

On a family of contractive maps

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The object of this paper is to study common fixed points of mappings of a complete metric space into itself. The results obtained are generalizations of theorems of Iséki and Ray. An example is given to show that our results are indeed extensions of the theorem of Iséki.

1. Introduction

Let (X, ρ) be a metric space. A mapping $T : X \rightarrow X$ is called an f -contraction iff for every $x, y \in X$, there exist real numbers $\alpha_i = \alpha_i(x, y)$ ($i = 1, \dots, 5$) such that

$$\rho[Tx, Ty] \leq f(\alpha_1 \rho[x, y], \alpha_2 \rho[x, Tx], \alpha_3 \rho[y, Ty], \alpha_4 \rho[x, Ty], \alpha_5 \rho[y, Tx]) ,$$

$$\sup\{f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) : x, y \in X\} = \bar{\lambda} \in [0, 1) ,$$

where the mapping $f : R^5 \rightarrow R$ is increasing, semihomogeneous [that is to say, for all $\delta \geq 0$ one has $f(\delta x_1, \delta x_2, \dots, \delta x_5) \leq \delta f(x_1, x_2, \dots, x_5)$]

and $g(x) = f(\alpha_1 x, \dots, \alpha_5 x^5)$ continuous in the point $x = 1$.

In [6] I proved the following result.

THEOREM T. *An f -contraction $T : X \rightarrow X$ on a T -orbitally complete [that is to say, every Cauchy sequence $\{T^n x \mid n \in N, x \in X\}$ converges in X] metric space X has a fixed point, and this is unique when*

$$\bar{\lambda}_0 \stackrel{\text{def}}{=} \sup\{(\alpha_1, 0, 0, \alpha_4, \alpha_5) : x, y \in X\} < 1 .$$

In a recent paper [2] Iséki proved the following:

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THEOREM I. *Let X be a complete metric space, T_n ($n = 1, 2, \dots$) a sequence of mappings of X into itself. Suppose that there are non-negative numbers a, b, c such that for $x, y \in X$,*

$$\rho[T_i x, T_j y] \leq a(\rho[x, T_i(x)] + \rho[y, T_j(y)]) + b(\rho[x, T_j(y)] + \rho[y, T_i(x)]) + c\rho[x, y],$$

where $2a + 2b + c < 1$. Then the sequence of mappings $\{T_n\}$ has a unique common fixed point.

(Let I be an infinite set of indices and let $F = \{T_\lambda : \lambda \in I\}$ be a family of maps which map X into itself. A point $\xi \in X$ is a common fixed point for F iff $\xi = T_\lambda \xi$ for each $T_\lambda \in F$.)

THEOREM 1. *Let X be a complete metric space, $\{T_n : n \in \mathbb{N}\}$ a sequence of mappings of X into itself. For $x, y \in X$, let there exist real numbers α_i ($i = 1, \dots, 5$) such that*

$$(1) \quad \rho[T_i x, T_j y] \leq f(\alpha_1 \rho[x, y], \alpha_2 \rho[x, T_i(x)], \alpha_3 \rho[y, T_j(y)], \frac{1}{2}\alpha_4 \rho[x, T_j(y)], \frac{1}{2}\alpha_5 \rho[y, T_i(x)]),$$

$$\sup\{f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) : x, y \in X\} = \delta \in [0, 1),$$

where the mapping $f : R^5 \rightarrow R$ is increasing, semihomogeneous and $g(x) = f(\alpha_1 x, \dots, \alpha_5 x^5)$ is continuous in the point $x = 1$. Then the sequence of mappings $\{T_n\}$ has a common fixed point $\xi \in X$, and this is unique when $\sup\{f(\alpha_1, 0, 0, \frac{1}{2}\alpha_4, \frac{1}{2}\alpha_5) : x, y \in X\} = \delta_0 < 1$, and for arbitrary $x_0 \in X$ the sequence

$$(2) \quad x_n = T_n(x_{n-1}), \quad (n = 1, 2, \dots),$$

converges to ξ .

Proof. We prove that (2) is a Cauchy sequence, where $x_0 \in X$ is arbitrary. By assertion (1) we have

$$\rho[x_n, x_{n+1}] = \rho[T_n(x_{n-1}), T_{n+1}(x_n)] \leq f(\alpha_1 \rho[x_{n-1}, x_n], \alpha_2 \rho[x_{n-1}, x_n], \alpha_3 \rho[x_n, x_{n+1}], \frac{1}{2} \alpha_4 \rho[x_{n-1}, x_{n+1}], \frac{1}{2} \alpha_5 \rho[x_n, x_n]) .$$

Now we use the following lemma, proved in [5], p. 198.

LEMMA. Let $f : R^{k+2} \rightarrow R$ ($k \in N$) be a monotonically increasing (with respect to every real argument) and semihomogeneous mapping, let $g(x) = f(a_0, a_1 x, \dots, a_{k+1} x^{k+1})$ be continuous in the point $x = 1$, and let the sequence $\{x_n\}$ of nonnegative real numbers satisfy the condition

$$x_{n+k} \leq f(a_0 x_n, a_1 x_{n+1}, \dots, a_k x_{n+k}, a_{k+1}^C) , \quad n = 1, 2, \dots ,$$

where k is a fixed natural number, $a_0, a_1, \dots, a_{k+1}, C$ are real constants, and $f(a_0, a_1, \dots, a_{k+1}) \in [0, 1)$. Then there exist positive numbers L and $\theta \in (0, 1)$ such that

$$(3) \quad x_n \leq L\theta^n \quad (n = 1, 2, \dots) , \quad L = \max_{i=1,2,\dots,k} \left\{ x_i \theta^{-i} \right\} .$$

Applying this lemma to the sequence $\{\rho[x_n, x_{n+1}]\}$, we obtain, according to (3),

$$\rho[x_n, x_{n+1}] \leq \theta^n \max_{i=1,\dots,k} \left\{ \rho[x_i, x_{i+1}] \theta^{-i} \right\} , \quad (n \in N, \theta \in (0, 1)) .$$

Hence, for $n, s \in N$,

$$\begin{aligned} \rho[x_n, x_{n+s}] &\leq \sum_{j=1}^s \rho[x_{n+j-1}, x_{n+j}] = \max_{i=1,\dots,k} \left\{ \rho[x_i, x_{i+1}] \theta^{-i} \right\} \sum_{j=1}^s \theta^{n+j-1} \\ &= \max_{i=1,\dots,k} \left\{ \rho[x_i, x_{i+1}] \theta^{-i} \right\} \theta^n (1-\theta)^{-1} , \quad (n \in N, \theta \in (0, 1)) , \end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence. Hence, by the completeness of X , $\{x_n\}$ converges to some point ξ in X . For the point $\xi \in X$,

$$\begin{aligned} \rho[\xi, T_n(\xi)] &\leq \rho[\xi, x_{m+1}] + \rho[x_{m+1}, T_n(\xi)] = \rho[\xi, x_{m+1}] + \rho[T_{m+1}(x_m), T_n(\xi)] \\ &\leq \rho[\xi, x_{m+1}] + f(\alpha_1 \rho[x_m, \xi], \alpha_2 \rho[x_m, x_{m+1}], \alpha_3 \rho[\xi, T_n(\xi)], \\ &\quad \frac{1}{2} \alpha_4 \rho[x_m, T_n(\xi)], \frac{1}{2} \alpha_5 \rho[\xi, x_{m+1}]) . \end{aligned}$$

Applying this lemma to the sequence $\{\rho[x_{m+1}, T_n(\xi)]\}$, we obtain, according to (3),

$$\rho[\xi, T_n(\xi)] \leq \rho[\xi, x_{m+1}] + L\theta^m \quad (m \in N, \theta \in (0, 1)) .$$

Letting $m \rightarrow \infty$, we have $\rho[\xi, T_n(\xi)] = 0$; that is, the point ξ is a common fixed point of all T_n .

To show that ξ is the unique common fixed point of all T_n , we consider a point y in X such that $T_n(y) = y$ for every n . Then we have

$$\rho[\xi, y] = \rho[T_n(\xi), T_n(y)] \leq f(\alpha_1\rho[\xi, y], \alpha_2\rho[\xi, T_n(\xi)], \alpha_3\rho[y, T_n(y)], \frac{1}{2}\alpha_4\rho[\xi, T_n(y)], \frac{1}{2}\alpha_5\rho[y, T_n(\xi)]) \leq \delta_0\rho[\xi, y] .$$

Hence $\rho[\xi, y] = 0$; that is, $\xi = y$. This completes the proof of Theorem 1.

Now we give an example of a family of maps satisfying the conditions of Theorem 1, for which the conditions of Theorem I do not hold.

EXAMPLE. Let $X = [0, 10]$ be the subset of reals with the usual metric and let $\{T_n\}$ be a sequence of functions which maps X into itself, defined as follows:

$$T_n(x) = \begin{cases} 0 & , x \in [0, 10/11) , \\ 10nx(1+11n)^{-1} & , x \in [10/11, 10] ; n = 0, 1, 2, \dots . \end{cases}$$

Then the conditions of Theorem I of Iséki [2] are not satisfied, since for $x = 10/11$, $y = 1$; while our conditions of Theorem 1 are satisfied for the mapping

$$f(t_1, \dots, t_5) \stackrel{\text{def}}{=} (t_1+t_2+t_3)^2(t_1+t_2+t_3-t_4)^{-1} + t_5 + t_6 \quad (t_1+t_2+t_3 > 2t_4, t_i \in R)$$

with $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 0$, $\alpha_4 = -3$, $\alpha_5 = \alpha_6 = 0$.

2.

In a recent paper [1], Ćirić proved the following:

THEOREM C. *Let $F = \{T_\lambda : \lambda \in I\}$ be a family of functions which maps a complete metric space (X, ρ) into itself and let $0 \leq q < 1$. If there exists some $T_{\lambda_0} \in F$ such that for each $T_\lambda \in F$ ($\lambda \neq \lambda_0$) there are positive integers i_λ and j_λ such that*

$$\rho \left[T_{\lambda_0}^{i_\lambda} x, T_{\lambda_0}^{j_\lambda} y \right] \leq q \cdot \max \left\{ \rho[x, y], \rho \left[x, T_{\lambda_0}^{i_\lambda} x \right], \rho \left[y, T_{\lambda_0}^{j_\lambda} y \right], \frac{1}{2} \left(\rho \left[x, T_{\lambda_0}^{j_\lambda} y \right] + \rho \left[y, T_{\lambda_0}^{i_\lambda} x \right] \right) \right\},$$

holds for all $x, y \in X$, then every $T_\lambda \in F$ has a unique fixed point in X , which is a unique common fixed point for F .

Let $F = \{T_\lambda : \lambda \in I\}$ be a family of functions each of which maps a metric space (X, ρ) into itself. If there exists some $T_{\lambda_0} \in F$ such that for each $T_\lambda \in F$ ($\lambda \neq \lambda_0$) there are positive integers i, j , and for every $x, y \in X$ there exist real numbers α_i ($i = 1, \dots, 5$) such that

$$(4) \quad \rho \left[T_{\lambda_0}^i(x), T_{\lambda_0}^j(y) \right] \leq f \left(\alpha_1 \rho[x, y], \alpha_2 \rho \left[x, T_{\lambda_0}^i(x) \right], \alpha_3 \rho \left[y, T_{\lambda_0}^j(y) \right], \frac{1}{2} \alpha_4 \rho \left[x, T_{\lambda_0}^j(y) \right], \frac{1}{2} \alpha_5 \rho \left[y, T_{\lambda_0}^i(x) \right] \right),$$

$$\sup \{ f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) : x, y \in X \} = \delta \in [0, 1),$$

where the mapping $f : R^5 \rightarrow R$ is increasing, semihomogeneous and $g(x) = f(\alpha_1 x, \dots, \alpha_5 x^5)$ continuous in the point $x = 1$, then mappings T_λ of this type will be referred to as having property (f) over X .

THEOREM 2. *Let $F = \{T_\lambda : \lambda \in I\}$ be a family of functions which maps a complete metric space (X, ρ) into itself, and let $T_\lambda \in F$ have*

property (f) over X . Then every $T_\lambda \in F$ has a fixed point in X , and it is unique when $\delta_0 < 1$, and it is then the unique common fixed point for F .

Proof. Let $T_\lambda \in F$ be arbitrary. For arbitrary $x \in X$ let us consider a sequence

$$(5) \quad x_0 = x, \quad x_1 = T_{\lambda_0}^i x_0, \quad x_2 = T_\lambda^j x_1, \dots, \quad x_{2n-1} = T_{\lambda_0}^i x_{2n-2},$$

$$x_{2n} = T_\lambda^j x_{2n-1}, \dots .$$

By assertion (4) we have

$$\begin{aligned} \rho[x_{2n}, x_{2n+1}] &= \rho\left[T_\lambda^j x_{2n-1}, T_{\lambda_0}^i x_{2n}\right] \leq \\ &\leq f\left(\alpha_1 \rho[x_{2n-1}, x_{2n}], \alpha_2 \rho\left[x_{2n-1}, T_\lambda^j x_{2n-1}\right], \alpha_3 \rho\left[x_{2n}, T_{\lambda_0}^i x_{2n}\right], \right. \\ &\quad \left. \frac{1}{2}\alpha_4 \rho\left[x_{2n-1}, T_{\lambda_0}^i x_{2n}\right], \frac{1}{2}\alpha_5 \rho\left[x_{2n}, T_\lambda^j x_{2n-1}\right]\right) = f\left(\alpha_1 \rho[x_{2n-1}, x_{2n}], \right. \\ &\quad \left. \alpha_2 \rho[x_{2n-1}, x_{2n}], \alpha_3 \rho[x_{2n}, x_{2n+1}], \frac{1}{2}\alpha_4 \rho[x_{2n-1}, x_{2n+1}], \frac{1}{2}\alpha_5 \rho[x_{2n}, x_{2n+1}]\right) . \end{aligned}$$

Then by a routine calculation one can show that (5), from the lemma, is a Cauchy sequence. Using completeness of X we have that $\lim_n x_n = \xi$, for

some $\xi \in X$. By assertion (4),

$$\begin{aligned} \rho\left[\xi, T_{\lambda_0}^i(\xi)\right] &\leq \rho[\xi, x_{2n}] + \rho\left[x_{2n}, T_{\lambda_0}^i(\xi)\right] \leq \rho[\xi, x_{2n}] + f\left(\alpha_1 \rho[x_{2n-1}, \xi], \right. \\ &\quad \left. \alpha_2 \rho[x_{2n-1}, x_{2n}], \alpha_3 \rho[\xi, T_{\lambda_0}^i(\xi)], \frac{1}{2}\alpha_4 \rho[x_{2n-1}, T_{\lambda_0}^i(\xi)], \frac{1}{2}\alpha_5 \rho[\xi, x_{2n}]\right) . \end{aligned}$$

Applying the lemma to the sequence $\left\{\rho\left[x_{2n}, T_{\lambda_0}^i(\xi)\right]\right\}$, we obtain according

to (3),

$$\rho\left[\xi, T_{\lambda_0}^i(\xi)\right] \leq \rho[\xi, x_{2n}] + L\theta^n \quad (n \in N, \theta \in (0, 1)) .$$

Letting $n \rightarrow \infty$, then we have $\rho\left[\xi, T_{\lambda_0}^i(\xi)\right] = 0$; that is, the point ξ

is a common fixed point of $T_{\lambda_0}^i$, and it is unique.

By Theorem 1 and assertion (4), ξ is a unique fixed point of $T_{\lambda_0}^i$ and T_{λ}^j . Since $T_{\lambda_0}^i(T_{\lambda_0} \xi) = T_{\lambda_0}^i(T_{\lambda_0}^i \xi) = T_{\lambda_0} \xi$, $T_{\lambda_0} \xi$ is also a fixed point of $T_{\lambda_0}^i$, and therefore $T_{\lambda_0} \xi = \xi$. Similarly it follows that $T_{\lambda} \xi = \xi$. So we have proved that ξ is a unique fixed point of T_{λ_0} and T_{λ} .

Now we shall show that ξ is a unique common fixed point for F . Let $T_{\lambda_1} \in F$ ($\lambda_0 \neq \lambda_1 \neq \lambda$), be arbitrary. Since $\xi = T_{\lambda_0} \xi$ implies $\xi = T_{\lambda_0}^i \xi$, by (4) and Theorem 1, ξ is a unique fixed point of T_{λ}^j . This implies that ξ is a unique fixed point of T_{λ_1} . This completes the proof of Theorem 2.

THEOREM 3. *Let X be a metric space, and let $\{T_n\}$ be a sequence of mappings which map X into itself. Let $T : X \rightarrow X$ be an f -contraction and let X be T -orbitally complete, and $\delta_0 < 1$. If each T_n has at least one fixed point u_n , and if we suppose that $\{T_n\}$ converges uniformly to T , then the sequence $\{u_n\}$ converges to the unique fixed point u of T .*

Proof. By Theorem T, $T : X \rightarrow X$ has a unique fixed point u . We have

$$\begin{aligned} \rho[u, u_n] &= \rho[Tu, T_n(u_n)] \leq \rho[Tu, T(u_n)] + \rho[T(u_n), T_n(u_n)] \leq \\ &\leq f(\alpha_1 \rho[u, u_n], \alpha_2 \cdot 0, \alpha_3 \rho[u_n, T(u_n)], \frac{1}{2} \alpha_4 \rho[u, T(u_n)], \\ &\quad \frac{1}{2} \alpha_5 \rho[u_n, u]) + \rho[T(u_n), T_n(u_n)]. \end{aligned}$$

Applying the lemma to the sequence $\{\rho[u, u_n]\}$ we obtain, according to (3),

$$\rho[u, u_n] \leq L\theta^n + \rho[T(u_n), T_n(u_n)] \quad (n \in N, \theta \in (0, 1)).$$

Since $\{T_n\}$ converges uniformly to T ,

$$\rho[u_n, T(u_n)] = \rho[T_n(u_n), T(u_n)] \rightarrow 0 \quad (n \rightarrow \infty)$$

and we have that $\lim \rho[u, u_n] = 0$, which completes the proof.

References

- [1] Ljubomir B. Ćirić, "On a family of contractive maps and fixed points", *Publ. Inst. Math. (Beograd) (NS)* 17 (31) (1974), 45-51.
- [2] Kiyoshi Iséki, "On common fixed points of mappings", *Bull. Austral. Math. Soc.* 10 (1974), 365-370.
- [3] Baidyanath Ray, "Some results on fixed points and their continuity", *Colloq. Math.* 27 (1973), 41-48.
- [4] Milan R. Tasković, "Monotonic mappings on ordered sets, a class of inequalities with finite differences and fixed points", *Publ. Inst. Math. (Beograd) (NS)* 17 (31) (1974), 163-172.
- [5] Milan R. Tasković. "Eine Abbildung von β -Typus", *Publ. Inst. Math. (Beograd) (NS)* 17 (32) (1974), 197-206.
- [6] Milan R. Tasković, "A generalization of Banach's contraction principle", submitted.

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