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UNIQUENESS OF COMPATIBLE QUASI-UNIFORMITIES

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ABSTRACT. It is shown that a topological space has a unique compatible quasi-uniformity if its topology is finite. Examples are given to show the converse is false for T_1 and for normal second countable spaces. Two sufficient conditions are given for a topological space to have a compatible quasi-uniformity strictly finer than the associated Császár-Pervin quasi-uniformity. These conditions are used to show that a Hausdorff, semi-regular or first countable T_1 space has a unique compatible quasi-uniformity if and only if its topology is finite. Császár and Pervin described, in quite different ways, quasi-uniformities which induce a given topology. It is shown that, for a given topological space, Császár and Pervin described the same quasi-uniformity.

1. **Introduction.** Fletcher [2] showed that every finite topological space has a unique compatible quasi-uniformity (which he called a quasi-uniform structure) and conjectured that a topological space (X, τ) has a unique compatible quasi-uniformity if and only if the topology τ is finite. In §2 of this paper it is shown that this conjecture is false, but that a finite topology does imply a unique compatible quasi-uniformity.

A sufficient condition for (X, τ) to have a compatible quasi-uniformity strictly finer than the Császár-Pervin quasi-uniformity (see §5) is found in §3. This is used to show that Fletcher's conjecture holds for either Hausdorff or semiregular spaces. In §4, another such sufficient condition is found and is used to show that Fletcher's conjecture is true for first countable T_1 spaces.

One rather obvious approach to the question of when a topological space has a unique compatible quasi-uniformity is to ask when the quasi-uniformities defined by Császár [1] and Pervin [7] differ. In §5, it is shown that the Császár and Pervin quasi-uniformities are always the same. Thus every topological space has a "canonical" compatible quasi-uniformity, which, in this paper, will be called the Császár-Pervin quasi-uniformity. Császár's description of this quasi-uniformity is used to show that a completely regular space (X, τ) has a compatible uniformity strictly coarser than the Császár-Pervin quasi-uniformity if and only if τ is infinite. Finally, another sufficient condition for (X, τ) to have a compatible quasi-uniformity strictly coarser than the Császár-Pervin quasi-uniformity is found in §6.

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Notation in this paper will, in general, be that used in Murdeshwar and Naimpally [5]. (X, τ) will denote a topological space, $P(\tau)$ will denote the associated Császár-Pervin quasi-uniformity, and $\tau\langle x\rangle$ will denote the set $\{G \in \tau \mid x \in G\}$, for $x \in X$. If $\mathfrak{U}$ is a quasi-uniformity on a set X, the topology on X induced by $\mathfrak{U}$ will be denoted by $\tau(\mathfrak{U})$. Hausdorff is not included in the terms regular or completely regular. The symbol "$\subset$" means "is a proper subset of". $\mathbf{N}$ denotes the set of positive integers and $\mathbf{R}$ denotes the set of real numbers with its usual topology. Finally, note that the set $\{S(G) \mid G \in \tau\}$ forms a subbasis for $P(\tau)$ (see [7]), where

$$S(A) = (A \times A) \cup ((X \backslash A) \times X)$$

for every $A \subseteq X$.

2. Some topological spaces with unique compatible quasi-uniformities.

THEOREM 2.1. *If τ is finite, then (X, τ) has a unique compatible quasi-uniformity.*

Proof. $P(\tau)$ is compatible and has a finite subbasis, hence is the maximum compatible quasi-uniformity, by [2, Theorem 3.3]. Now let $\mathfrak{U}$ be a quasi-uniformity on X such that $\tau(\mathfrak{U})=\tau$, and let $G \in \tau$. Let $G(x)=\bigcap \tau\langle x\rangle$, for each $x \in G$. Then $G(x) \in \tau$ and there exist $x_i \in G$, $1 \leq i \leq n$, such that $\{G(x) \mid x \in G\}=\{G(x_i) \mid 1 \leq i \leq n\}$. There exist V, $U_i \in \mathfrak{U}$ such that $U_i[x_i] \subseteq G(x_i)$, $1 \leq i \leq n$, and $V \circ V \subseteq \bigcap \{U_i \mid 1 \leq i \leq n\}$. Then $V[x_i]=G(x_i)$, so that, for $y \in G$,

$$V[y] \subseteq V \circ V[x_j] \subseteq U_j[x_j] \subseteq G(x_j) \subseteq G,$$

for some j, $1 \leq j \leq n$. Thus $V \subseteq S(G)$, so $S(G) \in \mathfrak{U}$. Hence $\mathfrak{U}=P(\tau)$.

The following examples show that an infinite topology does not imply multiple compatible quasi-uniformities, even for T_1 or normal second countable spaces.

EXAMPLE 2.2. Let X be an uncountable set and let τ denote the cofinite topology on X, i.e., $\phi \neq B \subseteq X$ implies $B \in \tau$ if and only if $X \backslash B$ is finite. Then (X, τ) is T_1, τ is infinite, and (X, τ) has a unique compatible quasi-uniformity.

Proof. Clearly, (X, τ) is T_1 and τ is infinite. Now let $\mathfrak{U}$ be a compatible quasi-uniformity. Note that $W[x] \in \tau$, for every $W \in \mathfrak{U}$ and $x \in X$. Let U, $V \in \mathfrak{U}$ such that $V \circ V \circ V \subseteq U$, and let A be a countably infinite subset of X. Now

$$\bigcap \{V[a] \mid a \in A\} \neq \phi,$$

so let $y \in \bigcap \{V[a] \mid a \in A\}$. Then $A \cap V[v] \neq \phi$, for all $v \in V[y]$, hence

$$V[y] \subseteq V \circ V \circ V[v] \subseteq U[v].$$

Thus $V[y] \times V[y] \subseteq U$. Now let $Q=X \times X$ if $V[y]=X$ and let

$$Q = \bigcap \{S(U[x]) \mid x \in X \backslash V[y]\}$$

if $V[y] \neq X$. Then $Q \cap S(V[y]) \subseteq U$ and $Q \cap S(V[y]) \in P(\tau)$, hence $U \in P(\tau)$. Thus $\mathfrak{U} \subseteq P(\tau)$.

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Now let $G \in \tau$, with $G \neq \phi$ (note that $S(\phi)=X\times X \in \mathfrak{U}$). For each $x \in G$, there exist U_x, $V_x \in \mathfrak{U}$ such that $U_x[x] \subseteq G$ and $V_x \circ V_x \subseteq U_x$. Let $a \in G$ and set $V=V_a$ if $V_a[a]=G$ and

$$V = V_a \cap (\bigcap \{U_x \mid x \in G\backslash V_a[a]\})$$

if $G \neq V_a[a]$. Then $V[y] \subseteq G$ for every $y \in G$, so $V \subseteq S(G)$. Hence $S(G) \in \mathfrak{U}$, since $V \in \mathfrak{U}$. Thus $P(\tau) \subseteq \mathfrak{U}$, so $\mathfrak{U}=P(\tau)$.

EXAMPLE 2.3. Let X denote the set of all ordinal numbers less than or equal to the first infinite ordinal ω_0, and let

$$\tau = \{\phi\} \cup \{[\alpha, \omega_0] \mid \alpha \in X, \alpha < \omega_0\},$$

where $[\alpha, \omega_0]=\{\beta \in X \mid \alpha \leq \beta \leq \omega_0\}$. Then (X, τ) is normal second countable, τ is infinite, and (X, τ) has a unique compatible quasi-uniformity.

Proof. Clearly τ is infinite and (X, τ) is second countable, and it is easy to verify that (X, τ) is normal. Now let $\mathfrak{U}$ be a compatible quasi-uniformity. Note that $[\gamma, \omega_0] \subseteq W[\gamma]$ for every $\gamma \in X$ and $W \in \mathfrak{U}$. Let U, $V \in \mathfrak{U}$ such that $V \circ V \subseteq U$. Then there exists $\alpha \in X$ such that $\alpha < \omega_0$ and $[\alpha, \omega_0] \subseteq V[\omega_0]$. Then

$$[\alpha, \omega_0] \subseteq V \circ V[\gamma] \subseteq U[\gamma]$$

for every $\gamma \in [\alpha, \omega_0]$, hence

$$S([\alpha, \omega_0]) \cap (\bigcap \{S(U[\beta]) \mid 0 \leq \beta \leq \alpha\}) \subseteq U.$$

Thus $U \in P(\tau)$, and so $\mathfrak{U} \subseteq P(\tau)$.

Now let $G \in \tau$ with $G \neq \phi (S(\phi)=X\times X \in \mathfrak{U})$, say $G=[\alpha, \omega_0]$. There exist U, $V \in \mathfrak{U}$ such that $U[\alpha] \subseteq G$ and $V \circ V \subseteq U$. Then $V[\alpha]=G$, so

$$V[\gamma] \subseteq V \circ V[\alpha] \subseteq U[\alpha] \subseteq G$$

for every $\gamma \in G$. Thus $V \subseteq S(G)$, hence $S(G) \in \mathfrak{U}$. This proves $P(\tau) \subseteq \mathfrak{U}$, hence $\mathfrak{U}=P(\tau)$.

3. **A sufficient condition for multiple compatible quasi-uniformities.** Just as in the case of uniformities, the set of quasi-uniformities on a set X forms a complete lattice when partially ordered by set inclusion (see [5, p. 17] and use the fact that $\{X\times X\}$ is a lower bound to get an infimum). The proof of the following proposition is essentially a duplication of the proof of the analogous statement for uniformities (see [8, p. 182, Theorem 20.21]), hence is omitted.

PROPOSITION 3.1. *Let $\mathfrak{U}(\alpha)$ be a quasi-uniformity on a set X, for each $\alpha \in A$, and let $\tau(\alpha)=\tau(\mathfrak{U}(\alpha))$. Then*

$$\bigvee \{\tau(\alpha) \mid \alpha \in A\} = \tau(\bigvee \{\mathfrak{U}(\alpha) \mid \alpha \in A\}).$$

If $\mathfrak{U}$ is a quasi-uniformity on X and $\tau(\mathfrak{U})=\tau$, then $\tau(P(\tau) \vee \mathfrak{U})=\tau$, by the preceding proposition. Thus (X, τ) fails to have a unique compatible quasi-uniformity precisely when it has one which is either strictly coarser or strictly finer than $P(\tau)$.

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THEOREM 3.2. *If τ contains an infinite increasing sequence, then (X, τ) has a compatible quasi-uniformity strictly finer than $P(\tau)$.*

Proof. Let $\{G_n \mid n \in \mathbf{N}\} \subseteq \tau$ such that $G_n \subset G_{n+1}$, and let $V = \bigcap \{S(G_n) \mid n \in \mathbf{N}\}$. Then $\{V\}$ forms a basis for a quasi-uniformity $\mathfrak{V}$ on X (see [5, Theorem 1.10]) and $\tau(\mathfrak{V}) \subseteq \tau$. Then $\tau(P(\tau) \vee \mathfrak{V}) = \tau$, by Proposition 3.1.

Now suppose $V \in P(\tau)$. Then there exists nonempty finite $\mathfrak{B} \subseteq \tau$ such that $U \subseteq V$, where $U = \bigcap \{S(B) \mid B \in \mathfrak{B}\}$. Let $x_1 \in G_1$, $x_{n+1} \in G_{n+1} \backslash G_n$, and $\mathfrak{B}_n = \mathfrak{B} \cap \tau\langle x_n \rangle$, for $n \in \mathbf{N}$. Then there exist $n, m \in \mathbf{N}$ such that $n < m$ and $\mathfrak{B}_n = \mathfrak{B}_m$. But then

$$U[x_n] = \bigcap \mathfrak{B}_n = \bigcap \mathfrak{B}_m = U[x_m]$$

so that $x_m \in U[x_n] \subseteq V[x_n] \subseteq G_n$, a contradiction. Thus $V \notin P(\tau)$, so $P(\tau) \vee \mathfrak{V}$ is strictly finer than $P(\tau)$.

(X, τ) is an R_1 *space* if, for any $x, y \in X$, $\{\bar{x}\} \neq \{\bar{y}\}$ implies x and y have disjoint neighborhoods. The proof of the following proposition is similar to the analogous statement for infinite Hausdorff spaces (see [3, p. 5]), and is omitted.

PROPOSITION 3.3. *Let (X, τ) be an R_1 space with τ infinite. Then (X, τ) contains an infinite discrete subspace.*

THEOREM 3.4. *An R_1 space (X, τ) has a unique compatible quasi-uniformity if and only if τ is finite.*

Proof. Assume τ is infinite. Then, by Proposition 3.3, there exist $x_n \in X$ and $G_n \in \tau$, $n \in \mathbf{N}$, such that $G_m \cap \{x_n \mid n \in \mathbf{N}\} = \{x_m\}$ and $\{x_n \mid n \in \mathbf{N}\}$ is infinite. Then $(\bigcup \{G_i \mid 1 \leq i \leq n\};\ n \in \mathbf{N})$ is an infinite increasing sequence in τ, so (X, τ) has a compatible quasi-uniformity strictly finer than $P(\tau)$, by Theorem 3.2, proving half the theorem. The other half follows from Theorem 2.1.

Since either of regular or Hausdorff implies R_1, the following is immediate from Theorem 3.4.

COROLLARY 3.5. *A Hausdorff (regular) space (X, τ) has a unique compatible quasi-uniformity if and only if $X(\tau)$ is finite.*

(X, τ) is *semiregular* if τ contains a basis of *regular-open* sets, i.e., of sets G such that $G = \operatorname{int}(\bar{G})$. For any topological space (X, τ), the set of regular-open sets forms a basis for a semiregular topology on X. This topology is denoted by τ_s and is called the *semiregularization* of τ. These concepts lead to the results in the remainder of this section.

PROPOSITION 3.6. *If the set $\mathfrak{B}$ of regular-open sets in (X, τ) is infinite, then $\mathfrak{B}$ contains an infinite increasing sequence.*

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Proof. Assume that $\mathfrak{V}$ is infinite, and let $\mathfrak{V}$ be partially ordered by set inclusion. Suppose that $\mathfrak{V}$ contains no infinite chain. Then $\mathfrak{V}$ contains infinitely many maximal chains, each of which is finite. Now let $\mathfrak{C}$ and $\mathfrak{D}$ be maximal chains in $\mathfrak{V}$, let C be the smallest element of $\mathfrak{C}\backslash\{\phi\}$ and let D be the smallest element of $\mathfrak{D}\backslash\{\phi\}$. Note that C and D exist since $X\in\mathfrak{C}$ and $X\in\mathfrak{D}$. Then $\phi\subseteq C\cap D\subseteq C$ and $\phi\subseteq C\cap D\subseteq D$. Therefore, by the definition of C and D, $C\cap D=\phi$, since $C\cap D\in\mathfrak{V}$. Thus $\mathfrak{V}$ contains an infinite subcollection of mutually disjoint sets. Let $\{B_n \mid n\in\mathbf{N}\}\subseteq\mathfrak{V}\backslash\{\phi\}$ such that $B_n\cap B_m=\phi$ for $n\neq m$. For each $m\in\mathbf{N}$, let $A_m=\bigcup\{B_n \mid n\in\mathbf{N}, n\leq m\}$ and let $G_m=\operatorname{int}(\bar{A}_m)$. Then $G_m\in\mathfrak{V}$ and $G_m\subseteq G_{m+1}$, for $m\in\mathbf{N}$. Now let n, $m\in\mathbf{N}$ with $n<m$. Then $B_m\subseteq G_m$, but $B_m\cap\bar{A}_n=\phi$, so $G_m\neq G_n$. Thus $\mathfrak{V}$ contains an infinite chain, in fact, an infinite increasing sequence, a contradiction of the supposition that $\mathfrak{V}$ contains no infinite chain. Therefore, $\mathfrak{V}$ contains an infinite chain, say $\mathfrak{H}$.

Now for each $H\in\mathfrak{H}$, either the collection of elements of $\mathfrak{H}$ which are subsets of H is infinite or the collection of elements of $\mathfrak{H}$ of which H is a subset is infinite. If the family of all elements $H\in\mathfrak{H}$ with the former property is infinite, then $\mathfrak{H}$ contains an infinite decreasing sequence; whereas, if the family of all elements $H\in\mathfrak{H}$ with the latter property is infinite, then $\mathfrak{H}$ contains an infinite increasing sequence; and one of these two families must be infinite. But $\mathfrak{V}$ contains an infinite increasing sequence if $\mathfrak{H}$ does, so suppose $\mathfrak{H}$ contains an infinite decreasing sequence, say $(H_n; n\in\mathbf{N})$. Since H_n is regular-open, for each $n\in\mathbf{N}$, $(\bar{H}_n; n\in\mathbf{N})$ is also an infinite decreasing sequence. Hence $(X\backslash\bar{H}_n; n\in\mathbf{N})$ is an infinite increasing sequence. But $X\backslash\bar{H}_n\in\mathfrak{V}$, for $n\in\mathbf{N}$. Thus $\mathfrak{V}$ contains an infinite increasing sequence, whenever $\mathfrak{V}$ is infinite.

THEOREM 3.7. *If the semiregularization τ_s of τ is infinite, then (X, τ) has a compatible quasi-uniformity strictly finer than $P(\tau)$.*

Proof. This follows from Proposition 3.6 and Theorem 3.2.

Since $\tau_s=\tau$ if (X, τ) is semiregular, the following corollary is an immediate consequence of Theorems 3.7 and 2.1.

COROLLARY 3.8. *A semiregular space has a unique compatible quasi-uniformity if and only if its topology is finite.*

4. **Another sufficient condition.** Let $X=\mathbf{N}$ and let $\tau=\{\phi\}\cup\{B_n \mid n\in\mathbf{N}\}$, where $B_n=\{x\in X \mid n\leq x\}$. Then the singleton

$$\{\bigcap\{S(B_n) \mid n\in\mathbf{N}\}\}=\{(m, n) \mid m, n\in\mathbf{N}, m\leq n\}$$

forms a basis for a quasi-uniformity $\mathfrak{V}$ such that $\tau(\mathfrak{V})=\tau$ and $P(\tau)\subset P(\tau)\vee\mathfrak{V}$. But τ contains no infinite increasing sequence, hence the condition of Theorem 3.2 is not necessary for (X, τ) to have a compatible quasi-uniformity strictly finer than

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$P(\tau)$. This example is covered by the sufficient condition of the next theorem, whose proof uses the following purely set-theoretic lemma. The proof of the lemma is straightforward and is omitted.

LEMMA 4.1. *Let $\mathfrak{B}$ be a nonempty collection of subsets of a set X. Then, with $\bigcap \phi = X$,*

$$\bigcap \{S(B) \mid B \in \mathfrak{B}\} = \bigcup \{((\bigcap \mathfrak{H})\backslash(\bigcup \mathfrak{K})) \times (\bigcap \mathfrak{H}) \mid \mathfrak{H} \cap \mathfrak{K} = \phi, \mathfrak{H} \cup \mathfrak{K} = \mathfrak{B}\}.$$

THEOREM 4.2. *If τ contains an infinite decreasing sequence with open intersection, then (X, τ) has a compatible quasi-uniformity strictly finer than $P(\tau)$.*

Proof. Let $\{G_n \mid n \in \mathbf{N}\} \subseteq \tau$ such that $G_{n+1} \subset G_n$ and $G \in \tau$, where $G = \bigcap \{G_n \mid n \in \mathbf{N}\}$. Let $G_0 = X$ and let

$$V = (G \times G) \cup (\bigcup \{(X \backslash G_n) \times G_{n-1} \mid n \in \mathbf{N}\}).$$

Then $\{V\}$ is a basis for a quasi-uniformity $\mathfrak{B}$ on X such that $\tau(\mathfrak{B}) \subseteq \leftrightarrow \subseteq \tau$. Then $\tau(P(\tau) \vee \mathfrak{B}) = \tau$, by Proposition 3.1.

Now suppose $V \in P(\tau)$. Then there exists a nonempty finite subset $\mathfrak{B} \subseteq \tau$ such that $\bigcap \{S(B) \mid B \in \mathfrak{B}\} \subseteq V$. Now $X = \bigcup \{(\bigcap \mathfrak{H})\backslash(\bigcup \mathfrak{K}) \mid \mathfrak{H} \cap \mathfrak{K} = \phi, \mathfrak{H} \cup \mathfrak{K} = \mathfrak{B}\}$ and $G_n \backslash G_{n+1} \neq \phi$, so, for each $n \in \mathbf{N}$, there is a partition $\{\mathfrak{H}_n, \mathfrak{K}_n\}$ of $\mathfrak{B}$ such that

$$(G_n \backslash G_{n+1}) \cap ((\bigcap \mathfrak{H}_n)\backslash(\bigcup \mathfrak{K}_n)) \neq \phi,$$

where $\bigcap \phi = X$. Since $\bigcap \{S(B) \mid B \in \mathfrak{B}\} \subseteq V$, Lemma 4.1 implies that

$$((\bigcap \mathfrak{H}_n)\backslash(\bigcup \mathfrak{K}_n)) \times (\bigcap \mathfrak{H}_n) \subseteq V,$$

for each $n \in \mathbf{N}$. Thus $(G_{n+1} \times (G_n \backslash G_{n+1})) \cap V \neq \phi$, if $G_{n+1} \cap ((\bigcap \mathfrak{H}_n)\backslash(\bigcup \mathfrak{K}_n)) \neq \phi$. But $(G_{n+1} \times (G_n \backslash G_{n+1})) \cap V = \phi$, by definition of V, so $G_{n+1} \cap ((\bigcap \mathfrak{H}_n)\backslash(\bigcup \mathfrak{K}_n)) = \phi$, for each $n \in \mathbf{N}$. Since $\mathfrak{B}$ is finite, there exist $m, n \in \mathbf{N}$ such that $n < m$ and $\mathfrak{H}_n = \mathfrak{H}_m$. But then the choice of $\{\mathfrak{H}_m, \mathfrak{K}_m\}$ implies that $G_m \cap ((\bigcap \mathfrak{H}_n)\backslash(\bigcup \mathfrak{K}_n)) \neq \phi$, a contradiction since $G_m \subseteq G_{n+1}$. Thus $V \notin P(\tau)$, so $P(\tau) \vee \mathfrak{B}$ is strictly finer than $P(\tau)$.

COROLLARY 4.3. *If τ contains an infinite decreasing sequence with closed intersection, then (X, τ) has a compatible quasi-uniformity strictly finer than $P(\tau)$.*

Proof. Let $\{G_n \mid n \in \mathbf{N}\} \subseteq \tau$ such that $G_{n+1} \subset G_n$ and $F = \bigcap \{G_n \mid n \in \mathbf{N}\}$ is closed. Then $\{G_n \backslash F \mid n \in \mathbf{N}\}$ defines an infinite decreasing sequence with empty, hence open, intersection. The corollary now follows from Theorem 4.2.

(X, τ) is an R_0 *space* if, for every $x, y \in X$, $\{\bar{x}\} \neq \{\bar{y}\}$ implies $\{\bar{x}\} \cap \{\bar{y}\} = \phi$. This concept leads to the following.

COROLLARY 4.4. *Let (X, τ) be an R_0 space such that $\{\bar{x}\}$ is a G_δ set for each $x \in X$. Then (X, τ) has a unique compatible quasi-uniformity if and only if τ is finite.*

Proof. By taking successive intersections, it is easy to show that $\{\bar{x}\}$ is the intersection of a decreasing sequence of open sets. Thus, either there is an $x \in X$ such that $\{\bar{x}\}$ is the intersection of an infinite decreasing sequence, or $\{\bar{x}\} \in \tau$ for every $x \in X$. The remainder of the proof is straightforward, using Corollary 4.3 and Theorem 2.1.

Since T_1 implies R_0, the following is an easy consequence of Corollary 4.4.

COROLLARY 4.5. *A first countable T_1 space has a unique compatible quasi-uniformity if and only if it is finite.*

5. **The Császár and Pervin quasi-uniformities.** Császár, as part of a more general theory, showed that every topological space has a compatible quasi-uniformity [1, 1st ed., p. 171, 2nd ed., p. 193]. Divested of the terminology and constructions not necessary for this particular case, the Császár quasi-uniformity is constructed as follows.

Let (X, τ) be a topological space. Let σ denote the set of all bounded functions $f:X\rightarrow\mathbf{R}$ such that for every $r \in \mathbf{R}$ and every $\varepsilon>0$, there exists $G \in \tau$ with

$$f^{-1}(]-\infty, r]) \subseteq G \subseteq f^{-1}(]-\infty, r+\varepsilon[).$$

For each $f \in \sigma$ and each $\varepsilon>0$, define

$$U(f, \varepsilon) = \{(x, y) \in X\times X \mid f(y)-f(x) < \varepsilon\}.$$

Then the set $\{U(f, \varepsilon) \mid f \in \sigma, \varepsilon>0\}$ forms a subbasis for a quasi-uniformity $\mathfrak{V}$ on X. But

$$\begin{aligned} U(f, \varepsilon)[x] &= \{y \in X \mid f(y)-f(x) < \varepsilon\} \\ &= f^{-1}(]-\infty, f(x)+\varepsilon[) \in \tau, \end{aligned}$$

for each $x \in X$, so $\tau(\mathfrak{V}) \subseteq \tau$. On the other hand, $\xi(X\backslash G) \in \sigma$, where $\xi(X\backslash G)$ is the characteristic function of $X\backslash G$, and

$$U(\xi(X\backslash G), 1)[x] = G$$

for all $x \in G$. Thus $\tau \subseteq \tau(\mathfrak{V})$, so $\tau=\tau(\mathfrak{V})$.

Note that $U(\xi(X\backslash G), 1)=S(G)$, so that $P(\tau) \subseteq \mathfrak{V}$. The following theorem shows that equality holds in this latter relation.

THEOREM 5.1. *Let $\mathfrak{V}$ denote the Császár quasi-uniformity associated with (X, τ). Then $\mathfrak{V}=P(\tau)$.*

Proof. As already observed, $P(\tau) \subseteq \mathfrak{V}$. Now let $f \in \sigma$, where σ is as indicated above, and let $\varepsilon>0$. Without loss of generality, assume $f(X) \subseteq [0, 1]$. Let $n \in \mathbf{N}$ such that $1/n<\varepsilon$. It is easy to verify that

$$\bigcap \left\{S\left(f^{-1}\left(\left[0, \frac{m}{n}\right[\right)\right) \,\middle|\, 1 \leq m < n\right\} \subseteq U(f, \varepsilon).$$

Thus $U(f, \varepsilon) \in P(\tau)$, since $f^{-1}([0, m/n[) \in \tau$. Hence $\mathfrak{V}=P(\tau)$.

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It is rather easy to verify that σ, as defined above, is precisely the set of bounded real-valued upper semicontinuous functions defined on (X, τ). Then $f \in \sigma$ implies $-f$ is a bounded real-valued lower semi-continuous function defined on (X, τ) and $f(y)-f(x)=(-f)(x)-(-f)(y)$. Thus the "natural" quasi-uniformity recently described by Nielsen and Sloyer [6] is simply the Császár quasi-uniformity, hence is precisely the Pervin quasi-uniformity. This latter fact was noted by Hunsaker and Lindgren [4].

Let γ denote the set of all bounded continuous functions $f:(X, \tau)\to\mathbf{R}$. Then $f, -f \in \sigma$, where σ is defined as above, and, with $U(f, \varepsilon)$ defined as above, $U(f, \varepsilon) \cap U(-f, \varepsilon)=V(f, \varepsilon)$, where $V(f, \varepsilon)=\{(x, y) \in X\times X \mid |f(x)-f(y)|<\varepsilon\}$. But it is a standard result that the set $\{V(f, \varepsilon) \mid f \in \gamma,\ \varepsilon>0\}$ forms a subbasis for a uniformity $\mathfrak{U}$ such that $\tau(\mathfrak{U})=\tau$, if (X, τ) is completely regular (see [8, p. 181, Theorem 20.20]). These considerations lead to the next theorem whose proof will use the following lemma. The proof of the lemma is straight-forward and is omitted.

Lemma 5.2. *Let $\mathfrak{B}$ be a subbasis for a topology τ on a set X. Then the set $\{S(B) \mid B \in \mathfrak{B}\}$ forms a subbasis for a quasi-uniformity $\mathfrak{U} \subseteq P(\tau)$ such that $\tau(\mathfrak{U})=\tau$. If $\mathfrak{B}$ is complemented (i.e., if $B \in \mathfrak{B}$ implies $X\backslash B \in \mathfrak{B}$), then $\mathfrak{U}$ is a uniformity and $\mathfrak{U}$ also has the set $\{S(B) \cap S(X\backslash B) \mid B \in \mathfrak{B}\}$ as a subbasis.*

Theorem 5.3. *A completely regular space (X, τ) has a compatible quasi-uniformity (strictly) coarser than $P(\tau)$ (if and only if τ is infinite).*

Proof. It is clear from the discussion preceding Lemma 5.2 that (X, τ) has a compatible uniformity $\mathfrak{U}$ coarser than $P(\tau)$. Then $\mathfrak{U}$ is strictly coarser than $P(\tau)$ unless $P(\tau)$ is a uniformity. From Theorem 2.1, $\mathfrak{U}$ is strictly coarser than $P(\tau)$ only if τ is infinite.

Now assume that τ is infinite and that $P(\tau)$ is a uniformity. Then, for $G \in \tau$,

$$\begin{aligned} S(X\backslash G) &= (X\backslash G)\times(X\backslash G) \cup (G\times X) \\ &= (X\times(X\backslash G)) \cup (G\times G) = S(G)^{-1} \in P(\tau). \end{aligned}$$

Thus $G \in \tau$ implies $X\backslash G \in \tau$, so $\{\bar{x}\} \in \tau$ for every $x \in X$. Then the set $\{\{\bar{x}\} \mid x \in X\}$ is an infinite partition of X and forms a basis for τ, so the set $\{S(\{\bar{x}\}) \cap S(\{\bar{x}\})^{-1} \mid x \in X\}$ forms a subbasis for a uniformity $\mathfrak{U}$ on X such that $\tau(\mathfrak{U})=\tau$, by Lemma 5.2, and $\mathfrak{U}$ is strictly coarser than $P(\tau)$. Thus the theorem is proved.

6. One more sufficient condition.

Theorem 6.1. *Let τ be a topology on a set X such that τ has a basis $\mathfrak{B}\neq\tau$ with $X \in \mathfrak{B}$ and $\mathfrak{B}$ closed under finite intersections and unions. Then the set $\{S(B) \mid B \in \mathfrak{B}\}$ forms a subbasis for a compatible quasi-uniformity $\mathfrak{U}$ strictly coarser than $P(\tau)$.*

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PROOF By Lemma 5.2, all that remains to be proved is that $P(\tau)\backslash\mathfrak{U}\neq\phi$. Let $G\in\tau\backslash\mathfrak{B}$. Note that $G\neq\phi$. If $\bigcap\{S(B_i)\mid 1\leq i\leq n\}\subseteq S(G)$, then

$$G=\bigcup\{\bigcap\{B_i\mid 1\leq i\leq n, x\in B_i\}\mid x\in G\}.$$

But this is a finite union of finite intersections, hence $B_m\notin\mathfrak{B}$, for some m, $1\leq m\leq n$, by definition of G. Thus $S(G)\in P(\tau)\backslash\mathfrak{U}$.

Note that if (X, τ) satisfies the conditions of Theorem 6.1, then τ contains an infinite increasing sequence, so that (X, τ) also has a compatible quasi-uniformity strictly finer than $P(\tau)$, by Theorem 3.2.

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