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UNFAITHFUL MINIMAL HEILBRONN CHARACTERS OF $L_2(q)$

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Abstract When a minimal Heilbronn character θ is unfaithful on a Sylow p-subgroup P of a finite group G, we know that G is quasi-simple, p is odd, P is cyclic, $N_G(P)$ is maximal and *either* $N_G(P)$ is the unique maximal subgroup containing $\Omega_1(P)$ *or* $G/Z(G) \cong L_2(q)$ for q an odd prime with p dividing $q-1$. In this paper we examine the exceptional case, where $G/Z(G) \cong L_2(q)$, explicitly constructing unfaithful minimal Heilbronn characters from the non-principal irreducible characters of G.

Keywords: Heilbronn characters; virtual characters; Artin's conjecture

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1. Introduction

Call a virtual character θ of a finite group G a *Heilbronn character* if its inner product with every monomial character of G is non-negative. If, moreover, θ restricts to a character of every proper subgroup and quotient of G (where we define the restriction of θ to G/N as the sum of the constituents of θ whose kernels contain N), but is not a character of G itself, then θ is said to be *minimal*. Heilbronn characters arise naturally in the study of Artin's Conjecture on the holomorphy of L-series, where a hypothetical minimal counterexample engenders a corresponding minimal Heilbronn character of the associated Galois group. Although motivated originally by this number theoretic application, the study of Heilbronn characters is of independent interest to both group theory and representation theory. Moreover, the analysis of fusion and strongly closed subgroups figures prominently among the techniques involved in this research, underscoring the relevance of these results to others in this special issue of the journal.

A natural subclass of Heilbronn characters to classify are those that are both minimal and *unfaithful*, where θ is said to be unfaithful if the set $\{g \in G \mid \theta(g) = \theta(1)\}$ is non-trivial. (It is important to note that this 'kernel', although a union of conjugacy classes, is not generally a subgroup of G; however, for any proper subgroup H of G, its intersection with H, denoted $\ker \theta|_H$, is not only a normal subgroup of H, but also strongly closed with respect to G.) It is easy to see that if a minimal Heilbronn character θ of a group G is unfaithful, then θ restricts to an unfaithful character of some Sylow p-subgroup P of G

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(if $\theta(g) = \theta(1)$, then θ is constant on the proper subgroup $\langle g \rangle$, hence unfaithful on a Sylow subgroup for every prime dividing $|g|$). Foote proved in [**2**] that in this case p is necessarily odd, and Ginsberg proved in [**3**] that, moreover, P is cyclic and $N_G(P)$ is *in general* the unique maximal subgroup of G containing $\Omega_1(P)$ (where $\Omega_1(P) = \langle x \in P \mid |x| = p \rangle$). An exception to this last condition occurs precisely when $G/Z(G) \cong L_2(q)$, $q \geqslant 5$ is prime, and p divides $q-1$. In this case, $\Omega_1(P)$ is contained in *two* maximal subgroups, $N_G(P)$ and a Borel subgroup $N_G(Q)$ for some equi-characteristic Sylow subgroup Q of G (and θ is unfaithful on both P and Q).

The groups $L_2(q)$ with q prime are unique among the groups of Lie type in that their equi-characteristic Sylow subgroups are cyclic, and it is precisely this property that distinguishes them in the context of unfaithful minimal Heilbronn characters. We consider herein the construction of unfaithful minimal Heilbronn characters from non-principal irreducible characters of $L_2(q)$, in both the exceptional and the general cases. Our main theorem is the following.

Theorem 1.1 (main theorem). *Let $G \cong L_2(q)$, where $q = r^a \geqslant 5$ and r is a prime. Let $p > 5$ be a prime dividing the order of G, and suppose that one of the following holds:*

(i) *p divides $q+1$ but p does not divide $r^b \pm 1$ for any proper divisor b of a;*

(ii) *$p = q = r$ is an odd prime; or*

(iii) *p divides $q-1$ and $q = r$ is an odd prime.*

Let P be a Sylow p-subgroup of G, let P_1 be any non-trivial subgroup of P and let $N = N_G(P) = P \rtimes H$. For $x \in P$, $h \in H$, define $\varphi\colon N \to N$ by $\varphi(xh) = x^{|P_1|}h$. Then if π is any non-principal irreducible character of G, the map $\theta\colon G \to \mathbb{C}$, given by

$$\theta(g) = \begin{cases} \pi(\varphi(n)) & \text{if } g \text{ is conjugate to } n \in N, \\ \pi(1) & \text{if case (iii) applies and } |g| = q = r, \\ \pi(g) & \text{otherwise}, \end{cases} \tag{1.1}$$

is a minimal Heilbronn character of G with $\ker \theta|_P = P_1$. When case (iii) applies, θ is unfaithful as well on a Sylow r-subgroup of G.

In § 2 we establish some preliminary lemmas and fix notation for the character tables of $L_2(q)$. We prove Theorem 1.1 in § 3, and in § 4 we calculate the norms of the unfaithful minimal Heilbronn characters of Theorem 1.1, proving the following.

Theorem 1.2. *In the notation of Theorem 1.1, Tables 2–4 and Lemma 2.8 (which defines α and β), the norm of θ is as specified in Table 1, with*

$$\delta = \begin{cases} 1 & \text{if } \pi = \psi_i \text{ and } i\beta \equiv 0 \bmod \frac{1}{4}(2,q)(q+1), \\ 1 & \text{if } \pi = \chi_j \text{ and } j\alpha \equiv 0 \bmod \frac{1}{4}(q-1), \\ 0 & \text{otherwise}. \end{cases}$$

Table 1. *Norms of unfaithful minimal Heilbronn characters of $L_2(q)$*

π	$\|\theta\|^2$ case (i)	$\|\theta\|^2$ case (ii)	$\|\theta\|^2$ case (iii)
η_i	$\frac{1}{4}(q-3)(\|P_1\|-1)+1$	$\frac{1}{2}(q+1)$	$\frac{1}{4}(q-1)(\|P_1\|+1)+1$
ξ_i	$\frac{1}{4}(q+1)(\|P_1\|-1)+1$	$\frac{1}{2}(q+3)$	$\frac{1}{4}(q+3)(\|P_1\|+1)$
ψ_i	$\dfrac{1}{(2,q)}(q-3)(\|P_1\|-1)+1+\delta$	$2q-3$	$(q-1)(\|P_1\|-1)+2q-3$
σ	$\dfrac{1}{(2,q)}(q-1)(\|P_1\|-1)+1$	$2q+1$	$(q+1)(\|P_1\|-1)+2q+1$
χ_j	$\dfrac{1}{(2,q)}(q+1)(\|P_1\|-1)+1$	$2q+5$	$(q+3)(\|P_1\|-1)+2q+5+\delta$

2. Preliminary lemmas and character tables

The complex character tables of the groups $L_2(q)$ are given in Tables 2–4, which are derived from [**1**, Chapter 38].

The hypotheses of Theorem 1.1 reflect the translation of the necessary conditions for the existence of an unfaithful minimal Heilbronn character of an arbitrary finite group, given in the main theorem of [**3**] and reprised at the outset of this paper, to the special case $G \cong L_2(q)$. The next lemma clarifies the equivalence of these different phrasings.

Lemma 2.1. *Let $G \cong L_2(q)$, where $q = r^a \geqslant 5$ and r is a prime. Let P be a Sylow p-subgroup of G for p an odd prime dividing the order of G. The following are equivalent.*

(1) *Either $N_G(P)$ is the unique maximal subgroup of G containing $\Omega_1(P)$, or $\Omega_1(P)$ is contained in both $N_G(P)$ and a Borel subgroup $N_G(Q)$ for some equi-characteristic Sylow subgroup Q of G.*

(2) *$p > 5$ and one of the following holds:*

  (i) *p divides $q+1$ but p does not divide $r^b \pm 1$ for any proper divisor b of a;*

  (ii) *$p = q = r$; or*

  (iii) *p divides $q-1$ and $q = r$ is an odd prime.*

Proof. The subgroups of $L_2(q)$ are well known, and are documented in [**4**, Theorem 6.5.1] (attributed to Dickson, Burnside, Moore and Wiman). Since

$$|G| = \tfrac{1}{2}q(q+1)(q-1)(2,q),$$

p divides q, $q+1$ or $q-1$. If $p = 3$ or 5, then P is contained in a subgroup isomorphic to A_4 or A_5, respectively. The conditions in (i) ensure that p does not divide the order of any proper subgroup $\mathrm{PGL}_2(r^b)$ or $\mathrm{PSL}_2(r^b)$, and these subgroups do not occur in cases (ii) and (iii). Since $N_G(P)$ is maximal, the result follows. □

Table 2. *Characters of $L_2(q)$, $q \equiv 1 \bmod 4$*

class:	1	t	a^l	b^m	c	d
size:	1	$\frac{1}{2}q(q+1)$	$q(q+1)$	$q(q-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$
1_G	1	1	1	1	1	1
ξ_1	$\frac{1}{2}(q+1)$	$(-1)^{(q-1)/4}$	$(-1)^l$	0	$-\frac{1}{2}(1+\sqrt{-q})$	$-\frac{1}{2}(1-\sqrt{-q})$
ξ_2	$\frac{1}{2}(q+1)$	$(-1)^{(q-1)/4}$	$(-1)^l$	0	$-\frac{1}{2}(1-\sqrt{-q})$	$-\frac{1}{2}(1+\sqrt{-q})$
ψ_i	$q-1$	0	0	$-(\zeta^{im}+\zeta^{-im})$	-1	-1
σ	q	1	1	-1	0	0
χ_j	$q+1$	$2(-1)^j$	$\rho^{jl}+\rho^{-jl}$	0	1	1

$$|t|=2, \quad |a|=\tfrac{1}{2}(q-1), \quad |b|=\tfrac{1}{2}(q+1), \quad |c|=|d|=q,$$
$$1 \leqslant i, m \leqslant \tfrac{1}{4}(q-1), \quad 1 \leqslant j, l \leqslant \tfrac{1}{4}(q-5),$$
$$\rho^{(q-1)/2}=\zeta^{(q+1)/2}=1, \text{ primitive roots of unity in } \mathbb{C}.$$

Table 3. *Characters of $L_2(q)$, $q \equiv 3 \bmod 4$*

class:	1	t	a^l	b^m	c	d
size:	1	$\frac{1}{2}q(q-1)$	$q(q+1)$	$q(q-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$
1_G	1	1	1	1	1	1
η_1	$\frac{1}{2}(q-1)$	$(-1)^{(q+5)/4}$	0	$(-1)^{m+1}$	$-\frac{1}{2}(1-\sqrt{-q})$	$-\frac{1}{2}(1+\sqrt{-q})$
η_2	$\frac{1}{2}(q-1)$	$(-1)^{(q+5)/4}$	0	$(-1)^{m+1}$	$-\frac{1}{2}(1+\sqrt{-q})$	$-\frac{1}{2}(1-\sqrt{-q})$
ψ_i	$q-1$	$2(-1)^{i+1}$	0	$-(\zeta^{im}+\zeta^{-im})$	-1	-1
σ	q	-1	1	-1	0	0
χ_j	$q+1$	0	$\rho^{jl}+\rho^{-jl}$	0	1	1

$$|t|=2, \quad |a|=\tfrac{1}{2}(q-1), \quad |b|=\tfrac{1}{2}(q+1), \quad |c|=|d|=q,$$
$$1 \leqslant i, j, l, m \leqslant \tfrac{1}{4}(q-3),$$
$$\rho^{(q-1)/2}=\zeta^{(q+1)/2}=1, \text{ primitive roots of unity in } \mathbb{C}.$$

Lemma 2.2. *Suppose $N = P \rtimes H$, where P is an abelian p-group for some prime p. Then for any $t \in \mathbb{Z}$, the map $\varphi\colon N \to N$ defined by*

$$\varphi(n) = x^t h, \quad \text{where } n = xh,\ x \in P,\ h \in H,$$

is a homomorphism with $\ker \varphi = \{x \in P \mid x^t = 1\}$.

Proof. Let $n_1, n_2 \in N$ with $n_1 = x_1h_1$, $n_2 = x_2h_2$, $x_i \in P$ and $h_i \in H$ for each $i = 1, 2$. Since $n_1n_2 = x_1(h_1x_2h_1^{-1})h_1h_2$,

$$\varphi(n_1n_2) = (x_1(h_1x_2h_1^{-1}))^t h_1h_2 = x_1^t(h_1x_2^th_1^{-1})h_1h_2 = \varphi(n_1)\varphi(n_2).$$

Table 4. *Characters of $L_2(q)$, q even*

class:	1	a^l	b^m	c
size:	1	$q(q+1)$	$q(q-1)$	q^2-1
1_G	1	1	1	1
ψ_i	$q-1$	0	$-(\zeta^{im}+\zeta^{-im})$	-1
σ	q	1	-1	0
χ_j	$q+1$	$\rho^{jl}+\rho^{-jl}$	0	1

$$|a| = q-1, \quad |b| = q+1, \quad |c| = q,$$
$$1 \leqslant j, l \leqslant \tfrac{1}{2}(q-2), \quad 1 \leqslant i, m \leqslant \tfrac{1}{2}q,$$
$$\rho^{q-1} = \zeta^{q+1} = 1, \text{ primitive roots of unity in } \mathbb{C}.$$

Hence, φ is a homomorphism. It is clear from the definition that $\ker\varphi \leqslant P$, and the result follows. □

Lemma 2.3. *If P is a cyclic p-group and h is a p'-element normalizing but not centralizing P, then every element of P is a simple commutator of the form $[x, h]$ for some $x \in P$.*

Proof. The map $x \mapsto [x, h]$ is injective from P into the subset of simple commutators in P; hence, this subset must be all of P. □

Lemma 2.4. *Let ζ be a primitive nth root of unity, and fix a positive integer s. Let K be the greatest integer strictly less than $\frac{1}{2}n$. Then*

$$\sum_{k=1}^{K}(\zeta^{sk}+\zeta^{-sk}) = \begin{cases} -1 & \text{if } s \not\equiv 0 \bmod n \text{ and } n \text{ is odd},\\ -1+(-1)^{s+1} & \text{if } s \not\equiv 0 \bmod n \text{ and } n \text{ is even},\\ n-1 & \text{if } s \equiv 0 \bmod n \text{ and } n \text{ is odd},\\ n-2 & \text{if } s \equiv 0 \bmod n \text{ and } n \text{ is even}.\end{cases}$$

Proof. Since $\zeta^{sn} = 1$, $\zeta^{-sk} = \zeta^{s(n-k)}$. Thus, we may write

$$\sum_{k=1}^{K}\zeta^{-sk} = \sum_{k=1}^{K}\zeta^{s(n-k)} = \sum_{k=n-K}^{n-1}\zeta^{sk}.$$

Observing that $n-K$ is $K+1$ when n is odd and $K+2$ when n is even,

$$\sum_{k=1}^{K}(\zeta^{sk}+\zeta^{-sk}) = \sum_{k=1}^{n-1}\zeta^{sk} - \begin{cases} 0 & \text{if } n \text{ is odd},\\ \zeta^{s(K+1)} & \text{if } n \text{ is even}.\end{cases}$$

If $s \not\equiv 0 \bmod n$, then ζ^s is a non-trivial nth root of unity, so $\sum_{k=1}^{n-1}\zeta^{sk} = -1$. Since $K+1 = \frac{1}{2}n$ when n is even, $\zeta^{K+1} = -1$, proving the first two cases. If, instead, $s \equiv 0 \bmod n$, then $\sum_{k=1}^{n-1}\zeta^{sk} = n-1$ and $\zeta^{s(K+1)} = 1$, completing the proof. □

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Remark 2.5. By inspection of Tables 2–4, it is easily seen that the number of conjugacy classes $[a^l]$ (respectively, $[b^m]$) is the greatest integer strictly less than $\frac{1}{2}|a|$ (respectively, $\frac{1}{2}|b|$). We may therefore apply Lemma 2.4 to sums of the values of the characters χ_j and ψ_i on the classes $[a^l]$ and $[b^m]$, respectively, as these values involve $|a|$th and $|b|$th roots of unity.

Corollary 2.6. *Let $G \cong L_2(q)$, and assume the notation of the appropriate complex character table (Table 2, 3 or 4). Then, if π is any non-principal irreducible character of G,*

$$\sum_{l=1}^{A} \pi(a^l) \leqslant 2 \quad \text{and} \quad \sum_{m=1}^{B} \pi(b^m) \leqslant 2,$$

where A and B are the number of conjugacy classes $[a^l]$ and $[b^m]$, respectively.

Proof. The result follows from Lemma 2.4 and inspection of the character tables. □

Lemma 2.7. *Assume the notation of Theorem 1.1 and Tables 2–4. Let $\mathcal{N}$ denote the set of elements of G on which θ is not by definition equal to π. Then*

- *$\mathcal{N}$ is the set of classes $[b^m]$ in case (i),*
- *$\mathcal{N}$ is the set of classes $[c]$ and $[d]$ in case (ii), and*
- *$\mathcal{N}$ is the set of classes $[a^l]$, $[c]$ and $[d]$ in case (iii).*

Proof. Aside from the class $[t]$, these are the classes involved in N, along with the classes corresponding to the characteristic prime in case (iii). The class $[t]$ is not contained in $\mathcal{N}$ since any involution in N is G-conjugate to an involution in H, and φ is the identity on H. □

Lemma 2.8. *In the notation of Theorem 1.1 and Tables 2–4, there exist $\alpha, \beta \in \mathbb{N}$ such that $\varphi(b^m) = b^{\beta m}$ (in case (i)) and $\varphi(a^l) = a^{\alpha l}$ (in case (iii)).*

Proof. The dihedral group N has a characteristic cyclic subgroup C generated by b in case (i) and by a in case (iii). In either case φ maps C into C; hence, $\varphi(b) = b^\beta$ for some constant β, and similarly $\varphi(a) = a^\alpha$. The result follows. □

In fact, the constants α and β are easily computed. In case (i), observe that $\langle b \rangle = P \times C$ for some cyclic subgroup C of H, so we may write $b = xh$ for $x \in P$ and $h \in C$. Then

$$\varphi(b) = x^{|P_1|}h = b^\beta = x^\beta h^\beta$$

as in Lemma 2.8. Observing that $|P|$ and $|C|$ are coprime, we may choose

$$\beta \equiv |P_1| \bmod |P| \quad \text{and} \quad \beta \equiv 1 \bmod |C|$$

by the Chinese Remainder Theorem. The computation of α in case (iii) is identical.

Lemma 2.9. *In the notation of Theorem 1.1, P_1 contains representatives of precisely $\frac{1}{2}(|P_1| - 1)$ distinct non-trivial G-conjugacy classes.*

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Proof. By a basic result due to Burnside, $x, y \in P_1$ are G-conjugate precisely when they are conjugate in N. Since N is dihedral and P_1 is contained in the characteristic cyclic subgroup of N, this occurs if and only if $x = y$ or an element of order 2 in N conjugates x into y, if and only if $x = y$ or $x = y^{-1}$. Hence, there are exactly two elements of P_1 in each non-trivial conjugacy class, yielding $\frac{1}{2}(|P_1| - 1)$ such classes. □

3. Proof of Theorem 1.1

Let $\Pi \colon G \to \mathrm{GL}(V)$ be an irreducible representation of G affording the character π. Then, by Theorem 2.2, $\Theta = \Pi|_N \circ \varphi$ is a representation of N into $\mathrm{GL}(V)$ with $\ker \Theta = P_1$. The map θ defined in (1.1) extends the character of N afforded by Θ to a class function on G. We proceed to show that θ is a minimal Heilbronn character of G with $\ker \theta|_P = P_1$.

Let M be a maximal subgroup of G. We argue that $\theta|_M$ is a character of M. If M is conjugate to N, then $\theta|_M$ is precisely the character afforded by the representation Θ. Suppose then that M is not conjugate to N, and suppose further that N is the unique maximal subgroup of G containing $\Omega_1(P)$. In this case we argue that $\theta|_M = \pi|_M$. Let $m \in M$. If m is not conjugate to any element of N, then $\theta(m) = \pi(m)$. Otherwise, replacing M by a conjugate (which is permissible since θ is a class function on G), we may assume $m \in N$. Write $m = xh$ for $x \in P$, $h \in H$.

Suppose that x and h commute. Since φ is the identity on H,

$$\theta(h) = \pi(h) \quad \text{for any } h \in H. \tag{3.1}$$

Hence, if $x = 1$, then $m \in H$ and $\theta(m) = \pi(m)$. Otherwise p divides the order of m, so $\Omega_1(P) \leqslant M$: a contradiction.

Thus, if $x \neq 1$, then $[x, h] \neq 1$. Then $[x, h^{-1}] \neq 1$ as well, so $x = [y, h^{-1}]$ for some $y \in P$ by Lemma 2.3. Thus,

$$x = [y, h^{-1}] = y^{-1}hyh^{-1},$$

which implies $xh = y^{-1}hy$, i.e. xh is conjugate to h in N. Since θ and π are class functions on N, it follows that $\theta(xh) = \theta(h)$, which is $\pi(h) = \pi(xh)$ by (3.1). Hence, $\theta|_M = \pi|_M$ is a character of M.

It remains to consider the case where M is a Borel subgroup of G containing $\Omega_1(P)$ and M is not conjugate to N (so case (iii) of the theorem applies). Here M is a Frobenius group $Q \rtimes C$ for some equi-characteristic Sylow subgroup Q of G, and, replacing M by a conjugate if necessary, $C = M \cap N$ is cyclic of order $\frac{1}{2}(q - 1)$ (see [**4**, Theorem 6.5.1]). By hypothesis, $q = r$ is prime, so $|m| = q$ for all non-trivial $m \in Q$, and in particular θ is constant on Q. Since we have established that $\theta|_N$ is a character of N, $\theta|_C$ is a character of $C \leqslant N$. Thus, $\theta|_M$ is a character of the Frobenius complement with the Frobenius kernel in its kernel, and is therefore a character of M.

We have shown that $\theta|_M$ is a character for every maximal subgroup M of G. It follows that θ restricts to a character of every elementary subgroup of G and, since θ is a G-class function, θ is a virtual character of G by Brauer's characterization of characters.

Moreover, θ is not a character of G since otherwise $1 \neq P_1 \leqslant \ker\theta \lhd G$: a contradiction ($G$ is simple).

It remains to show that $\langle\theta,\mu\rangle \geqslant 0$ for every monomial character μ of G (i.e. θ is a Heilbronn character). Let μ be a monomial character of G. If μ is induced from a linear character of a proper subgroup H of G, then since $\theta|_H$ is a character, $\langle\theta,\mu\rangle \geqslant 0$ by Frobenius reciprocity. Otherwise, μ is the principal character 1_G of G. We complete the proof by showing that $\langle\theta,1_G\rangle \geqslant 0$.

Let $\mathcal{N}$ be the set of elements of G on which θ is not by definition equal to π (see Lemma 2.7). Then

$$\begin{aligned}\langle\theta,1_G\rangle &= \frac{1}{|G|}\biggl(\sum_{g\in\mathcal{N}}\theta(g) + \sum_{g\notin\mathcal{N}}\pi(g)\biggr)\\ &= \langle\pi,1_G\rangle + \frac{1}{|G|}\biggl(\sum_{g\in\mathcal{N}}\theta(g) - \sum_{g\in\mathcal{N}}\pi(g)\biggr)\\ &= \frac{1}{|G|}\sum_{g\in\mathcal{N}}(\theta(g)-\pi(g)).\end{aligned}$$

Since $\langle\theta,1_G\rangle$ is integral, $\langle\theta,1_G\rangle \geqslant 0$ if and only if $\langle\theta,1_G\rangle > -1$; hence, it suffices to prove that

$$\sum_{g\in\mathcal{N}}(\pi(g)-\theta(g)) < |G|. \tag{3.2}$$

Referring to the complex character tables for $L_2(q)$ (Tables 2–4), we proceed by cases.

Case 1 (p divides $q+1$). Here $\mathcal{N}$ is the set of conjugacy classes $[b^m]$. Let B denote the number of such classes. Since each has size $q(q-1)$,

$$\begin{aligned}\sum_{g\in\mathcal{N}}(\pi(g)-\theta(g)) &= q(q-1)\biggl(\sum_{m=1}^{B}\pi(b^m) - \sum_{m=1}^{B}\theta(b^m)\biggr)\\ &\leqslant q(q-1)\biggl(2-\sum_{m=1}^{B}\theta(b^m)\biggr),\end{aligned}$$

by Corollary 2.6. Since $|\theta(b^m)| \leqslant 2$ for all values of m such that $b^m \notin P_1$, and $b^m \in P_1$ for at least one value of m,

$$\sum_{g\in\mathcal{N}}(\pi(g)-\theta(g)) \leqslant q(q-1)(2-\pi(1)+2B).$$

Observing that $2-\pi(1) < 0$, we obtain

$$\sum_{g\in\mathcal{N}}(\pi(g)-\theta(g)) < q(q-1)(2B)$$

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and, since $B \leqslant \frac{1}{2}|G|(q-1)(q+1)$,

$$\sum_{g\in\mathcal{N}}(\pi(g)-\theta(g)) < \frac{q}{q+1}|G|.$$

This establishes (3.2) in case 1.

Case 2 ($p = q = r$ is an odd prime). Here $\mathcal{N}$ consists only of the classes $[c]$ and $[d]$, both of size q^2-1. These classes are in the kernel of θ (since $P_1 = P$); hence,

$$\sum_{g\in\mathcal{N}}(\pi(g)-\theta(g)) = (q^2-1)(\pi(c)-\pi(1)) + (q^2-1)(\pi(d)-\pi(1)).$$

Since $\pi(1) > \pi(c)$ and $\pi(1) > \pi(d)$ for all non-principal $\pi \in \mathrm{Irr}(G)$, the sum above is negative and (3.2) follows.

Case 3 (p divides $q-1$ and $q = r$ is an odd prime). Here $\mathcal{N}$ is comprised of the conjugacy classes $[a^l]$ as well as the classes $[c]$ and $[d]$. Reasoning as in case 1, the sum over the classes $[a^l]$ is less than $|G|$, and we have seen that the sum over the classes $[c]$ and $[d]$ is negative. Hence, (3.2) holds in this case as well, completing the proof.

4. Proof of Theorem 1.2

Let $\mathcal{N}$ denote the set of elements of G on which θ is not by definition equal to π (see Lemma 2.7). Then

$$\begin{aligned}\|\theta\|^2 &= \frac{1}{|G|}\biggl(\sum_{g\in\mathcal{N}}|\theta(g)|^2 + \sum_{g\notin\mathcal{N}}|\pi(g)|^2\biggr)\\ &= \|\pi\|^2 + \frac{1}{|G|}\sum_{g\in\mathcal{N}}(|\theta(g)|^2-|\pi(g)|^2)\\ &= 1 + \frac{1}{|G|}\sum_{g\in\mathcal{N}}(|\theta(g)|^2-|\pi(g)|^2). \qquad (4.1)\end{aligned}$$

We proceed by cases.

Case 1 (p divides $q+1$). Here $\mathcal{N}$ is the set of classes $[b^m]$, each of size $q(q-1)$. By Lemma 2.9, $\theta(b^m) = \pi(1)$ for $\frac{1}{2}(|P_1|-1)$ such classes. Writing b as the product of commuting p- and p'-elements x and y, respectively, $\varphi(b^m)$ is an element of order 2 precisely when $x \in P_1$ and $|y| = 2$. If $|b|$ is odd, or, equivalently, if $q \equiv 0, 1 \bmod 4$, then there is no such element. Otherwise, $q \equiv 3 \bmod 4$ and there are $\frac{1}{2}(|P_1|-1)$ classes with $\theta(b^m) = \pi(t)$.

Suppose first that π is not one of the characters ψ_i, so π is constant in absolute value on the classes $[b^m]$. In particular, if $\varphi(b^m) \neq 1$ and $\varphi(b^m) \notin [t]$, then $|\theta(b^m)| = |\pi(b^m)|$. Thus, if $q \equiv 0, 1 \bmod 4$,

$$\|\theta\|^2 = 1 + \frac{q(q-1)}{|G|}\biggl(\frac{|P_1|-1}{2}\biggr)(|\pi(1)|^2 - |\pi(b^m)|^2).$$

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If $q \equiv 3 \bmod 4$, then

$$\|\theta\|^2 = 1 + \frac{q(q-1)}{|G|}\left(\frac{|P_1|-1}{2}\right)(|\pi(1)|^2 + |\pi(t)|^2 - 2|\pi(b^m)|^2).$$

Observing that

$$\frac{q(q-1)}{|G|}\left(\frac{|P_1|-1}{2}\right) = \frac{1}{(q,2)}\left(\frac{|P_1|-1}{q+1}\right),$$

the values for $\pi \neq \psi_i$ in Table 1 now follow by direct substitution.

Suppose next that π is one of the characters ψ_i, and choose β according to Lemma 2.8. Summing over conjugacy classes, (4.1) becomes

$$\|\theta\|^2 = 1 + \frac{q(q-1)}{|G|}\left(\sum_{m=1}^{B}|\psi_i(b^{\beta m})|^2 - \sum_{m=1}^{B}|\psi_i(b^m)|^2\right), \tag{4.2}$$

where B is the number of classes $[b^m]$. Since $|\psi_i(b^m)|^2 = 2 + \zeta^{2im} + \zeta^{-2im}$,

$$\begin{aligned}\sum_{m=1}^{B}|\psi_i(b^m)|^2 &= 2B + \sum_{m=1}^{B}(\zeta^{2im} + \zeta^{-2im}) \\ &= 2B + \begin{cases}-1 & \text{if } q \equiv 0,1 \bmod 4, \\ -2 & \text{if } q \equiv 3 \bmod 4,\end{cases}\end{aligned} \tag{4.3}$$

by Lemma 2.4 (since the bounds on i ensure $2i \not\equiv 0 \bmod B$, and $|b|$ is even precisely when $q \equiv 3 \bmod 4$).

We consider the remaining sum in (4.2):

$$\sum_{m=1}^{B}|\psi_i(b^{\beta m})|^2 = \sum_{b^{\beta m}=1}|\psi_i(b^{\beta m})|^2 + \sum_{b^{\beta m}\in[t]}|\psi_i(b^{\beta m})|^2 + \sum_{|b^{\beta m}|>2}|\psi_i(b^{\beta m})|^2.$$

Here we have divided the classes $[b^m]$ into those in the kernel of φ, those that φ maps to an involution and those that are mapped by φ to other classes $[b^m]$. Since $\psi_i(1) = q-1$,

$$\sum_{b^{\beta m}=1}|\psi_i(b^{\beta m})|^2 = \left(\frac{|P_1|-1}{2}\right)(q-1)^2. \tag{4.4}$$

Similarly,

$$\sum_{b^{\beta m}\in[t]}|\psi_i(b^{\beta m})|^2 = \begin{cases}0 & \text{if } q \equiv 0,1 \bmod 4, \\ 2(|P_1|-1) & \text{if } q \equiv 3 \bmod 4,\end{cases} \tag{4.5}$$

since these classes only arise when $q \equiv 3 \bmod 4$, in which case $|\psi_i(t)| = 2$.

The sum over the remaining conjugacy classes of $|\psi_i(b^{\beta m})|^2$ can be calculated by summing $2 + \zeta^{2i\beta m} + \zeta^{-2i\beta m}$ over *all* of the classes $[b^m]$ and then subtracting that

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expression summed over the classes already considered. Thus,

$$\sum_{|b^{\beta m}|>2} |\psi_i(b^{\beta m})|^2 = \sum_{m=1}^{B} (2+\zeta^{2i\beta m}+\zeta^{-2i\beta m}) - \bigg(\sum_{b^{\beta m}=1} (2+\zeta^{2i\beta m}+\zeta^{-2i\beta m}) + \sum_{b^{\beta m}\in[t]} (2+\zeta^{2i\beta m}+\zeta^{-2i\beta m}) \bigg).$$

Observe that $\zeta^{2\beta m} = 1$ whenever $b^{\beta m} = 1$ or $b^{\beta m} \in [t]$. Hence, by Lemma 2.4 and previous arguments,

$$\sum_{|b^{\beta m}|>2} |\psi_i(b^{\beta m})|^2 = 2B + \begin{cases} -1 & \text{if } 2i\beta \not\equiv 0 \bmod |b| \text{ and } q \equiv 0,1 \bmod 4 \\ -2 & \text{if } 2i\beta \not\equiv 0 \bmod |b| \text{ and } q \equiv 3 \bmod 4 \\ 2B & \text{if } 2i\beta \equiv 0 \bmod |b| \end{cases} - 2(|P_1|-1) - \begin{cases} 0 & \text{if } q \equiv 0,1, \bmod 4, \\ 2(|P_1|-1) & \text{if } q \equiv 3 \bmod 4. \end{cases} \tag{4.6}$$

Substituting from (4.3)–(4.6) into (4.2) (and after the obvious cancellations),

$$\|\theta\|^2 = 1 + \frac{q(q-1)}{|G|} \times \left[\left(\frac{|P_1|-1}{2}\right)(q-1)^2 + \begin{cases} -1 & \text{if } 2i\beta \not\equiv 0 \bmod |b| \text{ and } q \equiv 0,1 \bmod 4 \\ -2 & \text{if } 2i\beta \not\equiv 0 \bmod |b| \text{ and } q \equiv 3 \bmod 4 \\ 2B & \text{if } 2i\beta \equiv 0 \bmod |b| \end{cases} - 2(|P_1|-1) + \begin{cases} 1 & \text{if } q \equiv 0,1 \bmod 4 \\ 2 & \text{if } q \equiv 3 \bmod 4 \end{cases} \right]. \tag{4.7}$$

If $2i\beta \not\equiv 0 \bmod |b|$, then the sum of the two piecewise-defined components in (4.7) is zero. Otherwise, observing that

$$\frac{(q,2)}{2}(q+1) = \begin{cases} 2B+1 & \text{if } q \equiv 0,1 \bmod 4, \\ 2B+2 & \text{if } q \equiv 3 \bmod 4, \end{cases}$$

and

$$\frac{q(q-1)}{|G|} = \frac{1}{(2,q)}\frac{2}{q+1},$$

the right-hand side of (4.7) becomes

$$1 + \frac{1}{(2,q)}\frac{2}{q+1}\left[\left(\frac{|P_1|-1}{2}\right)((q-1)^2-4) + \begin{cases} 0 & \text{if } 2i\beta \not\equiv 0 \bmod |b| \\ \dfrac{(q,2)}{2}(q+1) & \text{if } 2i\beta \equiv 0 \bmod |b| \end{cases} \right].$$

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Simplifying, we obtain

$$\|\theta\|^2 = 1 + \frac{1}{(2,q)}(|P_1|-1)(q-3) + \begin{cases} 0 & \text{if } 2i\beta \not\equiv 0 \bmod |b|, \\ 1 & \text{if } 2i\beta \equiv 0 \bmod |b|, \end{cases}$$

which is the value given in Table 1.

Case 2 ($p = q = r$ is an odd prime). Here q is odd, $\mathcal{N}$ consists only of the classes $[c]$ and $[d]$ (of sizes $\frac{1}{2}(q^2-1)$), and $P_1 = P$. From (4.1),

$$\begin{aligned}\|\theta\|^2 &= 1 + \frac{q^2-1}{2|G|}(2|\pi(1)|^2 - |\pi(c)|^2 - |\pi(d)|^2) \\ &= 1 + \frac{1}{q}(2|\pi(1)|^2 - |\pi(c)|^2 - |\pi(d)|^2),\end{aligned}$$

and the values in Table 1 follow.

Case 3 (p divides $q-1$ and $q = r$ is an odd prime). Here $\mathcal{N}$ is the set of classes $[a^l]$, each of size $q(q+1)$, along with the classes $[c]$ and $[d]$, and φ maps $\frac{1}{2}(|P_1|-1)$ classes $[a^l]$ into the class of involutions precisely when $q \equiv 1 \bmod 4$. Recognizing that the norm of θ in case (ii) is a component of the calculation in this case, let $\|\theta_{(\mathrm{ii})}\|^2$ denote the case (ii) norm of θ.

If π is not one of the characters χ_j, then π is constant in absolute value on the classes $[a^l]$. Thus, from (4.1), if $q \equiv 1 \bmod 4$, then

$$\|\theta\|^2 = \frac{q(q+1)}{|G|}\left(\frac{|P_1|-1}{2}\right)(|\pi(1)|^2 + |\pi(t)|^2 - 2|\pi(a^l)|^2) + \|\theta_{(\mathrm{ii})}\|^2.$$

If $q \equiv 3 \bmod 4$, then

$$\|\theta\|^2 = \frac{q(q+1)}{|G|}\left(\frac{|P_1|-1}{2}\right)(|\pi(1)|^2 - |\pi(a^l)|^2) + \|\theta_{(\mathrm{ii})}\|^2.$$

Observing that

$$\frac{q(q+1)}{|G|}\left(\frac{|P_1|-1}{2}\right) = \frac{|P_1|-1}{q-1},$$

the values for $\pi \neq \chi_j$ in Table 1 now follow by direct substitution.

If instead π is one of the characters χ_j, then, proceeding as for ψ_i in case 1,

$$\|\theta\|^2 = \frac{q(q+1)}{|G|}\left(\sum_{l=1}^{A}|\chi_j(a^{\alpha l})|^2 - \sum_{l=1}^{A}|\chi_j(a^l)|^2\right) + \|\theta_{(\mathrm{ii})}\|^2,$$

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where A is the number of classes $[a^l]$. After an analogous argument we obtain

$$\|\theta\|^2 = \frac{q(q+1)}{|G|}\left[\left(\frac{|P_1|-1}{2}\right)(q+1)^2 + \begin{cases} -1 & \text{if } 2j\alpha \not\equiv 0 \bmod |a| \text{ and } q \equiv 3 \bmod 4 \\ -2 & \text{if } 2j\alpha \not\equiv 0 \bmod |a| \text{ and } q \equiv 1 \bmod 4 \\ 2A & \text{if } 2j\alpha \equiv 0 \bmod |a| \end{cases} \right.$$
$$\left. - 2(|P_1|-1) + \begin{cases} 1 & \text{if } q \equiv 3 \bmod 4 \\ 2 & \text{if } q \equiv 1 \bmod 4 \end{cases}\right] + \|\theta_{(\mathrm{ii})}\|^2. \tag{4.8}$$

Observing that

$$\frac{q-1}{2} = \begin{cases} 2A+1 & \text{if } q \equiv 3 \bmod 4, \\ 2A+2 & \text{if } q \equiv 1 \bmod 4, \end{cases}$$

(4.8) simplifies as before to the value given in Table 1. This completes the proof.

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