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MEASURES ON EFFECT ALGEBRAS

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Abstract

In this paper, by introducing the bounded variation measure defined on effect algebras, we present the equivalent conditions about uniformly strongly additive measures.

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1. Preliminaries

Foulis and Bennett in 1994 introduced the algebraic system $(L, \oplus, 0, 1)$ to model unsharp quantum logics, and $(L, \oplus, 0, 1)$ is said to be an *effect algebra* [1].

Let L be a set with two special elements 0, 1, and let $\perp$ be a subset of $L \times L$. If $(a, b) \in \perp$, write $a \perp b$. Let $\oplus : \perp \to L$ be a binary operation. Suppose that the following axioms hold.

(E1) If $a, b \in L$ and $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$ (commutative law).
(E2) If $a, b, c \in L$, $a \perp b$ and $(a \oplus b) \perp c$, then we have $b \perp c$, $a \perp (b \oplus c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associative law).
(E3) For every $a \in L$ there exists a unique $b \in L$ such that $a \perp b$ and $a \oplus b = 1$ (orthosupplementation law).
(E4) If $a \in L$ and $1 \perp a$, then $a = 0$ (zero-one law).

Then $(L, \oplus, 0, 1)$ is called an effect algebra.

If $a, b \in L$ and $a \perp b$, we say that a and b are othogonal. If $a \oplus b = 1$ we say that b is the orthosupplementation of a, and we write $b = a'$. Clearly $1' = 0$, $(a')' = a$, $a \perp 0$ and $a \oplus 0 = a$ for all $a \in L$. We say that $a \leq b$ if there exists $c \in L$ such that $a \perp c$ and $a \oplus c = b$. We know that $\leq$ is a partial order on L and satisfies the conditions that $0 \leq a \leq 1$, $a \leq b \Leftrightarrow b' \leq a'$ and $a \leq b' \Leftrightarrow a \perp b$ for $a, b \in L$.

If $a \leq b$, the element $c \in L$ such that $c \perp a$ and $c \oplus a = b$ is unique, and satisfies the condition $c = (a \oplus b')'$. It will be denoted $c = b \ominus a$.

Let $F = \{a_i : 1 \le i \le n\}$ be a finite subset of L. If $a_1 \perp a_2$, $(a_1 \oplus a_2) \perp a_3, \ldots$ and $(a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \perp a_n$, we say that F is orthogonal and we define

$$\bigoplus F = a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1} \oplus a_n = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n.$$

Now, if A is an arbitrary subset of L and $\mathcal{F}(A)$ is the family of all finite subsets of A, we say that A is orthogonal if F is orthogonal for every $F \in \mathcal{F}(A)$. If A is orthogonal, we define

$$\bigoplus A = \bigvee \Big\{ \bigoplus F : F \in \mathcal{F}(A) \Big\}.$$

Moreover, let $(a_i)_{i \in I}$ be an orthogonal subset of L. Then we know that [3]:

(1) if I is finite and $J \subseteq I$, then

$$\Big(\bigoplus_{i \in J} a_i \Big) \perp \Big(\bigoplus_{i \in I \setminus J} a_i \Big)$$

and

$$\bigoplus_{i \in I} a_i = \Big(\bigoplus_{i \in J} a_i \Big) \oplus \Big(\bigoplus_{i \in I \setminus J} a_i \Big);$$

(2) if $J \subseteq I$ and there exist $a = \bigoplus_{i \in I} a_i$, $b = \bigoplus_{i \in J} a_i$ and $c = \bigoplus_{i \in I \setminus J} a_i$, then $b \perp c$ and $a = b \oplus c$;

(3) if there exists $\bigoplus_{i \in M} a_i$ for all $M \subseteq I$ and $\{H_j : j \in J\}$ is a partition of I, then $A = \{\bigoplus_{i \in H_j} a_i : j \in J\}$ is an orthogonal subset of L, there exists $\oplus A$ and $\oplus A = \bigoplus_{i \in I} a_i$;

(4) if $(F_j)_{j \in J}$ is a family of finite and pairwise disjoint subsets of I, then the set $\{\bigoplus_{i \in F_j} a_i : j \in J\}$ is an orthogonal subset of L;

(5) if $b_i \in L$ and $b_i \le a_i$ for $i \in I$, then $(b_i)_{i \in I}$ is an orthogonal subset of L.

In what follows, let L be an effect algebra, X be a Banach space and $\mu : L \to X$ be a vector measure. The variation of μ is the nonnegative function $|\mu|$ whose value on an element $a \in L$ is given by

$$|\mu|(a) = \sup_{\Delta} \sum_{a_j \in \Delta} \|\mu(a_j)\|,$$

where $\Delta = \{a_1, a_2, \ldots, a_n\}$ such that $a_1 \oplus a_2 \oplus \cdots \oplus a_n = a$, $a_j \in L$ for all $j = 1, 2, \ldots, n$.

If $|\mu|(1) < \infty$, we call μ a measure of bounded variation.

The semivariation of μ is the nonnegative function $\|\mu\|$ whose value on an element $a \in L$ is defined by

$$\|\mu\|(a) = \sup\{|x^*\mu|(a) : x^* \in X^*, \ \|x^*\| \le 1\},$$

where $|x^*\mu|$ is the variation of the real-valued measure $x^*\mu$.

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If $\|\mu\|(1) < \infty$, we call μ a measure of bounded semivariation.

$\mu : L \to X$ is said to be strongly additive if, for any orthogonal sequence (a_n) of L, the series $\sum_{n=1}^{\infty} \mu(a_n)$ converges in X.

A family of strongly additive vector measures $\{\mu_\tau : \tau \in T\}$ is said to be uniformly strongly additive if, for any orthogonal sequence (a_n) of L, the series

$$\lim_{m} \left\| \sum_{n=m}^{\infty} \mu_\tau(a_n) \right\| = 0,$$

uniformly in $\tau \in T$.

$\mu : L \to X$ is said to be bounded if, for any orthogonal sequence (a_n) of L, $\{\mu(a_n)\}_{n=1}^{\infty}$ is bounded.

$\mu : L \to X$ is said to be strongly bounded if, for any orthogonal sequence (a_n) of L, $\lim_{n\to\infty} \mu(a_n) = 0$.

$\mu : L \to X$ is said to be countably additive if, for any orthogonal sequence (a_n) of L,

$$\mu\left(\bigoplus_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n).$$

Clearly, a strongly additive vector measure on effect algebras is strongly bounded and a strongly bounded vector measure on effect algebras is bounded.

2. Main results

PROPOSITION 1. *Let $\mu : L \to X$ be a vector measure. Then, for $a \in L$,*

$$\|\mu\|(a) = \sup_{\Delta} \left\{ \left\| \sum_{j} \varepsilon_j \mu(a_j) \right\| \right\},$$

where $\Delta = \{a_1, a_2, \ldots, a_n\}$ such that $a_1 \oplus a_2 \oplus \cdots \oplus a_n = a$, $a_j \in L$ for all $j = 1, 2, \ldots, n$ and $|\varepsilon_j| \le 1$.

PROOF. If $a = a_1 \oplus a_2 \oplus \cdots \oplus a_n$, $\{a_1, a_2, \ldots, a_n\}$ is a partition of a into orthogonal members of L and ε_j are scalars such that $|\varepsilon_j| \le 1$, then

$$\begin{aligned}
\left\| \sum_{j=1}^{m} \varepsilon_j \mu(a_j) \right\| &= \sup\left\{ \left| x^*\left(\sum_{j=1}^{m} \varepsilon_j \mu(a_j) \right) \right| : x^* \in X^*, \ \|x^*\| \le 1 \right\} \\
&\le \sup\left\{ \sum_{j=1}^{m} |\varepsilon_j x^* \mu(a_j)| : x^* \in X^*, \ \|x^*\| \le 1 \right\} \\
&\le \sup\left\{ \sum_{j=1}^{m} |x^* \mu(a_j)| : x^* \in X^*, \ \|x^*\| \le 1 \right\} \\
&\le \sup\{ |x^* \mu|(a) : x^* \in X^*, \ \|x^*\| \le 1 \} \\
&= \|\mu\|(a).
\end{aligned}$$

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On the other hand, let $x^* \in X^*$ with $\|x^*\| \le 1$ and $a = a_1 \oplus a_2 \oplus \cdots \oplus a_n$, $\{a_1, a_2, \ldots, a_n\}$ be a partition of a into orthogonal members of L. Then

$$\begin{aligned}\sum_{j=1}^{m} |x^*\mu(a_j)| &= \sum_{j=1}^{m}(\operatorname{sgn} x^*\mu(a_j))x^*\mu(a_j)\\ &= x^*\bigg(\sum_{j=1}^{m}(\operatorname{sgn} x^*\mu(a_j))\mu(a_j)\bigg)\\ &\le \bigg\|\sum_{j=1}^{m} \varepsilon_j\mu(a_j)\bigg\|\\ &\le \sup_{\Delta}\bigg\|\sum_{j} \varepsilon_j\mu(a_j)\bigg\|.\end{aligned}$$

This proves the result. □

PROPOSITION 2. *Let $\mu : L \to X$ be a vector measure. Then*

$$\sup\{\|\mu(h)\| : h \le e,\ h \in L\} \le \|\mu\|(e) \le 4\sup\{\|\mu(h)\| : h \le e,\ h \in L\}.$$

PROOF. For any $e \in L$,

$$\begin{aligned}\sup\{\|\mu(h)\| : h \le e,\ h \in L\} &= \sup\{\sup\{|x^*\mu(h)| : x^* \in X^*,\ \|x^*\| \le 1\} : h \le e,\ h \in L\}\\ &\le \|\mu\|(e).\end{aligned}$$

On the other hand, let $x^* \in X^*$ with $\|x^*\| \le 1$ and $e = e_1 \oplus e_2 \oplus \cdots \oplus e_n$, $\{e_1, e_2, \ldots, e_n\}$ be a partition of e into orthogonal members of L. Then

$$\begin{aligned}\sum_{i=1}^{n} |x^*\mu(e_i)| &= \sum_{i\in M^+} x^*\mu(e_i) - \sum_{i\in M^-} x^*\mu(e_i)\\ &= x^*\bigg(\sum_{i\in M^+} \mu(e_i)\bigg) - x^*\bigg(\sum_{i\in M^-} \mu(e_i)\bigg)\\ &\le 2\sup\{\|\mu(h)\| : h \le e,\ h \in L\},\end{aligned}$$

where

$$M^+ = \{i : x^*\mu(e_i) \ge 0,\ 1 \le i \le n\} \quad \text{and} \quad M^- = \{i : x^*\mu(e_i) < 0,\ 1 \le i \le n\}.$$

If X is a complex Banach space, it is easy to see that a similar estimate holds if the number 2 is replaced by the number 4.

Consequently, a vector measure is of bounded semivariation on L if and only if its range is bounded in X. □

THEOREM 3. *Let $\mu_\tau : L \to X$, $\tau \in T$, be a family of vector measures. The following statements are equivalent.*

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(I) *$\{\mu_\tau : \tau \in T\}$ is uniformly strongly additive.*
(II) *$\{x^*\mu_\tau : \tau \in T,\, x^* \in X^*,\, \|x^*\| \le 1\}$ is uniformly strongly additive.*
(III) *If (a_n) is a sequence of orthogonal members of L, then $\lim_{n\to\infty} \|\mu_\tau(a_n)\| = 0$ uniformly in $\tau \in T$.*
(IV) *If (a_n) is a sequence of orthogonal members of L, then $\lim_{n\to\infty} \|\mu_\tau\|(a_n) = 0$ uniformly in $\tau \in T$.*
(V) *$\{|x^*\mu_\tau| : \tau \in T,\, x^* \in X^*,\, \|x^*\| \le 1\}$ is uniformly strongly additive.*

PROOF. (I) ⇒ (II), (II) ⇒ (III), and (V) ⇒ (I) are obvious.

(III) ⇒ (IV) If not, there exist a $\delta > 0$ and an orthogonal sequence (a_n) of L such that $\sup_{\tau\in T} \|\mu_\tau\|(a_n) \ge 4\delta > 0$ holds for all $n \in N$. By Proposition 1, for every n there is an $h_n \in L$ such that $h_n \le a_n$ and $\sup_{\tau\in T} \|\mu_\tau\|(a_n) \le 4 \sup_{\tau\in T} \|\mu_\tau(h_n)\|$. The sequence (h_n) is orthogonal such that

$$\sup_{\tau\in T} \|\mu_\tau(h_n)\| \ge \delta > 0,$$

for every $n \in N$. This shows that (III) implies (IV).

(IV) ⇒ (V). Suppose that $\{|x^*\mu_\tau| : \tau \in T,\, x^* \in X^*,\, \|x^*\| \le 1\}$ is not uniformly strongly additive. Then there exist an orthogonal sequence (a_n) of L and a $\delta > 0$ such that, for all $m \in N$,

$$\sup\left\{\sum_{n=m}^{\infty} |x^*\mu_\tau|(a_n) : \tau \in T,\, x^* \in X^*,\, \|x^*\| \le 1\right\} \ge 2\delta > 0.$$

Thus there is an increasing sequence (m_j) of positive integers such that, for all j,

$$\begin{aligned}\sup&\left\{\sum_{n=m_j+1}^{m_{j+1}} |x^*\mu_\tau|(a_n) : \tau \in T,\, x^* \in X^*,\, \|x^*\| \le 1\right\}\\ &= \sup\left\{|x^*\mu_\tau|\left(\bigoplus_{n=m_j+1}^{m_{j+1}} a_n\right) : \tau \in T,\, x^* \in X^*,\, \|x^*\| \le 1\right\} \ge \delta > 0.\end{aligned}$$

Therefore, putting

$$h_j = \bigoplus_{n=m_j+1}^{m_{j+1}} a_n,$$

(h_n) is an orthogonal sequence of L such that

$$\sup\{\|\mu_\tau\|(h_j) : \tau \in T\} = \sup\{|x^*\mu_\tau|(h_j) : \tau \in T,\, x^* \in X^*,\, \|x^*\| \le 1\} \ge \delta > 0.$$

This leads to a contradiction. So (V) holds. □

COROLLARY 4. *Let $\mu : L \to X$ be a vector measure. The following statements are equivalent.*

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(I) *μ is strongly additive.*
(II) $\{x^*\mu : x^* \in X^*,\ \|x^*\| \le 1\}$ *is uniformly strongly additive.*
(III) *μ is strongly bounded, that is, if (a_n) is an orthogonal sequence of members of L, then $\lim_{n\to\infty} \mu(a_n) = 0$.*
(IV) *$\|\mu\|$ is strongly bounded, that is, if (a_n) is an orthogonal sequence of members of L, then $\lim_{n\to\infty} \|\mu\|(a_n) = 0$.*
(V) $\{|x^*\mu| : x^* \in X^*,\ \|x^*\| \le 1\}$ *is uniformly strongly additive.*
(VI) $\lim_n \mu(a_n)$ *exists for every nondecreasing monotone sequence (a_n) of L.*
(VII) $\lim_n \mu(a_n)$ *exists for every nonincreasing monotone sequence (a_n) of L.*

PROOF. The equivalence of (I)–(V) is clear from Theorem 3. And it is also clear that (VI) is equivalent to (VII).

(I) $\Rightarrow$ (VI). Let (a_n) be an orthogonal sequence of L satisfying $a_1 \le a_2 \le \cdots \le a_n$, and let $c_n = a_n \ominus a_{n-1}$. Then

$$\lim_n \mu(a_n) = \mu(a_1) + \lim_n \sum_{n=2}^{\infty} \mu(a_n \ominus a_{n-1})$$

exists since the sequence $(c_n)_{n=2}^{\infty}$ is an orthogonal sequence of L.

On the other hand, let $(a_n) \subseteq L$ be an orthogonal sequence, $b_k = \bigoplus_{n=1}^{k} a_n$ for $k \in N$. Then (b_k) is a nondecreasing sequence of L. Then

$$\lim_n \mu(a_n) = \lim_n \left[\mu\left(\bigoplus_{n=1}^{k} a_n\right) - \mu\left(\bigoplus_{n=1}^{k-1} a_n\right)\right] = 0.$$

This completes the proof. □

THEOREM 5. *Let $\mu : L \to X$ be a bounded vector measure. If L satisfies the finite chain condition, that is, no infinite subcollection of L can be orthogonal, then μ is countably additive.*

PROOF. Suppose that L satisfies the finite chain condition, and (a_n) is an orthogonal sequence of L; then $a_n = 0$ for all large $n \ge n_0$, $n_0 \in N$. Hence by finite additivity of μ,

$$\mu\left(\bigoplus_{n=1}^{\infty} a_n\right) = \mu\left(\bigoplus_{n=1}^{n_0} a_n\right) = \sum_{n=1}^{n_0} \mu(a_n) = \sum_{n=1}^{\infty} \mu(a_n),$$

and μ is countably additive. □

THEOREM 6. *If X is a Banach space containing no copy of c_0, $\mu : L \to X$ is a bounded vector measure, then μ is strongly additive.*

PROOF. Since μ is bounded, for every orthogonal sequence (a_n) of L the series $\{\sum_{n=1}^{m} \mu(a_n)\}_{m=1}^{\infty}$ is weakly unconditional Cauchy [2], that is, $\sum_{n=1}^{\infty} |x^*\mu(a_n)| < \infty$, for any $x^* \in X^*$. Therefore, $(\mu(a_j))_j$ is c_0-multiplier convergent. Since X contains no copy of c_0, then $\sum_{n=1}^{\infty} \mu(a_n)$ convergent. Thus μ is strongly additive. □

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