. \tag{2.9}$$

Substituting this value of p in $\partial F(c, 1, p)/\partial c = 0$ and further simplification gives

$$2c - 3c^3 + \sqrt{6(c^2 + 2)} = 0.$$

Taking the last term to the right-hand side and squaring on both the sides yields

$$9c^6 - 12c^4 - 2c^2 - 12 = 0. \tag{2.10}$$

This equation has exactly one root in $(0, 2)$, which can be shown using the well-known Sturm theorem for isolating real roots and hence for the sake of brevity we omit the details. By solving the equation (2.10) numerically, we obtain the approximate root 1.3584 in $(0, 2)$ and the corresponding value of p obtained from (2.9) is -0.4172. Thus, the extremum points of $F(c, 1, p)$ in the interior of R_2 lie in a small neighbourhood of the points $A_1 = (1.3584, 1, -0.4172)$ (on the plane $r = 1$). Clearly, $F(A_1) = 9.3689$. Since the function $F(c, 1, p)$ is uniformly continuous on R_2, the value of $F(c, 1, p)$ would not vary too much in the neighbourhood of the point A_1.

Next, we find the maximum value of $F(c, 1, p)$ on the boundary of R_2. Clearly, $F(0, 1, p) = 0$, $F(2, 1, p) = 8$,

$$F(c, 1, -1) = \begin{cases} 2c + c(10 - 3c^2) & \text{for } 0 \le c \le \sqrt{\dfrac{10}{3}}, \\ 2c - c(10 - 3c^2) & \text{for } \sqrt{\dfrac{10}{3}} < c \le 2 \end{cases}$$

and

$$F(c, 1, 1) = \begin{cases} 2c + c(2 - c^2) & \text{for } 0 \le c \le \sqrt{2}, \\ 2c - c(2 - c^2) & \text{for } \sqrt{2} < c \le 2. \end{cases}$$

By using elementary calculus,

$$\max_{0 \le c \le 2} F(c, 1, -1) = F\left(\frac{2\sqrt{3}}{3}, 1, -1\right) = \frac{16\sqrt{3}}{3} = 9.2376 \quad \text{and}$$
$$\max_{0 \le c \le 2} F(c, 1, 1) = F\left(\frac{2\sqrt{3}}{3}, 1, 1\right) = \frac{16\sqrt{3}}{9} = 3.0792.$$

Therefore,

$$\max_{(c,p) \in R_2} F(c, 1, p) \approx 9.3689.$$

On the face $p = -1$,

$$F(c, r, -1) = \begin{cases} \psi(c, r) + \eta_1(c, r) & \text{for } \eta_1(c, r) \ge 0, \\ \psi(c, r) - \eta_1(c, r) & \text{for } \eta_1(c, r) < 0, \end{cases}$$

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where $\eta_1(c,r) = c^3(3r^2 + 2r + 1) - 4cr(3r + 2)$ and $(c,r) \in R_3 := [0,2] \times [0,1]$. To find the maximum value of $F(c,r,-1)$ in the interior of R_3, we need to solve the pair of equations $\partial F(c,r,-1)/\partial c = 0$ and $\partial F(c,r,-1)/\partial r = 0$ in the interior of R_3, but it is important to note that $\partial F(c,r,-1)/\partial c$ and $\partial F(c,r,-1)/\partial r$ may not exist at points in $S_1 = \{(c,r) \in R_3 : \eta_1(c,r) = 0\}$. Solving this pair of equations,

$$\begin{aligned}\max_{(c,r)\in \operatorname{int} R_3 \setminus S_1} F(c,r,-1) &= F\big(\tfrac{1}{3}(\sqrt{82} - 8), \tfrac{1}{57}(\sqrt{82} - 5), -1\big) \\ &= \tfrac{4}{81}(41\sqrt{82} - 121) = 12.359.\end{aligned}$$

Now we find the maximum value of $F(c,r,-1)$ on the boundary of R_3 and on the set S_1. Note that

$$\max_{(c,r)\in S_1} F(c,r,-1) \le \max_{(c,r)\in R_3} \psi(c,r) = \tfrac{37}{3} = 12.33.$$

On the other hand, by using elementary calculus as before,

$$\max_{0\le r\le 1} F(0,r,-1) = \max_{0\le r\le 1} 12(1 - r^2) = F(0,0,-1) = 12, \quad \max_{0\le r\le 1} F(2,r,-1) = 8,$$

$$\max_{0\le c\le 2} F(c,0,-1) = \max_{(c,p)\in R_2} F(c,0,p) = F\big(\tfrac{2}{3}(3 - \sqrt{6}), 0, -1\big) = \tfrac{8}{9}(9 + \sqrt{6}) = 12.3546,$$

$$\max_{0\le c\le 2} F(c,1,-1) = F\Big(\frac{2\sqrt{3}}{3}, 1, -1\Big) = \frac{16\sqrt{3}}{3} = 9.2376.$$

Hence, by combining the above cases,

$$\begin{aligned}\max_{(c,r)\in R_3} F(c,r,-1) &= F\big(\tfrac{1}{3}(\sqrt{82} - 8), \tfrac{1}{57}(\sqrt{82} - 5), -1\big) \\ &= \tfrac{4}{81}(41\sqrt{82} - 121) = 12.359.\end{aligned}$$

On the face $p = 1$,

$$F(c,r,1) = \begin{cases} \psi(c,r) + \eta_2(c,r) & \text{for } \eta_2(c,r) \ge 0, \\ \psi(c,r) - \eta_2(c,r) & \text{for } \eta_2(c,r) < 0, \end{cases}$$

where $\eta_2(c,r) = c^3(3r^2 - 2r + 1) - 4cr(3r - 2)$ for $(c,r) \in R_3$. To find the maximum value of $F(c,r,1)$ in the interior of R_3, we need to solve the pair of equations $\partial F(c,r,1)/\partial c = 0$ and $\partial F(c,r,1)/\partial r = 0$ in the interior of R_3, but again it is important to note that $\partial F(c,r,1)/\partial c$ and $\partial F(c,r,1)/\partial r$ may not exist at points in the set $S_2 = \{(c,r) \in R_3 : \eta_2(c,r) = 0\}$. Solving this pair of equations,

$$\begin{aligned}\max_{(c,r)\in \operatorname{int} R_3 \setminus S_2} F(c,r,1) &= F\big(\tfrac{1}{3}(8 - \sqrt{46}), \tfrac{1}{75}(11 - \sqrt{46}), 1\big) \\ &= \tfrac{4}{81}(95 + 23\sqrt{46}) = 12.3947.\end{aligned}$$

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Now we find the maximum value of $F(c,r,1)$ on the boundary of R_3 and on the set S_2. By noting that (see earlier cases)

$$\max_{(c,r)\in S_2} F(c,r,1) \le \max_{(c,r)\in R_3} \psi(c,r) = \tfrac{37}{3} = 12.33,$$
$$\max_{0\le r\le 1} F(0,r,1) = 12, \quad \max_{0\le r\le 1} F(2,r,1) = 8,$$
$$\max_{0\le c\le 2} F(c,0,1) = \tfrac{8}{9}(9+\sqrt{6}) = 12.3546,$$
$$\max_{0\le c\le 2} F(c,1,1) = \frac{16\sqrt{3}}{9} = 3.0792$$

and combining all the cases,

$$\max_{(c,r)\in R_3} F(c,r,1) = F(\tfrac{1}{3}(8-\sqrt{46}), \tfrac{1}{75}(11-\sqrt{46}), 1)$$
$$= \tfrac{4}{81}(95+23\sqrt{46}) = 12.3947.$$

Let $S' = \{(c,r,p) \in R : \phi(c,r,p) = 0\}$. Then

$$\max_{(c,r,p)\in S'} F(c,r,p) \le \max_{(c,r)\in R_3} \psi(c,r) = \psi(0,\tfrac{1}{3}) = \tfrac{37}{3} = 12.33.$$

We prove that $F(c,r,p)$ has no maximum value at any interior point of $R \setminus S'$. Suppose that $F(c,r,p)$ has a maximum value at an interior point of $R \setminus S'$. At such a point $\partial F(c,r,p)/\partial c = \partial F(c,r,p)/\partial r = \partial F(c,r,p)/\partial p = 0$. Note that the partial derivatives may not exist at points in S'. From $\partial F(c,r,p)/\partial p = 0$ (for points in the interior of $R \setminus S'$), a straightforward but laborious calculation gives

$$p = \frac{3c^2r^2 + c^2 - 12r^2}{6c^2 r}.$$

Substituting this value of p in $\partial F(c,r,p)/\partial r = 0$ and simplifying,

$$(4-c^2)r(\sqrt{6(c^2+2)} - 6) = 0.$$

This equation has no solution in the interior of $R \setminus S'$ and hence $F(c,r,p)$ has no maximum in the interior of $R \setminus S'$.

On combining all the above cases,

$$\max_{(c,r,p)\in R} F(c,r,p) = F(\tfrac{1}{3}(8-\sqrt{46}), \tfrac{1}{75}(11-\sqrt{46}), 1)$$
$$= \tfrac{4}{81}(95+23\sqrt{46}) = 12.3947$$

and hence, from (2.8),

$$|\gamma_3| \le \tfrac{1}{972}(95+23\sqrt{46}) = 0.2582. \tag{2.11}$$

We now show that the inequality (2.11) is sharp. An examination of the proof shows that equality holds in (2.11) if we choose $b_3 = 1$, $c_1 = c = \frac{1}{3}(8-\sqrt{46})$, $x = \frac{1}{75}(11-\sqrt{46})$ and $t = 1$ in (2.7). For such values of c_1, x and t, Lemma 2.3 gives

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$c_2 = \frac{1}{27}(134 - 19\sqrt{46})$ and $c_3 = \frac{2}{243}(721 - 71\sqrt{46})$. A function $P \in \mathcal{P}$ having the first three coefficients c_1, c_2 and c_3 as above is given by

$$\begin{aligned} P(z) &= (1-2\lambda)\frac{1+z}{1-z} + \lambda\frac{1+uz}{1-uz} + \lambda\frac{1+\overline{u}z}{1-\overline{u}z} \\ &= 1 + \frac{1}{3}(8 - \sqrt{46})z + \frac{1}{27}(134 - 19\sqrt{46})z^2 + \frac{2}{243}(721 - 71\sqrt{46})z^3 + \cdots, \end{aligned} \tag{2.12}$$

where $\lambda = \frac{1}{10}(-4 + \sqrt{46})$ and $u = \alpha + i\sqrt{1-\alpha^2}$ with $\alpha = \frac{1}{18}(-1 - \sqrt{46})$. Hence, equality holds in (2.11) for a function f which is defined by $zf'(z) = g(z)P(z)$, where $g(z) = z/(1-z^2)$ and $P(z)$ is given by (2.12). This completes the proof. □

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