Canad. Math. Bull. Vol. **67** (4), 2024, pp. 1069–1080
http://dx.doi.org/10.4153/S000843952400050X

A class of Hausdorff–Berezin operators on the unit ball

Alexey Karapetyants and Adolf Mirotin

Abstract. The article introduces and studies Hausdorff–Berezin operators on the unit ball in a complex space. These operators are a natural generalization of the Berezin transform. In addition, the class of such operators contains, for example, the invariant Green potential, and some other operators of complex analysis. Sufficient and necessary conditions for boundedness in the space of p – integrable functions with Haar measure (invariant with respect to involutive automorphisms of the unit ball) are given. We also provide results on compactness of Hausdorff–Berezin operators in Lebesgue spaces on the unit ball. Such operators have previously been introduced and studied in the context of the unit disc in the complex plane. Present work is a natural continuation of these studies.

1 Introduction

The Hausdorff–Berezin operators on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$ were introduced in the paper [15] and then the study of such operators was continued (see, for example, [13]). Some analogues of these operators were also studied, which were called the Hausdorff–Zhu operators (see papers [7, 14]). The Hausdorff–Berezin operators appear as a natural generalization of the classical Berezin transform, and in addition to this, the class of such operators contains some other operators of complex analysis, including a maximal operator constructed from pseudohypertrophic discs in $\mathbb{D}$, for more details, see [15].

In multidimensional complex analysis, the Berezin transform also plays an important role. Recall that, on the unit ball, the Berezin transform is defined by (see, e.g., [8, 24, 25])

$$\mathbb{B}f(z) = \int_{\mathbb{B}^n} f(\varphi_z(w))\,\mathrm{dA}(w) = \int_{\mathbb{B}^n} f(w)|k_z(w)|^2\,\mathrm{dA}(w). \tag{1}$$

Here, $\mathrm{dA}(z)$ is the volume measure, normalized so that $\mathrm{A}(\mathbb{B}^n) = 1$, $\varphi_z(w)$ stand for involutive automorphisms of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, see (5), and $k_z(\cdot)$ are the normalized reproducing kernels of the classical Bergman space $A^2(\mathbb{B}^n, \mathrm{dA})$; see formula (8).

Received by the editors May 7, 2024; revised July 28, 2024; accepted July 30, 2024.
AMS subject classification: **47G10, 47B38, 46E30.**
Keywords: Hausdorff operators, Berezin transform, Haar measure, Möbius group.

https://doi.org/10.4153/S000843952400050X Published online by Cambridge University Press

It is very natural to introduce analogues of such operators in a ball and continue the study of such operators in this general setting. Given a measurable function K (an integral kernel) on the unit ball $\mathbb{B}^n$, we introduce and study the integral operators

$$\begin{aligned}\mathbb{K}f(z) &= \int_{\mathbb{B}^n} K(w)f(\varphi_z(w))\,\mathrm{dH}(w) \\ &= \int_{\mathbb{B}^n} K(\varphi_z(w))f(w)\,\mathrm{dH}(w), \quad z \in \mathbb{B}^n,\end{aligned} \tag{2}$$

where dH stands for the φ_z – invariant Haar measure

$$\mathrm{dH}(z) = \frac{\mathrm{dA}(z)}{(1-|z|^2)^{n+1}}, \quad z \in \mathbb{B}^n.$$

We call such operators Hausdorff–Berezin operators.

The multidimensional setting gives us one more interesting example of an operator in the Hausdorff–Berezin class, namely the invariant Green potential $\mathbb{G}$, which is defined as (see [24])

$$\mathbb{G}f(z) = \int_{\mathbb{B}^n} G(\varphi_z(w))\,f(w)\,\mathrm{dH}(w), \quad z \in \mathbb{B}^n, \tag{3}$$

with the kernel G being the Green's function for the invariant Laplacian $\widetilde{\Delta}$, or simply the invariant Green's function, and it is given by

$$G(z) = \frac{1}{2n}\int_{|z|}^{1} \frac{(1-t^2)^{n-1}}{t^{2n-1}}dt, \quad z \in \mathbb{B}^n. \tag{4}$$

The name Hausdorff in the title also comes into play due to some operator invariance, as can be seen from the formula (1). For the theory of Hausdorff operators, we mention first of all the paper [20] and also papers [4, 5, 16–19]. For Hausdorff operators in the complex analysis, we refer to [1, 2, 6].

The article is organized as follows: In Section 2, we collect some preliminary facts. In Section 3, we provide some preliminary information about Hausdorff–Berezin operators. Section 4 presents our main results and is devoted to establishing sufficient and necessary boundedness conditions for Hausdorff–Berezin operators within $L^p(\mathbb{D}, \mathrm{dH})$ spaces. Here, we dwell on the methods of the study of operators with homogeneous kernels that was earlier developed in a real variable settings. For operators with homogeneous kernels, we refer to the books [10, 12] and the review paper [11], see also [3] for a general setting. In Section 5, we provide results on compactness of Hausdorff–Berezin operators in $L^p(\mathbb{D}, \mathrm{dA})$.

2 Preliminaries

Let us agree that the norm in $L^p(\mathbb{D}, \mathrm{dH})$ will be denoted by $\|\cdot\|_p$. Let $a \in \mathbb{B}^n$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one-dimensional subspace $[a]$ generated by a, and Q_a is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [a]$. It is known that

$$P_a(z) = \frac{\langle z, a\rangle}{|a|^2} a, \quad z \in \mathbb{C}^n,$$

$$Q_a(z) = z - P_a(z) = z - \frac{\langle z, a\rangle}{|a|^2} a, \quad z \in \mathbb{C}^n.$$

The following map

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad a, z \in \mathbb{B}^n \tag{5}$$

defines the class of automorphisms of the unit ball, which is usually called symmetries or involutive automorphisms, i.e., involutions:

$$\varphi_a \circ \varphi_a(z) = z, \quad z \in \mathbb{B}^n.$$

Note that for $a = 0$, we assume $\varphi_0(z) = -z$. We will need the following lemma proven in [23] which is valid for a bounded symmetric domains $\Omega \in \mathbb{C}^n$, group of automorphisms $\mathrm{Aut}(\Omega)$ with G_0 be a subgroup $\{\psi \in \mathrm{Aut}(\Omega) : \psi(0) = 0\}$.

Lemma 1 [23, Lemma 2] *For any $a, b \in \Omega$, there exists a unitary $U \in G_0$ such that*

$$U\varphi_{\varphi_a(b)} = \varphi_b \circ \varphi_a. \tag{6}$$

Moreover, the unitary U is given by the formula

$$U = \varphi_b \circ \varphi_a \circ \varphi_{\varphi_a(b)}. \tag{7}$$

Recall that the function

$$K(z, w) = \frac{1}{(1 - \langle w, z\rangle)^{n+1}}, \qquad z, w \in \mathbb{B}^n,$$

is the Bergman reproducing kernel for the unit ball, and the normalized reproducing kernels $k_z(\cdot), z \in \mathbb{B}^n$, are given by the formula

$$k_z(w) = \frac{K(z, w)}{\|K(z, \cdot)\|_{L^2(\mathbb{B}^n, \mathrm{dA})}} = \frac{(1 - |z|^2)^{\frac{n+1}{2}}}{(1 - \langle w, z\rangle)^{n+1}}. \tag{8}$$

We will use the following known result (see [24, Theorem 1.12]).

Lemma 2 [24]. *Suppose $c \in \mathbb{R}$, $t > -1$, and*

$$I_{c,t}(z) = \int_{\mathbb{B}^n} \frac{(1 - |w|^2)^t}{|1 - \langle z, w\rangle|^{n+1+t+c}} \, \mathrm{dA}(w), \quad z \in \mathbb{B}^n.$$

(1) *If $c < 0$, then as a function of z, $I_{c,t}(z)$ is bounded from above and bounded from below on $\mathbb{B}^n$.*
(2) *If $c > 0$, then $I_{c,t} \approx (1 - |z|^2)^{-c}$ as $|z| \to 1^-$.*
(3) *If $c = 0$, then $I_{0,t} \approx -\ln(1 - |z|^2)$ as $|z| \to 1^-$.*

https://doi.org/10.4153/S000843952400050X Published online by Cambridge University Press

3 Hausdorff–Berezin operators

For any one-variable kernel (a function) K on the unit ball $\mathbb{B}^n$, the corresponding Hausdorff–Berezin operator is defined by formula

$$\mathbb{K}f(z) = \int_{\mathbb{B}^n} K(w) f(\varphi_z(w))\, \mathrm{dH}(w) = \int_{\mathbb{B}^n} K(\varphi_z(w)) f(w)\, \mathrm{dH}(w),$$

if thus integral makes sense. Here, dH stands for the invariant Haar measure

$$\mathrm{dH}(z) = \frac{\mathrm{dA}(z)}{(1-|z|^2)^{n+1}}, \qquad z \in \mathbb{B}^n.$$

We will use $\mathfrak{K}$ to denote the class of all Hausdorff–Berezin operators.

Along with the operator $\mathbb{K}$, we consider the conjugate operator

$$\mathbb{K}^* f(z) = \int_{\mathbb{B}^n} \overline{K(\varphi_w(z))} f(w)\, \mathrm{dH}(w) = \int_{\mathbb{B}^n} \overline{K(\varphi_{\varphi_z(\xi)}(z))} f(\varphi_z(\xi))\, \mathrm{dH}(\xi).$$

Lemma 3 *The class $\mathfrak{K}$ coincides with the class of operators of the type*

(9) $$\mathcal{K}f(z) \equiv \int_{\mathbb{B}^n} \widetilde{K}(z,w) f(w)\, \mathrm{dH}(w),$$

where the integral kernel $\widetilde{K}$ is invariant in the sense that

(10) $$\widetilde{K}(z,w) = \widetilde{K}(0, -\varphi_z(w)), \qquad z, w \in \mathbb{D}.$$

Proof If we define the two-variable integral kernel by the rule

$$\widetilde{K}(z,w) = K(\varphi_z(w)),$$

then it satisfies (10), because

$$\widetilde{K}(z,w) = K(\varphi_z(w)) = K(\varphi_0(-\varphi_z(w))) = \widetilde{K}(0, -\varphi_z(w)).$$

Conversely, every operator $\mathcal{K}$ with the kernel $\widetilde{K}$ satisfying (10) has the form of the operator $\mathbb{K}$ with

$$K(\varphi_z(w)) = \widetilde{K}(0, -\varphi_z(w))$$

by definition. ∎

Remark 1 Note that in one-dimensional complex case, i.e., $\mathbb{B}^n = \mathbb{D}$ the more strict invariance condition

$$\widetilde{K}(\varphi_a(z), \varphi_a(w)) = \widetilde{K}(z,w), \qquad a, z, w \in \mathbb{B}^n,$$

is equivalent to the case that initial one-variable kernel K is radial, see [15]. We state the problem of finding an analog of such a condition as an open question for the case $\mathbb{B}^n, n > 1$.

4 Boundedness of Hausdorff–Berezin operators in $L^p(\mathbb{B}^n, \mathrm{dH})$

In this section, we consider the boundedness of our Hausdorff–Berezin operators on the spaces $L^p(\mathbb{B}^n, \mathrm{dH})$. We start with the case $p = 1$. Let

$$\kappa = \sup_{z\in\mathbb{B}^n} \int_{\mathbb{B}^n} |K(\varphi_w(z))| \, \mathrm{dH}(w). \tag{11}$$

If K is radial, then the formula (11) reads as

$$\kappa = \int_{\mathbb{B}^n} |K(w)| \, \mathrm{dH}(w) = \int_0^1 \frac{|K(r)|}{(1-r^2)^{n+1}} 2nr^{2n-1} dr.$$

We use $\|\mathbb{K}\|$ to define the operator norm of the operator $\mathbb{K}$ acting in $L^p(\mathbb{B}^n, \mathrm{dH})$.

Theorem 4 *Assume that $\kappa < \infty$. Then the operator $\mathbb{K}$ is bounded on $L^1(\mathbb{B}^n, \mathrm{dH})$ and its operator norm on $L^1(\mathbb{B}^n, \mathrm{dH})$ satisfies $\|\mathbb{K}\| \leqslant \kappa$.*

Proof The proof is immediate by Minkowski's inequality. In fact,

$$\begin{aligned}\|\mathbb{K}f\|_1 &= \int_{\mathbb{B}^n} \left| \int_{\mathbb{B}^n} K(\varphi_z(w)) f(w) \, \mathrm{dH}(w) \right| \mathrm{dH}(z) \\ &\leqslant \int_{\mathbb{B}^n} |f(w)| \, \mathrm{dH}(w) \int_{\mathbb{B}^n} |K(\varphi_z(w))| \, \mathrm{dH}(z) \leqslant \kappa \|f\|_1\end{aligned}$$

as desired. ∎

Let us consider the case $1 < p < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. Given $\sigma \in \mathbb{R}$, let us write

$$\kappa_1(p,\sigma) = \sup_{z\in\mathbb{B}^n} \int_{\mathbb{B}^n} |k_\zeta(z)|^{\frac{\sigma}{p}} |K(\zeta)| \, \mathrm{dH}(\zeta), \tag{12}$$

and

$$\kappa_2(q,\sigma) = \sup_{z\in\mathbb{B}^n} \int_{\mathbb{B}^n} |k_\zeta(z)|^{\frac{\sigma}{q}} \left|K\left(\varphi_{\varphi_z(\zeta)}(z)\right)\right| \, \mathrm{dH}(\zeta). \tag{13}$$

If K is radial, then due to (7), the formula in (13) reads as

$$\kappa_2(q,\sigma) = \sup_{z\in\mathbb{B}^n} \int_{\mathbb{B}^n} |k_\zeta(z)|^{\frac{\sigma}{q}} |K(\zeta)| \, \mathrm{dH}(\zeta).$$

Theorem 5 *Let $1 < p < \infty$. If there exists $\sigma_0 \in \mathbb{R}$ such that*

$$\kappa_1(p,\sigma_0) < \infty, \qquad \kappa_2(q,\sigma_0) < \infty, \tag{14}$$

then the operator $\mathbb{K}$ is bounded in $L^p(\mathbb{B}^n, \mathrm{dH})$ and its operator norm on $L^p(\mathbb{B}^n, \mathrm{dH})$ satisfies the estimate

$$\|\mathbb{K}\| \leqslant \inf \left\{ \kappa_1(p,\sigma)^{\frac{1}{q}} \kappa_2(q,\sigma)^{\frac{1}{p}} \right\},$$

where infimum is taken with respect to all those $\sigma = \sigma_0$ for which (14) holds.

https://doi.org/10.4153/S000843952400050X Published online by Cambridge University Press

Proof Denote $\tau_\sigma(z) = (1-|z|^2)^\sigma$, $\sigma \in \mathbb{R}$. By Hölder's inequality, we obtain

$$\begin{aligned} |\mathbb{K}f(z)| &= \left| \int_{\mathbb{B}^n} \tau_\sigma(w)^{\frac{1}{pq}} \tau_\sigma(w)^{-\frac{1}{pq}} K(\varphi_z(w)) f(w)\, \mathrm{dH}(w) \right| \\ &\leqslant \left(\int_{\mathbb{B}^n} \tau_\sigma(w)^{\frac{1}{p}} |K(\varphi_z(w))|\, \mathrm{dH}(w) \right)^{\frac{1}{q}} \\ &\quad \times \left(\int_{\mathbb{B}^n} \tau_\sigma(w)^{-\frac{1}{q}} |K(\varphi_z(w))| |f(w)|^p\, \mathrm{dH}(w) \right)^{\frac{1}{p}}. \end{aligned}$$

Note that

$$\begin{aligned} \int_{\mathbb{B}^n} \tau_\sigma(w)^{\frac{1}{p}} |K(\varphi_z(w))|\, \mathrm{dH}(w) &= \int_{\mathbb{B}^n} \tau_\sigma(\varphi_z(\zeta))^{\frac{1}{p}} |K(\zeta)|\, \mathrm{dH}(\zeta) \\ &= (1-|z|^2)^{\frac{\sigma}{p}} \int_{\mathbb{B}^n} |k_\zeta(z)|^{\frac{2\sigma}{(n+1)p}} |K(\zeta)|\, \mathrm{dH}(\zeta) \\ &\leqslant \kappa_1(p, 2\sigma/(n+1))\, (1-|z|^2)^{\frac{\sigma}{p}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\kappa_1(p, 2\sigma/(n+1))^{-\frac{p}{q}} \|\mathbb{K}f\|_p^p \\ &\quad \le \int_{\mathbb{B}^n} (1-|z|^2)^{\frac{\sigma}{q}}\, \mathrm{dH}(z) \int_{\mathbb{B}^n} \tau_\sigma(w)^{-\frac{1}{q}} |K(\varphi_z(w))| |f(w)|^p\, \mathrm{dH}(w) \\ &\quad = \int_{\mathbb{B}^n} |f(w)|^p\, \mathrm{dH}(w) \int_{\mathbb{B}^n} (1-|z|^2)^{\frac{\sigma}{q}} \tau_\sigma(w)^{-\frac{1}{q}} |K(\varphi_z(w))|\, \mathrm{dH}(z) \\ &\quad = \int_{\mathbb{B}^n} |f(w)|^p\, \mathrm{dH}(w) \int_{\mathbb{B}^n} (1-|\varphi_w(\zeta)|^2)^{\frac{\sigma}{q}} \tau_\sigma(w)^{-\frac{1}{q}} \left| K\left(\varphi_{\varphi_w(\zeta)}(w)\right)\right| \mathrm{dH}(\zeta) \\ &\quad = \int_{\mathbb{B}^n} |f(w)|^p\, \mathrm{dH}(w) \int_{\mathbb{B}^n} |k_\zeta(w)|^{\frac{2\sigma}{(n+1)q}} \left| K\left(\varphi_{\varphi_w(\zeta)}(w)\right)\right| \mathrm{dH}(\zeta) \\ &\quad \leqslant \kappa_2(q, 2\sigma/(n+1)) \|f\|_p^p. \end{aligned}$$

Here, to justify the change of the order of integration, we used Fubini's theorem. Finally, collecting the above estimates, we obtain

$$\|\mathbb{K}f\|_p \le \kappa_1(p, 2\sigma/(n+1))^{\frac{1}{q}} \kappa_2(q, 2\sigma/(n+1))^{\frac{1}{p}} \|f\|_p.$$

This finishes the proof. ∎

As a corollary, we formulate below the corresponding boundedness result for the conjugate operator $\mathbb{K}^*$ (we will need it to prove Theorem 7).

Corollary 6 *Let $1 < q < \infty$. If there exist $\sigma \in \mathbb{R}$ such that (14) holds, then the operator $\mathbb{K}^*$ is bounded on $L^q(\mathbb{B}^n, \mathrm{dH})$ and its norm on $L^q(\mathbb{B}^n, \mathrm{dH})$ satisfies*

$$\|\mathbb{K}^*\| \leqslant \inf\{\kappa_1(p,\sigma)^{\frac{1}{q}} \kappa_2(q,\sigma)^{\frac{1}{p}}\},$$

where infimum is taken with respect to all those σ for which (14) holds.

For a nonnegative kernel (function) K, let us write

$$\varkappa = \inf_{z\in\mathbb{B}^n} \int_{\mathbb{B}^n} K\left(\varphi_\zeta(z)\right) \mathrm{dH}(\zeta),$$
$$\varkappa_1(p,\sigma) = \inf_{z\in\mathbb{B}^n} \int_{\mathbb{B}^n} |k_\zeta(z)|^{\frac{\sigma}{p}} K(\zeta)\, \mathrm{dH}(\zeta),$$
$$\varkappa_2(q,\sigma) = \inf_{z\in\mathbb{B}^n} \int_{\mathbb{B}^n} |k_\zeta(z)|^{\frac{\sigma}{q}} K\left(\varphi_{\varphi_z(\zeta)}(z)\right) \mathrm{dH}(\zeta).$$

If K is also radial, then

$$\varkappa_2(q,\sigma) = \inf_{z\in\mathbb{B}^n} \int_{\mathbb{B}^n} |k_\zeta(z)|^{\frac{\sigma}{q}} K(\zeta)\, \mathrm{dH}(\zeta).$$

Theorem 7 *Let the kernel K be nonnegative. Suppose that the operator $\mathbb{K}$ is bounded on $L^p(\mathbb{D},\mathrm{dH})$ with $1 \leqslant p < \infty$. Then the following statements hold:*

(1) *If $p = 1$, then $\varkappa < \infty$.*
(2) *If $1 < p < \infty$, then $\varkappa_1(p,\sigma) < \infty$ and $\varkappa_2(q,\sigma) < \infty$ for any $\sigma > \frac{2n}{n+1}$.*

Proof First, suppose that the operator $\mathbb{K}$ is bounded in $L^1(\mathbb{D},\mathrm{dH})$. Then for $\phi(z) = (1-|z|^2)^{n+1}$, we have

$$\begin{aligned}\|\mathbb{K}\phi\|_1 &= \int_{\mathbb{B}^n}\int_{\mathbb{B}^n} K(\varphi_z(w))\, \mathrm{dA}(w)\, \mathrm{dH}(z)\\ &= \int_{\mathbb{B}^n} \mathrm{dA}(w) \int_{\mathbb{B}^n} K(\varphi_z(w))\, \mathrm{dH}(z) \geqslant \varkappa.\end{aligned}$$

Suppose that the operator $\mathbb{K}$ is bounded in $L^p(\mathbb{B}^n,\mathrm{dH})$ for some $1 < p < \infty$. Then for all $\phi \in L^p(\mathbb{B}^n,\mathrm{dH})$ and $\psi \in L^q(\mathbb{D},\mathrm{dH})$, we have

$$\left|\int_{\mathbb{B}^n} (\mathbb{K}\phi)(z)\, \psi(z)\, \mathrm{dH}(z)\right| \leqslant \|\mathbb{K}\|\, \|\phi\|_p \|\psi\|_q.$$

Let

$$\phi(z) = (1-|z|^2)^{\frac{\sigma}{p}}, \quad \psi(z) = (1-|z|^2)^{\frac{\sigma}{q}}, \quad \sigma > n.$$

We obtain

$$\begin{aligned}&\int_{\mathbb{B}^n} (\mathbb{K}\phi)(z)\, \psi(z)\, \mathrm{dH}(z)\\ &= \int_{\mathbb{B}^n} (1-|z|^2)^{\frac{\sigma}{q}}\, \mathrm{dH}(z) \int_{\mathbb{B}^n} (1-|\varphi_z(w)|)^{\frac{\sigma}{p}} K(w)\, \mathrm{dH}(w)\\ &= \int_{\mathbb{B}^n} (1-|z|^2)^{\sigma}\, \mathrm{dH}(z) \int_{\mathbb{B}^n} |k_w(z)|^{\frac{2\sigma}{(n+1)p}} K(w)\, \mathrm{dH}(w)\\ &\geqslant \varkappa_1(p, 2\sigma/(n+1)) \int_{\mathbb{B}^n} (1-|z|^2)^{\sigma}\, \mathrm{dH}(z).\end{aligned}$$

The integral $\int_{\mathbb{B}^n}(1-|z|^2)^{\sigma}\, \mathrm{dH}(z)$ is finite if and only if $\sigma > n$. This implies that $\varkappa_1(p,\sigma) < \infty$ for any $\sigma > \frac{2n}{n+1}$. By the same arguments applied to the conjugate operator $\mathbb{K}^*$, we obtain that $\varkappa_2(q,\sigma) < \infty$ for any $\sigma > \frac{2n}{n+1}$. ■

https://doi.org/10.4153/S000843952400050X Published online by Cambridge University Press

At the conclusion of this section, let us consider the important example of the kernel K given by the formula $K(z) = (1-|z|^2)^\alpha$. First, we prove the following technical lemma.

Lemma 8 *Suppose* $K(z) = (1-|z|^2)^\alpha$, $\alpha \in \mathbb{R}$, *and* $1 < p < \infty$ *with* $1/p + 1/q = 1$. *Suppose that* $\kappa, \kappa_1(p,\sigma)$ *and* $\kappa_2(q,\sigma)$ *are the corresponding to K numbers, as defined above. Then:*

(a) $\kappa < \infty$ *if and only if* $\alpha > n$.

(b) $\kappa_1(p,\sigma) < \infty$ *if and only if*

$$\alpha > \max\left\{\frac{n+1}{2}\frac{\sigma}{p}, n - \frac{n+1}{2}\frac{\sigma}{p}\right\}.$$

(c) $\kappa_2(q,\sigma) < \infty$ *if and only if*

$$\alpha > \max\left\{\frac{n+1}{2}\frac{\sigma}{q}, n - \frac{n+1}{2}\frac{\sigma}{q}\right\}.$$

Proof The proof is straightforward: one needs to substitute the kernel $K(z) = (1-|z|^2)^\alpha$ into (11), (12), and (13), and then apply Lemma 2. ∎

As a corollary of Lemma 8, we obtain the following result on the boundedness of the operator $\mathbb{K}$ induced by the positive kernel $K(z) = (1-|z|^2)^\alpha$ on $L^p(\mathbb{D}, \mathrm{dH})$. This result shows that for this kernel, the sufficient condition $\kappa < \infty$ in Theorem 4 and the sufficient conditions (14) with $\sigma = \frac{2n}{n+1}$ stated in Theorem 5 turn out to be necessary.

Theorem 9 *Let* $1 \leqslant p < \infty$ *and* $\mathbb{K}$ *be the Hausdorff–Berezin operator with kernel* $K(z) = (1-|z|^2)^\alpha$. *Then* $\mathbb{K}$ *is bounded in* $L^p(\mathbb{B}^n, \mathrm{dH})$ *if and only if*

$$\alpha > n \max\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}. \tag{15}$$

Proof The sufficiency of (15) follows from Theorems 4 and 5 and Lemma 8 under the choice $\sigma = \frac{2n}{n+1}$.

To prove necessity, we choose the minimizing function

$$f_\beta(z) = (1-|z|^2)^{\frac{\beta}{p}}, \qquad \beta \in \mathbb{R}.$$

It is obvious that $f_\beta \in L^p(\mathbb{B}^n, \mathrm{dH})$ if and only if $\beta > n$, and

$$\begin{aligned}\mathbb{K}f_\beta(z) &= \int_{\mathbb{B}^n} (1-|\varphi_z(w)|^2)^\alpha (1-|w|^2)^{\frac{\beta}{p}} \,\mathrm{dH}(w)\\ &= (1-|z|^2)^\alpha \int_{\mathbb{B}^n} \frac{(1-|w|^2)^{\alpha+\frac{\beta}{p}}}{|1-\langle w,z\rangle|^{2\alpha}} \,\mathrm{dH}(w).\end{aligned}$$

It is clear that there should be $\alpha > n - \frac{\beta}{p}$. Otherwise, the integral on the right side of the above equality is infinite for any $z \in \mathbb{B}^n$. From Lemma 2, we see that

$$\mathbb{K}f_\beta(z) \asymp (1-|z|^2)^{\frac{\beta}{p}}, \quad |z| \to 1^-,$$

if and only if $\alpha > \beta/p$. Since $\beta > n$ can be chosen arbitrarily close to n, this implies that $\alpha \geqslant n/p$. The case $\alpha = n/p$ is excluded since in that case from Lemma 2, we obtain

$$\mathbb{K}f_\beta(z) \approx (1-|z|^2)^{\frac{n}{p}}, \quad |z| \to 1^-.$$

Now the rest of the proof follows by Lemma 8. ∎

5 Compactness of Hausdorff–Berezin operators

Before we formulate the next theorem, recall the definition of a positive-definite kernel. Let X be a nonempty set, and let $\mathbf{K}$ be a function on $X \times X$ such that

$$\sum_{j=1}^{N}\sum_{k=1}^{N} \mathbf{K}(x_j, x_k)\xi_j\overline{\xi_k} \geq 0 \text{ for any } x_1, \dots, x_N \in X, \xi_j \in \mathbb{C}, \text{ and any } N.$$

Then $\mathbf{K}$ is called a positive-definite kernel on X. In this case, $\mathbf{K}(x,x) \geq 0$ for all x. Let, in addition, X be a locally compact space which is equipped with regular Borel measure μ, and let $\mathbf{K}$ be continuous. The well-known sufficient condition for an integral operator B on $L^2(\mu)$ (the space of μ – measurable quadratically summable functions with respect to the measure μ) with a kernel $\mathbf{K}$ to be in a trace class states that this is the case if $\mathbf{K}$ is a positive-definite kernel on X and

$$I := \int_X \mathbf{K}(x,x)d\mu(x) < \infty.$$

In such a case, the trace equals to: $\mathrm{tr}B = I$ (see, e.g., Theorem 2.12 in [22] or arguments before Theorem XI.31 in [21]).

Theorem 10 *The following statements hod true.*

(i) *Assume that* $1 \leq p < \infty$*, and there are real numbers* σ *and* r *such that*

$$\sigma < p, \quad (1-\sigma/p)p' < r, \quad (1/p + 1/p' = 1),$$

and that the following conditions are satisfied:

$$\int_{\mathbb{B}^n} \frac{|K(\varphi_z(w))|^r}{(1-|w|^2)^{r(n+1)}} \mathrm{dA}(w) < c_1 < \infty \ \text{ for a.e. } z \in \mathbb{B}^n; \tag{16}$$

$$\int_{\mathbb{B}^n} |K(\varphi_z(w))|^\sigma \mathrm{dA}(z) < c_2 < \infty \ \text{ for a.e. } w \in \mathbb{B}^n, \tag{17}$$

with some constants c_1, c_2*. Then* $\mathbb{K}$ *is a compact operator in* $L^p(\mathbb{B}^n, \mathrm{dA})$*.*

(ii) *The operator* $\mathbb{K}$ *is a Hilbert–Schmidt operator in* $L^2(\mathbb{B}^n, \mathrm{dA})$ *if and only if*

$$\frac{K(\varphi_z(w))}{(1-|w|^2)^{n+1}} \in L^2(\mathrm{dA}\otimes\mathrm{dA}).$$

(iii) *Let* $K(\varphi_z(w))/(1-|w|^2)^{n+1}$ *be a positive-definite kernel on* $\mathbb{B}^n$ *and* $K(0) = 0$*. Then* $\mathbb{K}$ *is a trace class operator in* $L^2(\mathbb{B}^n, \mathrm{dA})$ *and* $\mathrm{tr}\,\mathbb{K} = 0$*.*

https://doi.org/10.4153/S000843952400050X Published online by Cambridge University Press

Proof To prove statement (i), we are going to apply [9, Chapter XI, Theorem 3] to the operator $\mathbb{K}$ in the form

$$\mathbb{K}f(z) = \int_{\mathbb{B}^n} K(\varphi_z(w)) f(w)\,\mathrm{dH}(w) = \int_{\mathbb{B}^n} \frac{K(\varphi_z(w))}{(1-|w|^2)^{n+1}} f(w)\,\mathrm{dA}(w).$$

To this end, we identify the space $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and $\mathbb{B}^n$ with the unit ball D of $\mathbb{R}^{2n}$. Then the point $z = s_1 + \imath s_2 \in \mathbb{B}^n$ corresponds to the point $s = (s_1, s_2) \in \mathbb{R}^{2n}$ and the point $w = t_1 + \imath t_2 \in \mathbb{B}^n$ corresponds to the point $t = (t_1, t_2) \in \mathbb{R}^{2n}$. Moreover, let the function $K(\varphi_z(w))/(1-|w|^2)^{n+1}$ corresponds to $\mathbf{K}(s,t)$ in the aforementioned theorem. Further, we put $D' = D$ and $q = p$ in this theorem. Since $dt = |\mathbb{B}^n|\mathrm{dA}(w)$ ($|\mathbb{B}^n|$ stands for the Euclidean volume of $\mathbb{B}^n$), the condition (16) implies the validity of the condition 1) from [9, Chapter XI, Theorem 1] because

$$\int_{\mathbb{B}^n} |\mathbf{K}(s,t)|^r dt = |\mathbb{B}^n| \int_{\mathbb{B}^n} \frac{|K(\varphi_z(w))|^r}{(1-|w|^2)^{r(n+1)}} \mathrm{dA}(w) < \infty \text{ for a.e. } z \in \mathbb{B}^n.$$

Similarly, the condition (17) implies the validity of the condition 2) from [9, Chapter XI, Theorem 1]. The remaining condition of this theorem is also satisfied due to our assumptions.

The statement (ii) is a corollary of the well-known result (see, e.g., [22, Theorem 2.11]).

Now, to prove (iii), we use the mentioned above sufficient condition for an integral operator to be in a trace class. Since $\mathbf{K}(s,t)$ is a positive-definite kernel, it suffices to note that the second condition holds, since $\mathbf{K}(s,s) = K(0) = 0$. This completes the proof. ∎

Since dH is invariant, choosing $r = 1$, we get the following corollary.

Corollary 11 *Let $K \in L^1(\mathrm{dH})$. If $1 < \sigma < p$ and (17) holds, then $\mathbb{K}$ is compact in $L^p(\mathbb{B}^n, \mathrm{dA})$.*

Proof The condition (16) is valid for $r = 1$, since for all $z \in \mathbb{B}^n$,

$$\begin{aligned}\int_{\mathbb{B}^n} \frac{|K(\varphi_z(w))|}{(1-|w|^2)^{n+1}} \mathrm{dA}(w) &= \int_{\mathbb{B}^n} |K(\varphi_z(w))| \mathrm{dH}(w)\\ &= \int_{\mathbb{B}^n} |K(w)| \mathrm{dH}(w) =: c_1 < \infty.\end{aligned}$$

It remains to note that in our case $(1-\sigma/p)p' < 1$. ∎

Example 12 The Hausdorff–Berezin operator with kernel

$$K(z) = (1-|z|^2)^\alpha$$

is a Hilbert–Schmidt operator in $L^2(\mathbb{B}^n, \mathrm{dA})$ if and only if $\alpha > n + 1/2$.

Acknowledgments A.K. and A.M. acknowledge the support of the Ministry of Education and Science of Russia (Agreement No. 075-02-2024-1427). Adolf Mirotin is partially supported by the State Program of Scientific Research of Republic of Belarus, project 20211776.

Data availability statement The authors confirm that all data generated or analyzed during this study are included in this article.

Conflict of interest This work does not have any conflict of interest.

References

[1] L. Aizenberg and E. Liflyand, *Hardy spaces in Reinhardt domains, and Hausdorff operators*. Illinois J. Math. **53**(2009), 1033–1049.
[2] L. Aizenberg, E. Liflyand, and A. Vidras, *Hausdorff operators in Hardy spaces on Cartan type domains in* $\mathbb{C}^n$. In: Complex analysis and dynamical systems VI, Part 2, Contemporary Mathematics, 667, Israel Mathematical Conference Proceedings, American Mathematical Society, Providence, RI, 2016, pp. 27–46.
[3] Z. Avetisyan and A. Karapetyants, *Homogeneous operators and homogeneous integral operators*. Math. Methods Appl. Sci. **46**(2023), no. 1, 811–829.
[4] G. Brown and F. Móricz, *Multivariate Hausdorff operators on the spaces* $L^p(\mathcal{R}^n)$. J. Math. Anal. Appl. **271**(2002), 443–454.
[5] J. Chen, D. Fan, and S. Wang, *Hausdorff operators on Euclidean spaces*. Appl. Math. J. Chinese Univ. (Ser. B) (4) **28**(2014), 548–564.
[6] P. Galanopoulos and M. Papadimitrakis, *Hausdorff and quasi-Hausdorff matrices on spaces of analytic functions*. Canad. J. Math. **58**(2006), 548–579.
[7] S. Grudsky, A. Karapetyants, and A. Mirotin, *Estimates for singular numbers of Hausdorff–Zhu operators and applications*. Math. Methods Appl. Sci. **46**(2023), no. 8, 9676–9693. https://doi.org/10.1002/mma.9080
[8] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*, Springer, New York, 2000.
[9] L. V. Kantorovich and G. P. Akilov, *Functional analysis*. 2nd ed., Pergamon Press, Oxford, 1982.
[10] N. K. Karapetiants and S. G. Samko, *Equations with an involutive operators and their applications*, Rostov University Publishing House, Rostov-on-Don, 1988 (in Russian).
[11] N. K. Karapetiants and S. G. Samko, *Multi-dimensional integral operators with homogeneous kernels*. Fract. Calc. Appl. Anal. **2**(1999), 67–96.
[12] N. K. Karapetiants and S. G. Samko, *Equations with an involutive operators and their applications*, Birkhauser, Boston, 2001.
[13] A. Karapetyants and A. Mirotin, *Hausdorff–Berezin operators on weighted spaces*. Math. Methods Appl. Sci. **46**(2023), no. 13, 14260–14277. https://doi.org/10.1002/mma.9318
[14] A. Karapetyants and A. R. Mirotin, *A class of Hausdorff–Zhu operators*. Anal. Math. Phys. **12**(2022), 79. https://doi.org/10.1007/s13324-022-00681-x
[15] A. Karapetyants, S. Samko, and K. Zhu, *A class of Hausdorff–Berezin operators on the unit disc*. Complex Anal. Oper. Theory **13**(2019), no. 8, 3853–3870. https://doi.org/10.1007/s11785-019-00934-x
[16] A. Lerner and E. Liflyand, *Multidimensional Hausdorff operator on the real hardy space*. J. Aust. Math. Soc. **83**(2007), 79–86.
[17] E. Liflyand, *Boundedness of multidimensional Hausdorff operators on* $H^1(\mathbb{R}^n)$. Acta Sci. Math. (Szeged) **74**(2008), 845–851.
[18] E. Liflyand, *Hausdorff operators on hardy spaces*. Eurasian Math. J. **4**(2013), 101–141.
[19] E. Liflyand and A. Miyachi, *Boundedness of the Hausdorff operators in* H^p *spaces*, $0 < p < 1$. Studia Math. **194**(2009), no. 3, 279–292.
[20] E. Liflyand and F. Móricz, *The Hausdorff operator is bounded on the real Hardy space* $H^1(\mathbb{R})$. Proc. Amer. Math. Soc. **128**(2000), 1391–1396.
[21] M. Reed and B. Simon, *Scattering theory, methods of modern mathematical physics*. Vol. 3. Academic Press, New York, 1979.
[22] B. Simon, *Trace ideals and their applications*. 2nd ed., American Mathematical Society, Providence, RI, 1995.

[23] K. Zhu, *On certain unitary operators and composition operators*. Proc. Sympos. Pure Math. **51**(1990), 371–385.
[24] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Springer, New York, 2004.
[25] K. Zhu, *Operator theory in function spaces*, AMS Mathematical Surveys and Monographs, 138, American Mathematical Society, Providence, RI, 2007.

Institute of Mathematics, Mechanics and Computer Sciences & Regional Mathematical Center, Southern Federal University, Rostov-on-Don, 344090, Russia
e-mail: karapetyants@gmail.com

Department of Mathematics and Programming Technologies, Francisk Skorina Gomel State University, Gomel, 246019, Belarus
and
Regional Mathematical Center, Southern Federal University, Rostov-on-Don, 344090, Russia
e-mail: amirotin@yandex.ru

https://doi.org/10.4153/S000843952400050X Published online by Cambridge University Press