Canad. Math. Bull.Vol. 35 (2), 1992 pp. 261–266

# UNIVERSAL COMPACT $T_1$-SPACES

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ABSTRACT. For every infinite cardinal $\kappa$ we construct a space $C_\kappa$ universal for all compact $T_1$-spaces of weight $\leq \kappa$. It follows, in particular, that there are only $2^\kappa$ topologically different compact $T_1$-spaces. We show that $C_\omega$ is universal for all second countable developable $T_1$-spaces. The existence of closely universal compact $T_1$-spaces is discussed.

1. **Introduction.** A space $K$ is said to be *universal* for a class $\mathcal{K}$ of topological spaces if $K \in \mathcal{K}$ and every space from $\mathcal{K}$ is homeomorphic to some subspace of $K$.

Let $\kappa$ be an arbitrary infinite cardinal number.

By the celebrated Tychonoff Theorem the cube $[0,1]^\kappa$ is universal for all compact $T_2$-spaces of weight $\leq \kappa$. Denote by $F$ the two-point space $\{0,1\}$ endowed with the topology consisting of the empty set, the set $\{0\}$ and the whole space. The Alexandroff cube $F^\kappa$ is universal for all $T_0$-spaces of weight $\leq \kappa$ (cf. [4. Theorem 2.3.26]). As $F^\kappa$ is compact, it is universal also for all compact $T_0$-spaces of weight $\leq \kappa$.

The above two theorems suggest the natural problem of whether the analogous result is valid for the class of compact $T_1$-spaces. The aim of this paper is to show that *the answer is positive*. We construct, for every $\kappa$, a space $C_\kappa$ which is universal for all compact $T_1$-spaces of weight $\leq \kappa$.

The paper is organized as follows.

The construction and the proof of universality of $C_\kappa$ are given in Section 1. We show that the spaces $C_\kappa$ are supercompact. Recall that a space $X$ is called *supercompact*, provided that there exists an open subbase $\mathcal{P}$ for $X$ such that every cover of $X$ by members of $\mathcal{P}$ has a two-element subcover. It follows from Alexander Subbase Theorem (cf. [4, Problem 3.12.2]) that every supercompact space is compact.

In section 2 we determine the number of compact $T_1$-spaces of given weight. Since all compact subspaces of a Hausdorff space are closed, the universality of $[0,1]^\kappa$ implies that there exists only $2^\kappa$ topologically different compact $T_2$-spaces of weight $\kappa$. On the other hand, it is easy to construct, for every $\kappa$, a family of $2^{2^\kappa}$ pairwise nonhomeomorphic compact $T_0$-spaces. This leads to the question about the largest possible number of topologically different compact $T_1$-spaces of weight $\kappa$. The existence of $C_\kappa$ implies easily that there are *only* $2^\kappa$ such spaces.

Received by the editors February 5, 1991.

AMS subject classification: Primary 54C25; secondary 54D10, 54D30, 54B10.

Key words and phrases: Universal, closely universal, compact $T_1$-space, supercompact, developable, $\kappa$-perfect

In section 3 we characterize those spaces which can be embedded into $C_\kappa$. In particular, it turns out that the space $C_\omega$ is universal for all second countable developable $T_1$-spaces. Various examples of such universal spaces have been constructed earlier in [2], [3] and [5]. Although those spaces (in opposition to $C_\omega$) are originally noncompact, they do have second countable and developable (Wallman) compactifications. We believe, however, that our construction is simpler.

Section 4 is devoted to the problem of existence of closely universal compact $T_1$-spaces. A space $K$ is called *closely universal* for a class $\mathcal{K}$ if $K \in \mathcal{K}$ and every space from $\mathcal{K}$ is homeomorphic to some *closed* subspace of $K$. We end the paper with an example exhibiting that closely universal compact $T_1$-spaces cannot be supercompact.

All undefined notions can be found in [4]. We do not distinguish, however, between compact and quasicompact spaces, i.e. no separation axioms are assumed in the definition of a compact space. We write $\kappa$ both for an *infinite* cardinal number and a set of respective cardinality. $|A|$ stands for cardinality of a set $A$.

In the sequel we shall frequently use the following simple

LEMMA. *If $X$ is a $T_1$-space and $\mathcal{P}$ is an open subbase for $X$, then every compact subset $C$ of $X$ is an intersection of finite unions of elements of $\mathcal{P}$.*

For the proof it suffices to show that for every point $x$ in $X - C$ the set $C$ has a finite cover by elements of $\mathcal{P}$ which do not contain $x$. This is, however, a direct consequence of compactness of $C$ and of the fact that $X$ is a $T_1$-space.

2. **The space $C_\kappa$.** For an infinite cardinal number $\kappa$ denote by $\Sigma(\kappa)$ the set of all finite sequences in $\kappa$, i.e. $\Sigma(\kappa) = \{\emptyset\} \cup \kappa \cup \kappa^2 \cup \cdots \cup \kappa^n \cup \cdots$.

For the sake of simplicity we shall denote by $(\sigma, s)$ the extension $(s_1, \ldots s_n, s)$ of $\sigma = (s_1, \ldots, s_n)$ and by $\sigma_\lceil$ the reduction $(s_1, \ldots, s_n)$ of $\sigma = (s_1, \ldots s_n, s_{n+1})$. Whenever we write $\sigma_\lceil$, it is tacitly assumed that $\sigma \neq \emptyset$.

Our space $C_\kappa$ is a subspace of Alexandroff cube $F^{\Sigma(\kappa)}$ consisting of all functions $\chi: \Sigma(\kappa) \longrightarrow \{0, 1\}$ which satisfy, for each $\sigma \in \Sigma(\kappa)$, the following condition:

$$(*) \qquad \chi(\sigma) = 0 \text{ if and only if } \chi(\sigma_\lceil) = 1 \text{ or } \chi(\sigma, s) = 1 \text{ for some } s \in \kappa.$$

THEOREM 1. *The space $C_\kappa$ is supercompact and is universal for all compact $T_1$-spaces of weight $\leq \kappa$.*

PROOF. For every $\sigma \in \Sigma(\kappa)$ let $W_\sigma = \{\chi \in C_\kappa : \chi(\sigma) = 0\}$. The form of the topology on $F^{\Sigma(\kappa)}$ implies that the family $\mathcal{P}_\kappa = \{W_\sigma : \sigma \in \Sigma(\kappa)\}$ is an open subbase for $C_\kappa$.

We prove our theorem by showing the following four facts.

FACT 1. $C_\kappa$ has weight $\leq \kappa$.

This is obvious since $|\Sigma(\kappa)| = \kappa$. Actually, it follows from Facts 1 and 4 (below) that $C_\kappa$ has weight equal to $\kappa$.

FACT 2. $C_\kappa$ is a $T_1$-space.

Let $\chi$ and $\chi'$ be two arbitrary different elements of $C_\kappa$. We may assume that $\chi(\sigma) = 0$ and $\chi'(\sigma) = 1$ for some $\sigma \in \Sigma(\kappa)$. Clearly the set $W_\sigma$ separates $\chi$ from $\chi'$. To separate $\chi'$ from $\chi$ observe that $\chi(\sigma_\lceil) = 1$ or $\chi(\sigma, s_0) = 1$ for some $s_0 \in \kappa$, while $\chi'(\sigma_\lceil) = 0 = \chi'(\sigma, s)$ for all $s \in \kappa$. It follows that either $W_{\sigma_\lceil}$ or $W_{(\sigma,s_0)}$ separates $\chi'$ from $\chi$.

FACT 3. $C_\kappa$ is supercompact.

To prove this we show that every cover $\mathcal{U}$ of $C_\kappa$ by elements of $\mathcal{P}_\kappa$ contains a two-element subcover. Let $\mathcal{U} = \{ W_\sigma : \sigma \in \Lambda\}$ for some $\Lambda \subseteq \Sigma(\kappa)$. It follows directly from the condition $(*)$ that $C_\kappa = W_\sigma \cup W_{(\sigma,s)}$ for every $\sigma \in \Sigma(\kappa)$ and each $s \in \kappa$. Therefore it suffices to show that there must exist $\sigma \in \Sigma(\kappa)$ and $s \in \kappa$ such that both $\sigma$ and $(\sigma, s)$ belong to $\Lambda$.

Suppose, on the contrary, that $(\sigma, s) \notin \Lambda$ for all $\sigma \in \Lambda$ and $s \in \kappa$. We will show that the family $\{ W_\sigma : \sigma \in \Lambda\}$ cannot be a cover of $C_\kappa$. To this end we construct a function $\chi \in C_\kappa$ such that $\chi(\sigma) = 1$ for all $\sigma \in \Lambda$.

The values $\chi(\sigma)$ are defined inductively with respect to the length of sequences $\sigma$. Let $\chi(\emptyset) = 0$ if $\emptyset \notin \Lambda$, otherwise put $\chi(\emptyset) = 1$. For $\sigma \neq \emptyset$ we take

$$\chi(\sigma) = \begin{cases} 0, & \text{if } \chi(\sigma_\lceil) = 1 \text{ or } (\sigma, s) \in \Lambda \text{ for some } s \in \kappa, \\ 1, & \text{otherwise.} \end{cases}$$

Observe that if $\sigma \in \Lambda$ then $\chi(\sigma) = 1$ for all $\sigma \in \Lambda$. In fact, $\chi(\sigma) = 1$ means that $\chi(\sigma_\lceil) = 0$ and $(\sigma, s) \notin \Lambda$ for all $s \in \kappa$. The former follows from the fact that $(\sigma_\lceil, s) = \sigma$ for some $s \in \kappa$; the latter is given by our assumption on $\Lambda$. This and the definition of $\chi$ imply directly that $\chi$ satisfies the condition $(*)$ and therefore is in $C_\kappa$.

FACT 4. Every compact $T_1$-space $X$ of weight $\leq \kappa$ can be embedded into $C_\kappa$.

To construct a respective embedding consider an open subbase $\mathcal{P}$ for $X$ such that $|\mathcal{P}| \leq \kappa$, $X \in \mathcal{P}$ and $X - U = \bigcap\{ V \in \mathcal{P} : U \cup V = X\}$ for every $U \in \mathcal{P}$. The existence of such a subbase for $X$ follows directly from the Lemma—it suffices to take any open subbase which has cardinality $\leq \kappa$ and is closed for finite unions.

Arrange $\mathcal{P}$ as $\{ U_s : s \in \kappa\}$ and then reindex $\mathcal{P}$ inductively by the elements of $\Sigma(\kappa)$ in the following way. Put $U_\emptyset = X$ and for $\sigma = (\sigma_\lceil, s)$ take $U_\sigma = U_s$ provided that $U_{\sigma_\lceil} \cup U_s = X$; otherwise put $U_\sigma = X$.

To every point $x$ in $X$ there corresponds a function $\chi_x: \Sigma(\kappa) \longrightarrow \{0, 1\}$ such that $\chi_x(\sigma) = 0$ if and only if $x \in U_\sigma$. It follows directly from the definition of the sets $U_\sigma$ that $\chi_x \in C_\kappa$ for all $x \in X$. The mapping $h: X \longrightarrow C_\kappa$ given by $h(x) = \chi_x$ is the required embedding. To see this it suffices to note that $h^{-1}(W_\sigma) = U_\sigma$ for all $\sigma \in \Sigma(\kappa)$.

The proof of Theorem 1 is complete.

REMARK 1. It can be proved (cf. [7]) that for any $\kappa$ there exists no space universal for all $T_1$-spaces of weight $\leq \kappa$.

REMARK 2. Some similar selection of a suitable subspace of $F^{\Sigma(\kappa)}$ leads to a universal regular space of weight $\kappa$ (cf. [7]).

https://doi.org/10.4153/CMB-1992-037-1 Published online by Cambridge University Press

3. **The number of compact spaces.** It follows directly from the Lemma that every $T_1$-space of weight $\kappa$ can have only $2^\kappa$ compact subspaces. This and Theorem 1 imply immediately the following:

THEOREM 2. *For every $\kappa$ there are at most $2^\kappa$ topologically distinct compact $T_1$-spaces.*

REMARK 1. The above estimation is the best possible. In fact, as proved in [6], for every $\kappa$ there exists a family of $2^\kappa$ pairwise nonhomeomorphic connected compact $T_2$-spaces. Moreover, if $\kappa > \omega$, then the family in question can consist of connected compact linearly ordered spaces (cf. [8]).

REMARK 2. For the class of compact $T_0$-spaces the estimation analogous to that of Theorem 1 is not valid. Actually, for every $\kappa$ there exists a family consisting of $2^{2^\kappa}$ pairwise nonhomeomorphic compact $T_0$-spaces of weight $\kappa$.

To obtain such a family observe first that for every space $X$ of density $\leq \kappa$ there are at most $2^\kappa$ continuous mappings from $X$ to the cube $[0,1]^\kappa$. It follows that every subspace of $[0,1]^\kappa$ cannot be homeomorphic to more than $2^\kappa$ of the other subspaces and, therefore, there exists a family $\mathcal{X}$ consisting of $2^{2^\kappa}$ pairwise nonhomeomorphic subspaces of $[0,1]^\kappa$.

Choose a point $* \notin [0,1]^\kappa$ and for every $X \in \mathcal{X}$ consider the space $X \cup \{*\}$ endowed with the topology consisting of $X \cup \{*\}$ and all open subsets of $X$. It is easy to see that $\{X \cup \{*\} : X \in \mathcal{X}\}$ forms the required family.

4. **What can be embedded into $C_\kappa$.** Let us say that a topological space $X$ is $\kappa$-*perfect* if every open set in $X$ is a union of $\leq \kappa$ closed sets. The answer to the question in the title is given by the following:

THEOREM 3. *For a $T_1$-space $X$ of weight $\leq \kappa$ the following conditions are equivalent:*

*(a) $X$ is homeomorphic to some subspace of $C_\kappa$,*
*(b) $X$ has a $T_1$-compactification of weight $\leq \kappa$,*
*(c) $X$ has a closed network of cardinality $\leq \kappa$,*
*(d) $X$ is $\kappa$-perfect.*

PROOF. Implications (a)$\Rightarrow$(b) and (c)$\Rightarrow$(d) are obvious.

To prove (b)$\Rightarrow$(c) let $cX$ be a $T_1$-compactification of $X$ with weight $\leq \kappa$ and let $\mathcal{B}$ be an open base for $cX$ such that $|\mathcal{B}| \leq \kappa$ and $cX - U = \bigcap\{V \in \mathcal{B} : U \cup V = cX\}$ for every $U \in \mathcal{B}$ (see the Lemma again). The family $\{X - U : U \in \mathcal{B}\}$ is easily seen to be a required closed network for $X$.

To prove (d)$\Rightarrow$(a) assume that $X$ is a $\kappa$-perfect space. It is not a difficult task to find a (sub)base $\mathcal{P}$ for $X$ such that $|\mathcal{P}| \leq \kappa$, $X \in \mathcal{P}$ and $X - U = \bigcap\{V \in \mathcal{P} : U \cup V = X\}$ for every $U \in \mathcal{P}$. Yet, such a subbase for $X$ is all we need to construct an embedding of $X$ into $C_\kappa$ (compare the proof of Theorem 1).

The equivalence of (a) and (d) can be reformulated as follows.

COROLLARY 1. *The space $C_\kappa$ is universal for all $\kappa$-perfect $T_1$-spaces of weight $\leq \kappa$.*

Since a second countable $T_1$-space is developable if and only if it is perfect ($= \omega$-perfect), we have the following:

COROLLARY 2. *The space $C_\omega$ is universal for all second coutable developable $T_1$-spaces.*

REMARK 1. It can be shown that there exists no universal space for second countable developable Hausdorff spaces. Note, however, that $[0,1]^\omega$ is universal for all second countable developable regular spaces.

REMARK 2. The equivalence (b)$\Leftrightarrow$(c) of Theorem 3 has been obtained earlier in [1].

5. **On closely universal compact spaces.** Since compact subspaces of the Tychonoff cube $[0,1]^\kappa$ are closed, it is closely universal for all compact $T_2$-spaces of weight $\leq \kappa$. In the class of compact $T_0$-spaces closely universal spaces do not exist. In order to see this compare Remark 2 of Section 2 and notice that a space of weight $\kappa$ can have at most $2^\kappa$ closed subspaces.

The above two observations lead directly to the following

PROBLEM. Is there, for every (or some) $\kappa$, a closely universal space for all compact $T_1$-spaces of weight $\leq \kappa$?

The problem remains open. As it follows from the example given at the end of the paper, the universal spaces $C_\kappa$ cannot serve as closely universal. Meanwhile, they enable us to give the following:

THEOREM 4. *For every $\kappa$ there exists a $T_1$-space $S_\kappa$ of weight $\kappa$ such that every compact $T_1$-space of weight $\leq \kappa$ is homeomorphic to some closed subspace of $S_\kappa$.*

PROOF. The family $\mathcal{C}$ of all compact subspaces of $C_\kappa$ has only $2^\kappa$ elements. Arrange $\mathcal{C}$ as $\{C_x : x \in [0,1]^\kappa\}$ and let $S_\kappa$ be a subspace of $[0,1]^\kappa \times C_\kappa$ consisting of those pairs $(x,y)$ in which $y \in C_x$. Clearly $S_\kappa$ has weight $\kappa$.

Let $X$ be an arbitrary compact $T_1$-space of weight $\leq \kappa$. In view of Theorem 1 there exists $x \in [0,1]^\kappa$ such that the spaces $X$ and $C_x$ are homeomorphic. The space $X$ is therefore homeomorphic to the closed subspace $\{x\} \times C_x$ of $S_\kappa$.

REMARK. Observe that the spaces $S_\kappa$ are not compact and therefore do not provide a solution to our Problem. On the other hand, the answer would be positive if one could succeed in finding a $T_1$-compactification of weight $\kappa$ of $S_\kappa$ in which all the subspaces $\{x\} \times C_x$ remain closed.

The following example shows that closely universal compact $T_1$-spaces (if any) cannot be supercompact.

EXAMPLE. There is a second countable compact $T_1$-space $Y$ which is not homeomorphic to any closed subspace of any supercompact space.

Denote by $N$ the set of natural numbers with co-finite topology. Our space $Y$ is the space $N \times N$ with the diagonal $\Delta = \{(n,n) : n \in N\}$ as an additional closed set. It is easy to check that $Y$ is a second countable compact $T_1$-space. We show that $Y$ cannot be homeomorphic to a closed subspace of any supercompact space.

Let us start with the following observation.

If $X_0$ is a closed subspace of a space $X$ and every two nonempty open subsets of $X_0$ have nonempty intersection, then

(a) for every open subbase $\mathcal{P}$ for $X$ the complement $X - X_0$ can be expressed as a union of members of $\mathcal{P}$,

(b) if $X$ is supercompact then so is $X_0$.

For the proof of (a) observe that, since every finite family of nonempty open subsets of $X_0$ has nonempty intersection, there must exist, for every $x \in X - X_0$, a member $U$ of $\mathcal{P}$ with $x \in U$ and $U \cap X_0 = \emptyset$.

For the proof of (b) consider a binary open subbase $\mathcal{P}$ for $X$. *Binary* means here that every nontrivial open cover of $X$ by members of $\mathcal{P}$ contains a two-element subcover. With the use of (a), it is easy to verify that the family $\{U \cap X_0 : U \in \mathcal{P}\}$ forms a binary open subbase for $X_0$.

Now return to our space $Y$. Since every two nonempty open subsets of $Y$ have nonempty intersection, it suffices to show that $Y$ is not supercompact, i.e. no open subbase for $Y$ is binary.

Let $\mathcal{P}$ be an arbitrary open subbase for $Y$. By virtue of (a) there exists $\mathcal{U} \subseteq \mathcal{P}$ such that $Y - \Delta = \bigcup \mathcal{U}$. Similarly, there exist $\mathcal{V} \subseteq \mathcal{P}$ and $\mathcal{W} \subseteq \mathcal{P}$ such that $Y - (\{1\} \times N) = \bigcup \mathcal{V}$ and $Y - (N \times \{2\}) = \bigcup \mathcal{W}$. It is easy to see that the open cover $\mathcal{U} \cup \mathcal{V} \cup \mathcal{W} \subseteq \mathcal{P}$ of $Y$ contains no two-element subcover.

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