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On quotients of certain countable groups

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A group A is said to be SQ-universal if every countable group is embeddable in some quotient of A. It is well-known that the number of non-isomorphic factor groups of a countable SQ-universal group B is the power of the continuum. Since B has such an abundance of non-isomorphic quotients it is natural to ask what types of quotients B must have and what types B need not have. This note is concerned with these questions.

1. Introduction

The study of SQ-universal groups has received the attention of several authors in recent years (see [2] and the references cited there). Neumann [1] has observed that since there are $2^{\aleph_0}$ non-isomorphic finitely generated groups, a countable SQ-universal group must have $2^{\aleph_0}$ non-isomorphic quotients. By a similar argument it is not difficult to establish that *if* P *is a group-theoretic property which is preserved under taking subgroups and if there are* $2^{\aleph_0}$ *non-isomorphic finitely generated groups without* P *then any countable* SQ-*universal group has* $2^{\aleph_0}$ *non-isomorphic quotients without* P .

The above result gives information concerning the type of quotients a countable SQ-universal group *must* have. One can ask in the reverse

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direction what type of quotients such a group *need not* have. For example, it is known that there are countable (indeed finitely presented) SQ-universal groups which have no finite quotients apart from the trivial group [3]. A method is given here for constructing *two-generator* SQ-*universal groups which do not have any homomorphic images different from* 1 *satisfying a non-trivial law.*

In view of the fact that there are $2^{\aleph_0}$ non-isomorphic finitely generated groups containing a copy of the free group F_2 of rank 2, it is natural to ask whether there exist countable SQ-universal groups with every homomorphic image different from 1 containing a copy of F_2. I have not been able to answer this question.

2. The construction

Let $F = A * B$, where A and B are infinite cyclic groups generated by a and b respectively. The idea is to construct small cancellation quotients of F which have no homomorphic images different from 1 satisfying a non-trivial law, and then use small cancellation theory to establish SQ-universality. The reader unfamiliar with small cancellation theory should consult [3].

Every element of F can be written uniquely in the form

$$b^{\eta_0}a^{\xi_1}b^{\eta_1} \dots a^{\xi_l}b^{\eta_l} \quad (l \geq 0) ,$$

where the ξ_i are non-zero integers and the η_i are integers, non-zero except possibly for η_0 and η_l. The ξ_i will be called the *a-exponents* and the η_i the *b-exponents*.

Let $W_1, W_2, \dots$ be an enumeration of all the non-empty freely reduced words in variables $x_1, x_2, \dots$. Construct inductively a sequence $U_0, V_0, w_1, U_1, V_1, w_2, U_2, V_2, \dots$ of elements of F as follows.

Take $U_0 = aba^2b^2 \dots a^{10}b^{10}$ and $V_0 = a^{-1}ba^{-2}b^2 \dots a^{-10}b^{10}$.

Now suppose $r > 0$, and assume that

$$U_0,\ V_0,\ w_1,\ U_1,\ V_1,\ \ldots,\ w_{r-1},\ U_{r-1},\ V_{r-1}$$

have been constructed. Choose p to be greater than the moduli of all the a-exponents and all the b-exponents in U_{r-1}, V_{r-1}, and substitute $b^{-ip}a^p b^{ip}$ for x_i $(i = 1, 2, \ldots)$ in W_r. Let w_r be the normal form of the resulting word. Note that w_r begins and ends with b-symbols, and moreover all the exponents in w_r are divisible by p. Now let

$$U_r = a^{\alpha_1} w_r a^{\alpha_2} w_r \ldots a^{\alpha_s} w_r ,$$

$$V_r = a^{\gamma_1} b a^{\gamma_2} w_r \ldots a^{\gamma_s} w_r ,$$

where

(1) $s \geq 9$,

(2) all the numbers $|\alpha_1|, \ldots, |\alpha_s|, |\gamma_1|, \ldots, |\gamma_s|$ are distinct, and each is greater than all the a-exponents in w_r,

(3) $\sum \alpha_i = 1$, $\sum \gamma_i = 0$.

Let

$$G = \langle a, b;\ U_1, V_1, U_2, V_2, \ldots \rangle .$$

Then no homomorphic image of G different from 1 satisfies a non-trivial law. For suppose $W_r = 1$ were a law in a quotient $\bar{G}$ of G. Then in particular, w_r would be a consequence of the relators of $\bar{G}$. Thus w_r, U_r, V_r would all define the identity in $\bar{G}$, and so $a = b = 1$ in $\bar{G}$ by (3). Hence $\bar{G} = 1$.

The group G is SQ-universal. To show this, it suffices to verify that any two-generator group C can be embedded into a quotient of G, for it is well-known that any countable group can be embedded in a two-generator group. Suppose C is generated by h and k, and let $K = A * B * C$. Let

$$R = \{hU_0, kV_0, U_1, V_1, U_2, V_2, \ldots\} .$$

Taking account of (1) and (2), it is easily checked that the smallest symmetrized set containing R satisfies $C'(1/6)$, and so the factor group of K by the normal closure N of R is a small cancellation product of A, B, C . In particular, C is embedded into K/N . Now K/N is a homomorphic image of G . For in K/N the generators of C are expressed in terms of a and b , and so all occurrences of h and k can be eliminated using Tietze transformations.

References

[1] Peter M. Neumann, "The SQ-universality of some finitely presented groups", *J. Austral. Math. Soc.* **16** (1973), 1-6.

[2] Paul E. Schupp, "A survey of SQ-universality", *Conference on group theory*, University of Wisconsin-Parkside, 1972, 183-188 (Lecture Notes in Mathematics, **319**. Springer-Verlag, Berlin, Heidelberg, New York, 1973).

[3] Paul E. Schupp, "A survey of small cancellation theory", *Word problems: decision problems and the Burnside problem in group theory*, 569-589 (Studies in Logic and the Foundations of Mathematics, **71**. North-Holland, Amsterdam, London, 1973).

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