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S_3-FREE 2-FUSION SYSTEMS

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Abstract We develop a theory of 2-fusion systems of even characteristic, and use that theory to show that all S_3-free saturated 2-fusion systems are constrained. This supplies a new proof of Glauberman's Theorem on S_4-free groups and its various corollaries.

Keywords: finite groups; fusion systems; Sylow subgroups

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Introduction

Let G be a finite group. A *section* of G is a group of the form H/K, where $K \trianglelefteq H \leqslant G$. Given a positive integer n, write S_n for the symmetric group on a set of order n. The group G is S_n*-free* if G has no section isomorphic to S_n.

One of the striking results in finite group theory from the decade before the classification of the finite simple groups was the classification of the S_4-free finite simple groups. The first result in this direction was Thompson's proof (cf. [**17**]) that the Suzuki groups $\mathrm{Sz}(2^{2m+1})$ are the only non-abelian finite simple groups of order prime to 3. This problem was a natural first place for Thompson to test how well the methods he had developed in the N-group paper extend to groups with non-solvable local subgroups, since the only non-abelian composition factors in locals of a minimal counter-example to his theorem are Suzuki groups.

Next, in [**9**], Glauberman proved a triple factorization theorem for constrained S_4-free groups, and used this result to show that each S_4-free group has a non-trivial strongly closed abelian 2-subgroup. As a corollary to this result and Goldschmidt's Theorem in [**10**] classifying finite groups with such a subgroup, Glauberman classified the S_4-free and S_3-free non-abelian finite simple groups. Later, in [**16**], Stellmacher showed that in a constrained S_4-free group G there is a non-trivial characteristic subgroup of a Sylow 2-subgroup of G normal in G; this theorem can be used to give an alternate treatment of S_4-free groups.

Attempts to extend the Glauberman triple factorization to more general 2-constrained groups have to date been unsuccessful. On the other hand, other extensions of Thompson factorization (such as the so-called *amalgam method*) have proved effective; see, for example, [**5**].

In this paper we give yet another treatment of S_4-free groups, this time from the point of view of 2-fusion. Actually, we consider 2-fusion systems rather than groups, and prove that all saturated S_3-free 2-fusion systems are constrained. Then we derive our theorems about groups from this theorem about fusion systems and Goldschmidt's Theorem. Indeed, from our point of view, the main object of the paper is to put in place machinery to study fusion systems of even characteristic, building on earlier results in [**3**]. Given this machinery, the proof of the theorem on S_3-free fusion systems takes only a few paragraphs, making possible a very elegant treatment of S_4-free groups. We regard this work as a test case to evaluate the efficacy of our machinery to, first, analyse 2-fusion systems of even characteristic and, second, to use the theorems on fusion systems to prove results about finite groups.

One of the main tools in our approach is the use of the extensions of Thompson factorization mentioned above. But it also seems to be true that local analysis is sometimes easier in the category of saturated fusion systems than in the category of groups; deriving the S_4-Free Group Theorem from a result on fusion systems provides an example of this phenomenon.

It should be pointed out that, in [**13**], Onofrei and Stancu use Stellmacher's Theorem on S_4-free groups to prove a stronger result on S_3-free fusion systems than our main theorem. The virtue of our approach is that, once the general machinery on fusion systems is in place, a simple proof of the weaker theorem on fusion systems is possible, which, given Goldschmidt's Theorem, immediately leads to the theorems on groups. That is to say, we are using results on fusion systems to prove theorems about groups, rather than vice versa.

Fusion systems were defined and first studied by L. Puig, although Puig calls these objects *Frobenius categories* rather than fusion systems; see in particular [**14**, **15**]. Our introduction to the subject was from [**8**], and we adopt the notation and terminology found there.

The reader is directed to [**1**] for notation and terminology involving finite groups, and to [**8**, § 1 and Appendix A] or to [**6**] for notation and terminology involving fusion systems. Section 1 of this paper also contains some background on fusion systems.

Let $\mathcal{F}$ be a saturated fusion system on a finite 2-group S. We say that $\mathcal{F}$ is *S_3-free* if, for each subgroup U of S, the group $\mathrm{Aut}_{\mathcal{F}}(U)$ is S_3-free. Our result on S_3-free fusion systems is as follows.

Theorem 1. Let $\mathcal{F}$ be a saturated S_3-free fusion system on a finite 2-group S. Then $\mathcal{F}$ is constrained.

The result on fusion systems, together with Goldschmidt's Theorem, leads almost immediately to the following corollaries for groups.

Corollary 2. Assume that G is a finite S_4-free non-abelian finite simple group. Then G is a Goldschmidt group.

A finite simple group G is a *Goldschmidt group* if either G is a group of Lie type of Lie rank 1 in characteristic 2 or G has abelian Sylow 2-subgroups. The groups of the first type are the groups $L_2(2^n)$, $\mathrm{Sz}(2^n)$ and $U_3(2^n)$, $n > 1$. The non-abelian simple groups with abelian Sylow 2-subgroups (other than the groups $L_2(2^n)$) are the groups $L_2(q)$, $q \equiv \pm 3 \bmod 4$, ${}^2G_2(3^{2m+1})$, $m \geqslant 1$ and J_1.

Corollary 3. Assume that G is a finite S_3-free non-abelian finite simple group. Then G is $\mathrm{Sz}(2^{2m+1})$ or $L_2(3^{2m+1})$, with $m \geqslant 1$.

Corollary 4. Assume that G is a finite non-abelian simple group of order prime to 3. Then $G \cong \mathrm{Sz}(2^{2m+1})$ for some $m \geqslant 1$.

Section 1 contains a brief discussion of fusion systems. In §§ 2–4 we put in place machinery for studying saturated fusions systems $\mathcal{F}$ on finite 2-groups S with $\mathcal{F}$ of even characteristic (as defined in § 1). Then in § 5, we use this machinery to prove a version of Stellmacher's qrc-Lemma for fusion systems, together with a few supporting lemmas which make the qrc-Lemma more effective. Section 6 contains some lemmas on representations of finite groups over fields of even characteristic that are needed to apply the qrc-Lemma. Finally, the main theorem and its corollaries are proved in § 7.

1. Fusion systems

In this section p is a prime, S is a finite p-group and $\mathcal{F}$ is a saturated fusion system on S.

The definition of a *fusion system* appears in [**8**, Definition 1.1] and [**6**, Definition I.2.1]. The definition of a *saturated fusion system* appears in [**8**, Definition 1.2] and [**6**, Definition I.2.2]. Roughly speaking, a fusion system on S is a category whose objects are the subgroups of S, and such that the set $\hom_{\mathcal{F}}(P,Q)$ of morphisms between subgroups P and Q of S is a set of injective group homomorphisms from P into Q satisfying some weak axioms. If G is a finite group and $S \in \mathrm{Syl}_p(G)$, then $\mathcal{F}_S(G)$ is the fusion system on $\mathcal{F}$ such that $\hom_{\mathcal{F}_S(G)}(P,Q)$ consists of the conjugation maps $c_g\colon x \mapsto x^g$ for $g \in G$ with $P^g \leqslant Q$. Again, roughly speaking, $\mathcal{F}$ is saturated if it satisfies some axioms that are easily verified for $\mathcal{F}_S(G)$ using Sylow's Theorem.

The reader is referred to [**8**, § 1 and Appendix A] or [**6**, Part I] for notation, terminology and basic results about fusion systems. However, we also record some of this notation and terminology in this section, as well as introducing some new notation.

Let $P \leqslant S$, and let $P^{\mathcal{F}} = \{P\phi\colon \phi \in \hom_{\mathcal{F}}(P,S)\}$ be the set of $\mathcal{F}$-*conjugates* of P. Recall that P is *fully normalized*, *centric* if, for all $Q \in P^{\mathcal{F}}$, $|N_S(P)| \geqslant |N_S(Q)|$, $C_S(Q) \leqslant Q$, respectively. Write $\mathcal{F}^f$ for the set of non-trivial fully normalized subgroups of S.

Let $\mathcal{S} = \{\mathcal{F}_i\colon i \in I\}$ be a set of subcategories $\mathcal{F}_i$ of $\mathcal{F}$. Define the *subsystem of* $\mathcal{F}$ *generated by* $\mathcal{S}$ to be the smallest fusion system contained in $\mathcal{F}$ and containing each member of $\mathcal{S}$. Write $\langle \mathcal{S} \rangle$ for this subsystem. Thus, $\langle \mathcal{S} \rangle$ is the intersection of all fusion subsystems of $\mathcal{F}$ containing each member of $\mathcal{S}$.

Recall that, for $P \in \mathcal{F}^f$, $N_{\mathcal{F}}(P)$, $C_{\mathcal{F}}(P)$ is the fusion system $\mathcal{T}$ on $T = N_S(P)$, $C_S(P)$ such that, for $Q \leqslant T$, $\hom_{\mathcal{T}}(Q,T)$ consists of those $\phi \in \hom_{\mathcal{F}}(Q,T)$ such that ϕ extends to $\hat{\phi} \in \hom_{\mathcal{F}}(PQ,T)$ normalizing and centralizing P, respectively. By a result of Puig (appearing as [**8**, Proposition A.6] or [**6**, Theorem II.2.1]), $\mathcal{T}$ is a saturated fusion system on T.

We say P is *normal* in $\mathcal{F}$ if $\mathcal{F} = N_{\mathcal{F}}(P)$. There is a largest subgroup $O_p(\mathcal{F})$ of S normal in $\mathcal{F}$.

Furthermore, $\mathcal{F}$ is *constrained* if $\mathcal{F}$ has a normal centric subgroup. If $\mathcal{F}$ is constrained, then, by [**7**, Proposition C] (or [**6**, Theorem III.5.10]), there is a finite group G with $S \in \mathrm{Syl}_p(G)$, $F^*(G) = O_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$. Moreover, G is unique up to an isomorphism extending the identity map on S, and we refer to any such group as a *model* for $\mathcal{F}$. In particular, if $U \in \mathcal{F}^f$ with $N_{\mathcal{F}}(U)$ constrained, then there exists a model for $N_{\mathcal{F}}(U)$, which we denote by $G(U)$.

The system $\mathcal{F}$ is of *characteristic p-type* if, for each $1 \neq U \in \mathcal{F}^f$, $N_{\mathcal{F}}(U)$ is constrained. When $p = 2$, $\mathcal{F}$ is of *even characteristic* if $N_{\mathcal{F}}(U)$ is constrained for each $1 \neq U \trianglelefteq S$. Finally, $\mathcal{F}$ is a *local CK-system* if, for each $U \in \mathcal{F}^f$, $\mathrm{Aut}_{\mathcal{F}}(U)$ is a $\mathcal{K}$-group; here, a finite group H is a *$\mathcal{K}$-group* if each simple section of H is on the list $\mathcal{K}$ of 'known' simple groups appearing in the statement of the theorem classifying the finite simple groups.

Lemma 1.1. *Let $\mathcal{F}$ be a saturated constrained fusion system on a finite p-group S, and G a model for $\mathcal{F}$. Assume $\mathcal{E}$ is a saturated subsystem of $\mathcal{F}$ on S. Then there exists a unique overgroup H of S in G with $\mathcal{E} = \mathcal{F}_S(H)$. Indeed for any normal subgroup Q of G with $C_G(Q) \leqslant Q$, $H = \{g \in G\colon c_{g|Q} \in \mathrm{Aut}_{\mathcal{E}}(Q)\}$.*

Proof. Pick $Q \trianglelefteq G$ with $C_G(Q) \leqslant Q$; for example, we could choose $Q = O_p(G)$. Set $H = \{g \in G\colon c_{g|Q} \in \mathrm{Aut}_{\mathcal{E}}(Q)\}$. We claim that $\mathcal{F}_S(H) \leqslant \mathcal{E}$. Let $g \in H$ and $P \leqslant S$ with $P^g \leqslant S$. As $g \in H$, $c_g^* = \phi^*$ for some $\phi \in \mathrm{Aut}_{\mathcal{E}}(Q)$. As $P^g \leqslant S$, $P \leqslant N_\phi = \{x \in S\colon c_x\phi^* \in \mathrm{Aut}_S(Q)\}$. Therefore, as $\mathcal{E}$ is saturated, ϕ extends to $\varphi \in \hom_{\mathcal{E}}(P, S)$. As $\mathcal{E} \leqslant \mathcal{F}_S(G)$, $\varphi = c_h$ for some $h \in G$. Now $c_{h^{-1}g} = (c_h)^{-1}c_g = \varphi^{-1}c_g$ is the identity on Q, so as $C_G(Q) = Z(Q)$, $g = hz$ for some $z \in Z(Q)$. Then $c_g = c_h c_z \in \hom_{\mathcal{E}}(P, S)$, so $\hom_{\mathcal{F}_S(H)}(P, S) \subseteq \hom_{\mathcal{E}}(P, S)$, proving the claim. □

Conversely, let $\psi \in \hom_{\mathcal{E}}(P, S)$. As $Q \trianglelefteq G$, it follows from [**4**, Theorem 14.1] that $Q \trianglelefteq \mathcal{E}$. Therefore, as $\mathcal{E}$ is saturated, ψ extends to $\Psi \in \hom_{\mathcal{E}}(PQ, S)$. Next, $\Psi = c_y$ for some $y \in G$ and $y \in H$, so $\mathcal{E} \leqslant \mathcal{F}_S(H)$. Hence, $\mathcal{E} = \mathcal{F}_S(H)$.

Finally, suppose $K \leqslant G$ is a model for $\mathcal{E}$. Then $\mathrm{Aut}_K(Q) = \mathrm{Aut}_{\mathcal{E}}(Q) = \mathrm{Aut}_H(Q)$, so $K/Z(Q) = H/Z(Q)$, and hence $K = H$, establishing the uniqueness of H.

Lemma 1.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and set $Q = O_p(\mathcal{F})$. Then*

(i) *if $Q \leqslant Z(\mathcal{F})$, $\mathcal{F}/Q$ is saturated and $O_p(\mathcal{F}/Q) = 1$,*

(ii) *$C_{\mathcal{F}}(Q)$ is a saturated fusion system on $C_S(Q)$, and if $C_{\mathcal{F}}(Q)$ is constrained, then $\mathcal{F}$ is constrained.*

Proof. Assume $Q \leqslant Z(\mathcal{F})$. The fusion system $\mathcal{F}/Q$ on S/Q is defined in [**2**, § 8] and, by [**2**, 8.10], $\mathcal{F}/Q$ is saturated and, writing P for the preimage of $O_p(\mathcal{F}/Q)$ in S, $P \trianglelefteq \mathcal{F}$. Thus, $P \leqslant Q = O_p(\mathcal{F})$, so $O_p(\mathcal{F}/Q) = P/Q = 1$, establishing (i).

Set $\mathcal{C} = C_{\mathcal{F}}(Q)$. Recall that $\mathcal{C}$ is a saturated system on $T = C_S(Q)$. By [**4**, 10.2.1], $\mathcal{C} \trianglelefteq \mathcal{F}$. Then, by [**4**, 7.4 and 7.10], $Z = O_p(\mathcal{C}) \trianglelefteq \mathcal{F}$, so $Z \leqslant Q$. Therefore, $Z = Q \cap T = Z(Q)$. Assume that $\mathcal{C}$ is constrained. Then $C_T(Z) \leqslant Q$. But $C_T(Z) = T$, so $T \leqslant Q$. Thus, $C_S(Q) \leqslant Q$, so $\mathcal{F}$ is constrained, completing the proof of (ii). □

2. An equivalence relation and ordering

In this section we assume the following.

Hypothesis 2.1. *p is a prime, and $\mathcal{F}$ is a saturated fusion system on a finite p-group S. Write $\mathcal{U}$ for the set of non-trivial normal subgroups of S, and assume for each $U \in \mathcal{U}$ that $N_{\mathcal{F}}(U)$ is constrained.*

Notation 2.2. If G is a group and $H \leqslant G$, we write $\mathcal{O}_G(H)$ for the set of overgroups of H in G. Let ι be the identity map on S. Write $\mathcal{H}$ for the set of pairs (H, U) such that $U \in \mathcal{U}$ and $H \in \mathcal{O}_{G(U)}(S)$. Denote by $\mathcal{V}_S(H)$ the set of non-trivial H-invariant subgroups of S. Write $\mathfrak{C}$ for the set of constrained, saturated fusion subsystems of $\mathcal{F}$ on S.

Lemma 2.3. *For $i = 1, 2$, let $U_i \in \mathcal{U}$ and set $G_i = G(U_i)$. Then*

(i) *ι extends to an isomorphism $\iota_{1,2}\colon N_{G_1}(U_2) \to N_{G_2}(U_1)$,*

(ii) *$\iota_{1,2}$ is determined up to conjugacy by an element of $Z(S)$, so, for each $H \in \mathcal{O}_{N_{G_1}(U_2)}(S)$, the image $H\iota_{1,2}$ of H in G_2 is independent of the choice of $\iota_{1,2}$.*

Proof. Observe that, for $i = 1, 2$, $U_{3-i} \leqslant S = N_S(U_i)$, so (i) follows from [**3**, 2.2.4]. Furthermore, by [**6**, II.4.3], if μ and ν are two isomorphisms extending ι, then $\nu = c_z\mu$ for some $z \in Z(S)$. Thus, $H\nu = Hc_z\mu = H\mu$ as $z \in S \leqslant H$, so $H = Hc_z$. Thus, (ii) holds. □

Lemma 2.4. *Let $I = \{1, 2, 3\}$, and for $i \in I$ let $U_i \in \mathcal{U}_i$ and set $G_i = G(U_i)$. For $\sigma \in \mathrm{Sym}(I)$, set $G_{1\sigma,2\sigma,3\sigma} = N_{G_{1\sigma}}(U_{2\sigma}) \cap N_{G_{1\sigma}}(U_{3\sigma})$. Then*

(i) *$G_{1,2,3}\iota_{1,2} = G_{2,1,3}$,*

(ii) *$G_{1,2,3}\iota_{1,2}\iota_{2,3} = G_{3,1,2}$,*

(iii) *$(\iota_{1,2}\iota_{2,3})_{|G_{1,2,3}} = (c_z\iota_{1,3})_{|G_{1,2,3}}$ for some $z \in Z(S)$,*

(iv) *for each $H \in \mathcal{O}_{G_{1,2,3}}(S)$, $H\iota_{1,2}\iota_{2,3} = H\iota_{1,3}$.*

Proof. Set $X = G_{1,2,3}$. As $\iota_{1,2}$ extends ι, $X\iota_{1,2} \leqslant G_{2,1,3}$. By symmetry, $G_{2,1,3}\iota_{2,1} \leqslant X$, so (1) follows. Then by two applications of (i), $X\iota_{1,2}\iota_{2,3} = G_{2,1,3}\iota_{2,3} = G_{3,1,2}$, establishing (ii).

Set $\xi = \iota_{1,2}\iota_{2,3}$. By (i) and (ii), $\xi\iota_{1,3}^{-1}$ is an automorphism of $G_{1,2,3}$ extending ι, so (iii) follows from [**6**, II.4.3]. Then (iii) implies (iv), as in the proof of Lemma 2.3 (ii). □

Definition 2.5. Define a relation $\equiv$ on $\mathcal{H}$ by $(H_1, U_1) \equiv (H_2, U_2)$ if $U_2 \in \mathcal{V}_S(H_1)$ and $H_1\iota_{1,2} = H_2$. By Lemma 2.3 (ii), this definition is independent of the choice of $\iota_{1,2}$. We find in the next lemma that $\equiv$ is an equivalence relation on $\mathcal{H}$. Write $\mathfrak{H}$ for the set of equivalence classes of $\equiv$, and write $[(H, U)]$ for the equivalence class of $(H, U) \in \mathcal{H}$. Sometimes we write H for $[(H, U)]$, since the group H is determined up to an isomorphism extending ι, and, by Lemma 2.6 (iii), the class $[H, U]$ is independent of the choice of $U \in \mathcal{V}_S(H)$.

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Lemma 2.6.

(i) *The relation* $\equiv$ *is an equivalence relation.*

(ii) *If* $(H_1, U_1) \equiv (H_2, U_2)$ *then* $\mathcal{U}_S(H_1) = \mathcal{U}_S(H_2)$.

(iii) $(H_1, U_1) \equiv (H_2, U_2)$ *if and only if* $\mathcal{F}_S(H_1) = \mathcal{F}_S(H_2)$.

(iv) *The map* $H \mapsto \mathcal{F}_S(H)$ *is a bijection between* $\mathfrak{H}$ *and* $\mathfrak{C}$.

Proof. Suppose that $(H_1, U_1) \equiv (H_2, U_2)$. Then $H_1\iota_{1,2} = H_2$, so, as $\iota_{1,2}$ extends ι, (ii) follows and $\mathcal{F}_S(H_1) = \mathcal{F}_S(H_2)$.

Assume next that $\mathcal{F}_S(H_1) = \mathcal{F}_S(H_2) = \mathcal{E}$, and set $U_3 = O_p(\mathcal{E})$. Then $U_3 \in \mathcal{U}_S(H_i)$, so $(H_i, U_i) \equiv (H_i\iota_{i,3}, U_3)$, and hence, replacing H_i by $H_i\iota_{i,3}$, we may assume that $U_1 = U_2 = U_3$. Next, $\mathrm{Aut}_{H_1}(U_3) = \mathrm{Aut}_{\mathcal{E}}(U_3) = \mathrm{Aut}_{H_2}(U_3)$, so H_1 and H_2 are each the preimages in $G(U_3)$ of $\mathrm{Aut}_{\mathcal{E}}(U_3)$. This completes the proof of (iii).

Finally, (iii) implies (i), and says that $\varphi\colon H \mapsto \mathcal{F}_S(H)$ is a well-defined injection of $\mathfrak{H}$ into $\mathfrak{C}$. Suppose $\mathcal{D} \in \mathfrak{C}$. Then $U = O_p(\mathcal{D}) \in \mathcal{U}$, and $\mathcal{D} \leqslant N_{\mathcal{F}}(U)$, so $\mathcal{D}$ has a model $H \leqslant G(U)$ by Lemma 1.1. Then $\mathcal{D} = \mathcal{F}_S(H) = H\varphi$, so $\varphi\colon \mathfrak{H} \to \mathfrak{C}$ is a surjection, completing the proof of (iv) and the lemma. □

Definition 2.7. Define a relation $\leqslant$ on $\mathfrak{H}$ by $[(H_1, U_1)] \leqslant [(H_2, U_2)]$ if $U_2 \in \mathcal{U}_S(H_1)$ and $H_1\iota_{1,2} \leqslant H_2$. By Lemma 2.3 (ii), this definition is independent of the choice of $\iota_{1,2}$. Furthermore, the proof of part (i) of the next lemma shows that $U_2 \in \mathcal{U}_S(H_1)$ and $H_1\iota_{1,2} \leqslant H_2$ if and only if $\mathcal{F}_S(H_1) \leqslant \mathcal{F}_S(H_2)$; hence, by Lemma 2.6 (iii), the definition of the relation $\leqslant$ is independent of the choice of representatives for the equivalence classes.

Lemma 2.8.

(i) *For* $H_i \in \mathfrak{H}$, $i = 1, 2$, $H_1 \leqslant H_2$ *if and only if* $\mathcal{F}_S(H_1) \leqslant \mathcal{F}_S(H_2)$.

(ii) *The relation* $\leqslant$ *is a partial ordering of* $\mathfrak{H}$.

(iii) *The map* $H \mapsto \mathcal{F}_S(H)$ *is an isomorphism of the poset* $\mathfrak{H}$ *with* $\mathfrak{C}$ *partially ordered by inclusion.*

Proof. Assume $[(H_1, U_1)] \leqslant [(H_2, U_2)]$. Then $U_2 \in \mathcal{U}_S(H_1)$ and $H_1\iota_{1,2} \leqslant H_2$. Hence, by Lemma 2.6 (iii), $\mathcal{F}_S(H_1) = \mathcal{F}_S(H_1\iota_{1,2}) \leqslant \mathcal{F}_S(H_2)$.

Next suppose instead that $\mathcal{F}_S(H_1) \leqslant \mathcal{F}_S(H_2)$. As $U_2 \trianglelefteq H_2$, we also have $U_2 \trianglelefteq \mathcal{F}_S(H_1)$, so $U_2 \in \mathcal{U}_S(H_1)$. By Lemma 2.6 (iii), $\mathcal{F}_S(H_1\iota_{1,2}) = \mathcal{F}_S(H_1) \leqslant \mathcal{F}_S(H_2)$. Hence,

$$\mathrm{Aut}_{H_1\iota_{1,2}}(U_2) = \mathrm{Aut}_{\mathcal{F}_S(H_1)}(U_2) \leqslant \mathrm{Aut}_{\mathcal{F}_S(H_2)}(U_2) = \mathrm{Aut}_{H_2}(U_2),$$

so the preimage $H_1\iota_{1,2}$ in $G(U_2)$ of $\mathrm{Aut}_{H_1\iota_{1,2}}(U_2)$ is contained in the preimage H_2 of $\mathrm{Aut}_{H_2}(U_2)$. This establishes (i).

Observe that part (i) and Lemma 2.6 (iii) imply part (ii), while (i), (ii) and Lemma 2.6 (iv) imply (iii). □

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Definition 2.9. For $[(H_i, U_i)] \in \mathfrak{H}$, $i = 1, 2$, define $[(H_1, U_1)] \cap [(H_2, U_2)]$ (or just $H_1 \cap H_2$) to be $[(N_{H_1}(U_2)\iota_{1,2} \cap H_2, U_2)]$.

Define $\mathcal{F}_S(H_1) \wedge \mathcal{F}_S(H_2)$ to be the subfusion system of $\mathcal{F}$ on S generated by the maps in $\mathrm{Aut}_{N_{H_1}(U_2)}(P) \cap \mathrm{Aut}_{N_{H_2}(U_1)}(P)$, as P varies over the subgroups of S containing $C_S(P)$. By the next lemma and Lemma 2.6 (iii), the definition of $[(H_1, U_1)] \cap [(H_2, U_2)]$ is independent of the choice of representatives of the classes.

Lemma 2.10. *For $[(H_i, U_i)] \in \mathfrak{H}$, $i = 1, 2$, $\mathcal{F}_S(N_{H_1}(U_2)\iota_{1,2} \cap H_2) = \mathcal{F}_S(H_1) \wedge \mathcal{F}_S(H_2)$.*

Proof. Set $\mathcal{F}_i = \mathcal{F}_S(H_i)$, $\mathcal{E} = \mathcal{F}_1 \wedge \mathcal{F}_2$, and $\mathcal{D} = \mathcal{F}_S(N_{H_1}(U_2)\iota_{1,2} \cap H_2)$. Let $P \leqslant S$ with $C_S(P) \leqslant P$ and $\alpha \in \mathrm{Aut}_{\mathcal{D}}(P)$. Then $\alpha = c_{h|P}$ for some $h \in N_{H_1}(U_2)\iota_{1,2} \cap N_{H_2}(P)$. By Lemma 2.6 (ii), h acts on U_1 and U_2, so $\alpha \in \mathrm{Aut}_{N_{H_2}(U_1)}(P)$ and

$$\alpha \in \mathrm{Aut}_{N_{H_1\iota_{1,2}}(U_2)}(P) = \mathrm{Aut}_{\mathcal{F}_S(N_{H_1}(U_2)\iota_{1,2})}(P) = \mathrm{Aut}_{\mathcal{F}_S(N_{H_1}(U_2))}(P) = \mathrm{Aut}_{N_{H_1}(U_2)}(P),$$

by Lemma 2.6 (iii). Therefore,

$$\mathrm{Aut}_{\mathcal{D}}(P) \leqslant \mathrm{Aut}_{N_{H_1}(U_2)}(P) \cap \mathrm{Aut}_{N_{H_2}(U_1)}(P).$$

Furthermore, as $\mathcal{D}$ is saturated, Alperin's Fusion Theorem (cf. [**8**, A.10] or [**6**, II.3.5]) says that $\mathcal{D}$ is generated by the subcategories $\mathrm{Aut}_{\mathcal{D}}(P)$, as P varies over the subgroups of S containing $C_S(P)$. Therefore, $\mathcal{D} \leqslant \mathcal{E}$.

Conversely, assume $\beta \in \mathrm{Aut}_{N_{H_1}(U_2)}(P) \cap \mathrm{Aut}_{N_{H_2}(U_1)}(P)$. Then, as above, $\beta = c_{h_i}|_P$ for some $h_1 \in N_{H_1}(U_2)\iota_{1,2}$ and $h_2 \in N_{H_2}(U_1)$, so $h_1 h_2^{-1} \in C_{G(U_2)}(P) = Z(P)$, as $F^*(N_{G(U_2)}(P)) = O_p(N_{G(U_2)}(P))$ and $C_S(P) \leqslant P$. Thus, $h_1 = c_z h_2$ for some $z \in Z(P)$, so $h_1 \in N_{H_1}(U_2)\iota_{1,2} \cap N_{H_2}(U_1)$. Therefore, $\beta \in \mathrm{Aut}_{\mathcal{D}}(P)$, so $\mathcal{E} \leqslant \mathcal{D}$. □

Lemma 2.11. *Let $H_i \in \mathfrak{H}$, $1 \leqslant i \leqslant n$, for some $n \geqslant 2$. Then*

(i) *$H_1 \cap H_2 = H_2 \cap H_1$ is the greatest lower bound for H_1 and H_2 in the poset $\mathfrak{H}$,*

(ii) *there exists a greatest lower bound $H_1 \cap \cdots \cap H_n$ for $\{H_1, \ldots, H_n\}$ in $\mathfrak{H}$,*

(iii) *$(H_1 \cap H_2) \cap H_3 = H_1 \cap H_2 \cap H_3 = H_1 \cap (H_2 \cap H_3)$,*

(iv) *for $\mathcal{E}, \mathcal{D} \in \mathfrak{C}$, $\mathcal{E} \wedge \mathcal{D}$ is the greatest lower bound for $\mathcal{E}$ and $\mathcal{D}$ in $\mathfrak{C}$.*

Proof. Set $\mathcal{F}_i = \mathcal{F}_S(H_i)$. By definition, $\mathcal{F}_1 \wedge \mathcal{F}_2 \leqslant \mathcal{F}_i$ for $i = 1, 2$, so by Lemmas 2.8 (i) and 2.10, $H_1 \cap H_2 \leqslant H_i$. Suppose $H_3 \leqslant H_i$ for $i = 1, 2$. Then, by Lemma 2.8 (i), $\mathcal{F}_S(H_3) \leqslant \mathcal{F}_S(H_i)$, so, using Alperin's Fusion Theorem (cf. [**8**, A.10] or [**6**, II.3.5]), $\mathcal{F}_S(H_3) \leqslant \mathcal{F}_1 \wedge \mathcal{F}_2$. Then, by Lemmas 2.8 (i) and 2.10, $H_3 \leqslant H_1 \cap H_2$. Therefore, $H_1 \cap H_2$ is the greatest lower bound for H_1 and H_2 in $\mathfrak{H}$. By symmetry, $H_2 \cap H_1$ is also the greatest lower bound, so $H_1 \cap H_2 = H_2 \cap H_1$, establishing (i). Then (i) and elementary lattice theory imply (ii) and (iii). Finally, Lemma 2.8 (iii) and (i) imply (iv). □

Lemma 2.12. *Let $H_i = [(H_i, U_i)] \in \mathfrak{H}$, for $i = 1, 2$, and assume that $U_3 \in \mathcal{U}_S(H_1) \cap \mathcal{U}_S(H_2)$. Set $G_3 = G(U_3)$. Then we have the following.*

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(i) $H_i \leqslant [(G_3, U_3)]$ for $i = 1, 2$.

(ii) Set $H_3 = \langle H_1\iota_{1,3}, H_2\iota_{2,3}\rangle$. Then $\mathcal{F}_S(H_3)$ is the smallest member $\mathcal{F}_S(H_1) \vee \mathcal{F}_S(H_2)$ of $\mathfrak{C}$ containing $\mathcal{F}_S(H_i)$ for $i = 1, 2$.

(iii) $H_1 \vee H_2 = [(H_3, U_3)]$ is the least upper bound for H_1 and H_2 in $\mathfrak{H}$.

(iv) If $\mathcal{E}, \mathcal{D} \in \mathfrak{C}$ have a common non-trivial normal subgroup, then $\mathcal{E} \vee \mathcal{D}$ is the least upper bound for $\mathcal{E}$ and $\mathcal{D}$ in $\mathfrak{C}$.

(v) If $[(K, U)] \in \mathfrak{H}$ and $K_i \in \mathcal{O}_K(S)$ for $1 \leqslant i \leqslant n$ with $K = \langle K_1, \ldots, K_n\rangle$, then $[(K, U)]$ is the least upper bound $K_1 \vee \cdots \vee K_n$ for $[(K_1, U)], \ldots, [(K_n, U)]$ in the poset $\mathfrak{H}$. In particular, if $H \in \mathfrak{H}$ with $[(K_i, U)] \leqslant H$ for each $1 \leqslant i \leqslant n$, then $[(K, U)] \leqslant H$.

Proof. By hypothesis, $U_3 \in \mathcal{U}_S(H_i)$ for $i = 1, 2$, so (i) follows. By construction, $H_i \leqslant H_3$ for $i = 1, 2$, so $\mathcal{F}_i = \mathcal{F}_S(H_i) \leqslant \mathcal{F}_3 = \mathcal{F}_S(H_3)$ by Lemma 2.8 (i). Thus, $\mathcal{F}_3$ is an upper bound for $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathfrak{C}$.

Suppose that $\mathcal{F}_4$ is an upper bound for $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathfrak{C}$. By Lemma 2.8 (iii), $\mathcal{F}_4 = \mathcal{F}_S(H_4)$ for some $H_4 \in \mathfrak{H}$, and $H_i\iota_{i,4} \leqslant H_4$. Thus, $H_5 = \langle H_1\iota_{1,4}, H_2\iota_{2,4}\rangle \leqslant H_4$ and, by Lemma 2.8 (iii), $\mathcal{F}_i \leqslant \mathcal{F}_S(H_5) = \mathcal{F}_5 \leqslant \mathcal{F}_4$. As $U_3 \in \mathcal{U}_S(H_i)$, also $U_3 \in \mathcal{U}_S(H_5)$ by Lemma 2.6 (ii). By Lemma 2.4 (iv), $H_i\iota_{i,4}\iota_{4,3} = H_i\iota_{i,3}$, so $H_5\iota_{4,3} = H_3$, and hence $\mathcal{F}_3 = \mathcal{F}_5$ by Lemma 2.6 (iii). Thus, $\mathcal{F}_3 = \mathcal{F}_5 \leqslant \mathcal{F}_4$, completing the proof of (ii). Then, applying the isomorphism of Lemma 2.8 (iii), (ii) implies (iii) and (iv).

Assume the hypothesis of (v). Proceeding by induction on n, we may take $n = 2$. Then, by (iii), $K = K_1 \vee K_2$. Hence, if $K_i \leqslant H$ for $i = 1, 2$, then $K \leqslant H$, completing the proof of (v). □

3. Another partial ordering of $\mathfrak{H}$

In this section we continue to assume Hypothesis 2.1, and adopt the notation established in § 2, such as Notation 2.2, and Definitions 2.5, 2.7 and 2.9.

Definition 3.1. Let $Z = \Omega_1(Z(S))$ and pick $1 \neq E \leqslant Z$. Given $H_1 = [(H_1, U_1)] \in \mathfrak{H}$, define

$$V(H_1) = V_E(H_1) = \langle E^{H_1}\rangle \leqslant S.$$

Observe that as $F^*(H_1) = O_p(H_1)$, $V(H_1) \leqslant Z(O_p(H_1))$, so indeed $V(H_1) \leqslant S$. Also, if $(H_1, U_1) \equiv (H_2, U_2)$, then

$$V(H_1) = V(H_1)\iota_{1,2} = \langle E^{H_1}\rangle\iota_{1,2} = \langle E\iota_{1,2}^{H_1\iota_{1,2}}\rangle = \langle E^{H_2}\rangle = V(H_2),$$

so the definition of $V(H_1)$ is independent of the choice of representative for the equivalence class.

In addition, $U_3 = V(H_1) \in \mathcal{U}_S(H_1)$ and $(H_1, U_1) \equiv (H_1\iota_{1,3}, U_3)$. Therefore, our convention in this section will be that our canonical representative for $\mathfrak{h} \in \mathfrak{H}$ will be (H, U), where $U = V(\mathfrak{h})$ and H is the unique subgroup of $G(U)$ such that $(H, U) \in \mathfrak{h}$.

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Write $\mathfrak{H}_E$ for the set of $H \in \mathfrak{H}$ such that $H = G(V(H))$.

Define a relation $\lesssim = \lesssim_E$ on $\mathfrak{H}$ by $H_1 \lesssim H_2$ if $H_1 = (N_{H_2}(U_1)\iota_{2,1} \cap H_1)C_{H_1}(U_1)$.

For $H \in \mathfrak{H}$, let $\mathcal{O}(H) = \{K \in \mathfrak{H} \colon H \leqslant K\}$ and denote by $\mathcal{M}(H)$ the set of maximal members of $\mathcal{O}(H)$ under the partial ordering $\leqslant$ on $\mathfrak{H}$ in Definition 2.7. Note that $\mathcal{M}(H) \subseteq \mathfrak{H}_E$. Also, $S \leqslant H$ for each $H \in \mathfrak{H}$, so $\mathcal{O}(S) = \mathfrak{H}$, and $\mathcal{M}(S)$ is the set of maximal members of the poset $\mathfrak{H}$ under the ordering $\leqslant$.

We find in Lemma 3.6 that the new relation $\lesssim$ is a partial ordering of $\mathfrak{H}_E$. On the other hand, the set $\mathcal{O}(H)$ of 'overgroups' of H and the set $\mathcal{M}(H)$ of 'maximal overgroups' of H are defined with respect to the old partial ordering $\leqslant$ on $\mathfrak{H}$ from Definition 2.7. If $\mathcal{F} = \mathcal{F}_S(G)$ is the fusion system of some group G, then the ordering $\leqslant$ corresponds to the inclusion relation on 2-local subgroups of G containing S, while $\lesssim$ corresponds to the relation $\lesssim$ on such subgroups appearing in [**5**, Definition A.5.2]. Lemma 3.8 records some useful relationships between the two partial orders.

Lemma 3.2. *Let $H_i \in \mathfrak{H}$, for $i = 1, 2$. Then $H_1 \lesssim H_2$ if and only if*

$$\mathrm{Aut}_{\mathcal{F}_S(H_1) \wedge \mathcal{F}_S(H_2)}(V(H_1)) = \mathrm{Aut}_{\mathcal{F}_S(H_1)}(V(H_1)).$$

Proof. Let $V = V(H_1)$, and recall from Definition 3.1 that, by convention, $V = U_1$. Set $H = N_{H_2}(U_1)\iota_{2,1} \cap H_1$ and, for $i = 1, 2$, set $\mathcal{F}_i = \mathcal{F}_S(H_i)$. By Lemma 2.10, $\mathcal{F}_1 \wedge \mathcal{F}_2 = \mathcal{F}_S(H)$, so $\mathrm{Aut}_H(V) = \mathrm{Aut}_{\mathcal{F}_1 \wedge \mathcal{F}_2}(V)$. Therefore, as $\mathrm{Aut}_{H_1}(V) = \mathrm{Aut}_{\mathcal{F}_1}(V)$, it follows that

$$\begin{aligned} \mathrm{Aut}_{\mathcal{F}_1 \wedge \mathcal{F}_2}(V) = \mathrm{Aut}_{\mathcal{F}_1}(V) &\iff \mathrm{Aut}_H(V) = \mathrm{Aut}_{H_1}(V) \\ &\iff H_1 = HC_{H_1}(V) \\ &\iff H_1 \lesssim H_2. \end{aligned}$$

□

Lemma 3.3. *Suppose $H_i \in \mathfrak{H}$, for $i = 1, 2$, with $H_2 \in \mathfrak{H}_E$. Then*

(i) $N_{H_2}(U_1)\iota_{2,1} \cap H_1 = N_{H_1}(U_2)$,

(ii) *$H_1 \lesssim H_2$ if and only if $H_1 = N_{H_1}(U_2)C_{H_1}(U_1)$.*

Proof. As $H_2 \in \mathfrak{H}_E$, $H_2 = G(U_2)$, so $N_{H_2}(U_1)\iota_{2,1} = N_{G(U_1)}(U_2)$ by Lemma 2.3 (i). Thus, (i) follows, and (i) implies (ii). □

Lemma 3.4. *If $H_i \in \mathfrak{H}$, for $i = 1, 2$, with $H_1 \lesssim H_2$, then*

(i) $U_1 = V(H_1) \leqslant V(H_2) = U_2$,

(ii) $C_{H_2}(U_2) \leqslant C_{H_2}(U_1)$,

(iii) *if $H_2 \in \mathfrak{H}_E$, then $C_{H_2}(U_1)\iota_{2,1} \cap H_1 = N_{H_1}(U_2) \cap C_{H_1}(U_1)$, so $C_{H_2}(U_2)\iota_{2,1} \cap H_1 \leqslant C_{H_1}(U_1)$.*

Proof. As $H_1 \lesssim H_2$, $H_1 = (N_{H_2}(U_1)\iota_{2,1} \cap H_1)C_{H_1}(U_1)$. Thus,

$$\begin{aligned} U_1 &= \langle E^{H_1}\rangle \\ &= \langle E^{N_{H_2}(U_1)\iota_{2,1}\cap H_1}\rangle \leqslant \langle E^{N_{H_2}(U_1)\iota_{2,1}}\rangle \\ &= \langle E^{N_{H_2}(U_1)}\rangle\iota_{2,1} \\ &= \langle E^{N_{H_2}(U_1)}\rangle \leqslant \langle E^{H_2}\rangle \\ &= U_2, \end{aligned}$$

establishing (i). Then (i) implies (ii).

Assume $H_2 \in \mathfrak{H}_E$. By Lemma 3.3 (i), $N_{H_2}(U_1)\iota_{2,1} \cap H_1 = N_{H_1}(U_2)$, so, as $\iota_{2,1}$ extends ι, the first statement in (iii) follows. The second statement of (iii) follows from the first statement and (ii). □

Lemma 3.5. *Assume $H_i \in \mathfrak{H}_E$ for $i = 1, 2$. Then the following are equivalent:*

(i) $H_1 = H_2$;

(ii) $V(H_1) = V(H_2)$;

(iii) $H_1 \lesssim H_2 \lesssim H_1$.

Proof. Trivially, (i) implies (iii). As $H_i \in \mathfrak{H}_E$, $H_i = G(U_i)$, so (ii) implies (i). Finally, by Lemma 3.4 (i), (iii) implies (ii). □

Lemma 3.6. *$\lesssim$ is a partial ordering of $\mathfrak{H}_E$.*

Proof. Trivially, $\lesssim$ is reflexive. By Lemma 3.5, $\lesssim$ is antisymmetric. Assume that $H_1 \lesssim H_2 \lesssim H_3$. By Lemma 3.3 (ii), $H_2 = N_{H_2}(U_3)C_{H_2}(U_2)$, and by Lemma 3.4 (iii), $C_{H_2}(U_2) \leqslant N_{H_2}(U_1)$. Then

$$N_{H_2}(U_1) = N_{H_2}(U_1) \cap N_{H_2}(U_3)C_{H_2}(U_2) = (N_{H_2}(U_1) \cap N_{H_2}(U_3))C_{H_2}(U_2),$$

so, by Lemma 2.3 (i),

$$N_{H_1}(U_2) = N_{H_2}(U_1)\iota_{2,1} = (N_{H_2}(U_1) \cap N_{H_2}(U_3))\iota_{2,1}C_{H_2}(U_2)\iota_{2,1} \leqslant N_{H_1}(U_3)C_{H_1}(U_1),$$

and hence $H_1 \lesssim H_3$, so that $\lesssim$ is transitive. □

Lemma 3.7. *Let $H_i \in \mathfrak{H}$ for $i = 1, 2$. Then*

(i) *If $H_1 \leqslant H_2$, $H_1 \lesssim H_2$,*

(ii) *If $H_i \lesssim H_{3-i}$ for $i = 1$ and 2, $V(H_1) = V(H_2)$.*

Proof. Part (i) is immediate from the definitions, while part (ii) follows from Lemma 3.4 (i). □

Lemma 3.8. *Let M be maximal in $\mathcal{M}(S)$ with respect to $\lesssim$. Then we have the following.*

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(i) $\{M\} = \mathcal{M}(X)$ *for each* $X \in \mathfrak{H}$ *with* $X \leqslant M$ *and* $M = XC_M(V(M))$.

(ii) *Let* $R = C_S(V(M))$. *Then* $R = O_p(N_M(R))$, $\mathrm{Aut}_M(V(M)) = \mathrm{Aut}_{N_M(R)}(V(M))$ *and* $\mathcal{M}(N_M(R)) = \{M\}$.

(iii) *Let* $H \in \mathfrak{H}_E$. *Then* H *is maximal in* $\mathfrak{H}_E$ *with respect to* $\lesssim$ *if and only if* H *is maximal in* $\mathcal{M}(S)$ *with respect to* $\lesssim$.

Proof. Assume that H is maximal in either $\mathfrak{H}_E$ or $\mathcal{M}(S)$ with respect to $\lesssim$, and set $U = V(H)$. Suppose $X \in \mathfrak{H}$ with $X \leqslant H$ and $H = XC_H(U)$. Then $U = \langle E^H \rangle = \langle E^X \rangle = V(X)$. Suppose $M_2 \in \mathcal{M}(X)$. Then $X \lesssim M_2$ by Lemma 3.7 (i), so $X = N_X(V(M_2))C_X(V(X))$ by Lemma 3.3 (ii). Hence,

$$H = XC_H(U) = N_X(V(M_2))C_X(V(X))C_H(U) = N_H(V(M_2))C_H(U),$$

so $H \lesssim M_2$ by Lemma 3.3 (ii). Therefore, $H = M_2$ by maximality of H and Lemma 3.6. This proves (i), and shows that if H is maximal in $\mathfrak{H}_E$ with respect to $\lesssim$, then, for $M_3 \in \mathcal{M}(H)$, $H = M_3$. Thus, the implication H maximal in $\mathfrak{H}_E$ with respect to $\lesssim$ implies H maximal in $\mathcal{M}(S)$ with respect to $\lesssim$ of (iii) is also established.

Let $V = V(M)$. By a Frattini argument, $M = N_M(R)C_M(V)$, so $\mathrm{Aut}_M(V) = \mathrm{Aut}_{N_M(R)}(V)$. Now $V = \langle E^M \rangle$, so as $M = N_M(R)C_M(V)$, $V = \langle E^{N_M(R)} \rangle$. Hence, it follows from [**5**, B.2.14] that $O_2(N_M(R)) \leqslant C_S(V) = R$, so $R = O_2(N_M(R))$. Then, by (i), $\{M\} = \mathcal{M}(N_M(R))$, which completes the proof of (ii).

Let $M \lesssim K$ with K maximal in $\mathfrak{H}_E$ with respect to $\lesssim$. By the first paragraph of the proof, $K \in \mathcal{M}(S)$, so $M = K$ as M is maximal in $\mathcal{M}(S)$ with respect to $\lesssim$. This completes the proof of (iii). □

4. Further results on $\mathfrak{H}$

If G is a finite group with Sylow p-subgroup T, a *minimal parabolic* of G over T is an overgroup H of T in G such that T is not normal in H, and T is contained in a unique maximal subgroup of H.

In this section we assume the following hypothesis.

Hypothesis 4.1. *Assume Hypothesis 2.1, and in addition assume* $\mathcal{F} = \langle N_{\mathcal{F}}(U) \colon U \in \mathcal{U} \rangle$ *and* $\mathcal{F}$ *is not constrained. For* $H \in \mathfrak{H}$, *let* $\mathcal{X}(H)$ *be the set of minimal parabolics of* H *over* S. *We say that* H *is a uniqueness group if* $|\mathcal{M}(H)| = 1$.

Lemma 4.2. $O_p(\mathcal{F}) = 1$.

Proof. If $U = O_p(\mathcal{F}) \neq 1$ then $U \in \mathcal{U}$, so by Hypothesis 2.1, $\mathcal{F} = N_{\mathcal{F}}(U)$ is constrained, contrary to Hypothesis 4.1. □

Lemma 4.3. *Let* $M \in \mathcal{M}(S)$ *and* $H \in \mathfrak{H}$. *Then*

(i) *there exists* $K \in \mathfrak{H}$ *with* $K \nleq M$,

(ii) *either* $N_K(S) \nleq M$ *or there exists* $X \in \mathcal{X}(K)$ *with* $X \nleq M$,

(iii) $H \leqslant M$ *if and only if* $\mathcal{U}_S(M) \cap \mathcal{U}_S(H) \neq \varnothing$.

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Proof. Assume (i) fails. Then, by Lemma 2.8 (iii), for each $U \in \mathcal{U}$, $N_{\mathcal{F}}(U) \leqslant \mathcal{F}_S(M)$. But then by Hypothesis 4.1, $\mathcal{F} = \langle N_{\mathcal{F}}(U) \colon U \in \mathcal{U}\rangle \leqslant \mathcal{F}_S(M)$, so $1 \neq O_p(M) \trianglelefteq \mathcal{F}$, contrary to Lemma 4.2. This establishes (i).

By McBride's Lemma (cf. [**5**, B.6.3]), $K = \langle N_K(S), \mathcal{X}(K)\rangle$, so (ii) follows from Lemma 2.12 (v).

If $H \leqslant M$ then $O_p(M) \in \mathcal{U}_S(M) \cap \mathcal{U}_S(H)$. Conversely, if $\mathcal{U}_S(M) \cap \mathcal{U}_S(H) \neq \varnothing$, then by Lemma 2.12 (i) there exists $K \in \mathfrak{H}$ with $M, H \leqslant K$. But then $H \leqslant K = M$ by maximality of M. □

Lemma 4.4. *Let $M \in \mathcal{M}(S)$,and let $H \in \mathfrak{H}$. Then we have the following.*

(i) *Assume $Y \leqslant M$ is a uniqueness group. Then $H \leqslant M$ if and only if $\mathcal{U}_S(Y) \cap \mathcal{U}_S(H) \neq \varnothing$.*

(ii) *Set $R = C_S(V(M))$. Then $R = O_2(N_M(R))$.*

Proof. Assume the hypothesis of (i) holds. One implication in (i) follows from Lemma 4.3 (iii). Suppose $\mathcal{U}_S(Y) \cap \mathcal{U}_S(H) \neq \varnothing$. Then, by Lemma 2.12 (i), there exists $M' \in \mathcal{M}(S)$ with $Y, H \leqslant M'$. Then $M' = M$, as Y is a uniqueness group, completing the proof of (i).

By [**5**, B.2.14], $O_2(N_M(R)) \leqslant C_S(V) = R$, so (ii) holds. □

Lemma 4.5. *Let M be maximal in $\mathcal{M}(S)$ with respect to $\lesssim$, and let $H \in \mathfrak{H}$. Set $R = C_S(V(M))$ and $Y = N_M(R)$. Then*

(i) *$\mathcal{M}(Y) = \{M\}$ and $R = O_2(Y)$,*

(ii) *$H \leqslant M$ if and only if $\mathcal{U}_S(Y) \cap \mathcal{U}_S(H) \neq \varnothing$.*

Proof. Part (i) follows from Lemma 3.8 (ii). Then (i) and Lemma 4.4 (i) imply (ii). □

See [**5**, B.2.2] for the definition of the *Thompson subgroup* $J(S)$ and the *Baumann subgroup* $\mathrm{Baum}(S)$ of a *p*-group S.

Lemma 4.6. *Let $H \in \mathfrak{H}$. Then the following are equivalent:*

(i) *$J(S)$ centralizes $V(H)$;*

(ii) *$\mathrm{Baum}(S)$ centralizes $V(H)$;*

(iii) *$H = N_H(\mathrm{Baum}(S))C_H(V(H))$;*

(iv) *$H \lesssim G(\mathrm{Baum}(S))$.*

Proof. Set $V = V(H)$, $D = \Omega_1(Z(J(S)))$, $B = \mathrm{Baum}(S)$ and $K = G(B)$. If $J(S)$ centralizes V, then $V \leqslant D$, so $B = C_S(D) \leqslant C_S(V)$. Thus, (i) implies (ii).

Next, as B is weakly closed in S, (ii) implies (iii) by a Frattini argument.

Assume (iii). By Lemma 2.3 (i), there exists an isomorphism $\xi \colon N_{G(V(K))}(V) \to N_{G(V)}(V(K))$ extending ι. We may take $K = N_{G(V(K))}(B)$, so $N_K(V) = N_{G(V(K))}(V) \cap$

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$N_{G(V(K))}(B)$, and hence $N_K(V)\xi = N_{G(V)}(V(K)) \cap N_{G(V)}(B)$. Then $N_H(B) \leqslant N_K(V)\xi \cap H$, so $H \lesssim K$; that is (iii) implies (iv).

Finally, assume (iv) and set $H^* = H/C_H(V)$. Then $H = (N_K(V)\xi \cap H)C_H(V)$ and $B \trianglelefteq N_K(V)\xi \cap H$, so $B^* \leqslant O_p(H^*) = 1$ as $E \leqslant Z$. Therefore, $J(S) \leqslant B \leqslant C_H(V)$, so (iv) implies (i). □

Lemma 4.7. *Assume $J(S)$ centralizes each member of $\{V(H) : H \in \mathfrak{H}\}$. Then*

(i) *$\mathcal{M}(G(\mathrm{Baum}(S))) = \{M\}$, where $M = G(V(G(\mathrm{Baum}(S))))$,*

(ii) *M is the unique maximal member of $\mathcal{M}(S)$ under $\lesssim$.*

Proof. Set $B = \mathrm{Baum}(S)$ and $K = G(B)$. By Lemma 4.6, for each $H \in \mathfrak{H}$, $H \lesssim K$. On the other hand, for $M' \in \mathcal{M}(K)$, Lemma 3.7 says that $K \lesssim M'$ and $V(K) = V(M')$. Thus, (i) follows from Lemma 3.5. Furthermore, for $M'' \in \mathcal{M}(S)$, $M'' \lesssim K \lesssim M$, so if $M'' \neq M$, then M'' is *not* maximal in $\mathcal{M}(S)$ with respect to $\lesssim$ by Lemma 3.6, establishing (ii). □

5. Systems of even characteristic

In this section we assume one of the following two hypotheses.

Hypothesis 5.1. *Assume Hypothesis 4.1 with $p = 2$.*

Hypothesis 5.2. *Assume Hypothesis 2.1 with $p = 2$. Assume $(H_i, V_i) \in \mathcal{H}$ for $i = 1, 2$ and set $R_i = O_2(H_i)$. Assume in addition that*

(i) *$\Phi(V_1) = 1$, $O_2(\mathrm{Aut}_{H_1}(V_1)) = 1$ and $R_1 \in \mathrm{Syl}_2(C_{H_1}(V_1))$,*

(ii) *H_2 is a minimal parabolic over S,*

(iii) *$\mathcal{U}_S(H_1) \cap \mathcal{U}_S(H_2) = \varnothing$.*

Lemma 5.3. *Assume Hypothesis 5.1 holds. Assume $M \in \mathcal{M}(S)$ and $G(S) \leqslant M$. Let $\mathfrak{H}_M = \{H \in \mathfrak{H} : H \not\lesssim M\}$, and for $H \in \mathfrak{H}_M$ let $\mathcal{X}_M(H) = \{X \in \mathcal{X}(H) : X \not\lesssim M\}$ and $\mathcal{X}^*_M(H)$ be the minimal members of $\mathcal{X}_M(H)$ under inclusion. Then we have the following.*

(i) *$\mathfrak{H}_M \neq \varnothing$ and for each $H \in \mathfrak{H}_M$, $\mathcal{X}_M(H) \neq \varnothing$.*

(ii) *Let $H \in \mathfrak{H}_M$ and $X \in \mathcal{X}_M(H)$. Set $R = C_S(V(M))$ and $Y = N_M(R)$. If Y is a uniqueness group, then the pair $(Y, V(M))$, $(X, V(X))$ satisfies Hypothesis 5.2.*

(iii) *Let $H \in \mathfrak{H}_M$ and $X \in \mathcal{X}^*_M(H)$. Then $N_X(V(M))$ is the unique maximal overgroup of S in X.*

Proof. Part (i) is a consequence of parts (i) and (ii) of Lemma 4.3.

Assume the hypothesis of (ii) with Y a uniqueness group. Then the pair $(Y, V(M))$, $(X, V(X))$ satisfies conditions (i) and (iii) of Hypothesis 5.2 by Lemma 4.4 and, as X is a minimal parabolic, the pair satisfies condition (ii) of Hypothesis 5.2. Thus, (ii) holds.

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Assume that $X \in \mathcal{X}_M^*(H)$ and set $V = V(M)$. Then $S \leqslant N_X(V) < X$, so $N_X(V) \leqslant Y_X$, the unique maximal overgroup of S in X. Suppose $N_X(V) \neq Y_X$. Then as $M = G(V)$, $Y_X \in \mathfrak{H}_M$, so by (i) there exists $X' \in \mathcal{X}_M(Y_X)$. Then $X' \leqslant Y_X < X$, contrary to the minimality of X. This establishes (iii). □

If G is a finite group and V is a faithful $\mathbb{F}_2G$-module then the parameters $q(G,V)$ and $\hat{q}(G,V)$ are defined in [**5**, Definitions B.1.1 and B.4.1].

Lemma 5.4 (Stellmacher qrc-Lemma for fusion systems). *Assume Hypothesis 5.2 holds, and set $q = q(H_1/C_{H_1}(V_1), V_1)$ and $U = \langle V_1^{H_2}\rangle$. Then one of the following holds.*

(i) $V_1 \nleqslant R_2$.

(ii) $q \leqslant 1$.

(iii) *The dual of V_1 is an FF-module for $H_1/C_{H_1}(V_1)$.*

(iv) *$q \leqslant 2$, U is abelian and H_2 has $c \geqslant 2$ non-central chief factors on U. If $q = 2$, then $c = 2$ and, for each non-central H_2-chief factor W on U, $q(H_2/C_{H_2}(W), W) = 1$.*

(v) *$R_1 \cap R_2 \trianglelefteq H_2$, U is abelian, H_2 has one non-central chief factor W,*

$$q(H_2/C_{H_2}(W), W) = 1,$$

$[U, O^2(H_2)] \leqslant Z(R_2)$, $O^2(H_2) = [O^2(H_2), J(R_1)]$ and $J(R_1) = J(S)$.

Proof. This follows from the qrc-Lemma for groups, which appears in [**5**, D.1.5]. For example, set $H_{1,2} = N_{H_1}(V_2) \cap N_{H_2}(V_1)\iota_{2,1}$, and form the amalgam $\alpha = (\alpha_i \colon H_{1,2} \to H_i \colon i = 1, 2)$, where α_1 is the inclusion map, and $\alpha_2 = \iota_{1,2} \colon H_{1,2} \to H_2$. Then we can form the free amalgamated product G of α and identify H_J, $J \subseteq \{1, 2\}$, with the corresponding subgroups of G to obtain [**5**, Hypothesis D.1.1], the hypothesis of D.1.5 in [**5**].

Note the second statement in (iv) appears in [**5**, D.1.3]; more precisely, the fact that $q(H_2/C_{H_2}(W), W) = 1$ appears in the last paragraph of the proof of that lemma. Similarly, the corresponding statement in (v) appears in [**5**, D.1.4]. □

Lemma 5.5. *Assume Hypothesis 5.2 holds, with $V_1 \nleqslant R_2$. Assume in addition that $N_{H_2}(V_1)$ is the unique maximal overgroup of S in H_2 and $q(H_1/C_{H_1}(V_1), V_1) > 1$. Let $\Gamma = \Gamma(H_2, V_1)$ be the set of subgroups $\langle V_1, V_1^g\rangle$ of H_2 that are not 2-groups, and let $\Gamma_* = \Gamma_*(H_2, V_1)$ be the set of minimal members of Γ.*

(i) *$\Gamma \neq \varnothing$. Pick $X \in \Gamma_*$ and set $U_1 = V_1 \cap R_2$.*

(ii) *$N_X(V_1)$ is the unique maximal overgroup M of V_1 in X, and V_1 is weakly closed in M with respect to X. Pick $g \in X - M$ and set $U = U_1U_1^g$, and $I = U_1 \cap U_1^g$.*

(iii) *$X = \langle V_1, V_1^g\rangle$, $I \leqslant Z(X)$ and $U \trianglelefteq X$. Set $\bar{X} = X/I$ and $m = m(\bar{U}_1)$.*

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(iv) $\bar{U} = \bar{U}_1 \oplus \bar{U}_1^g$ *is elementary abelian of rank* $2m$, *for some positive integer* m.

(v) *Let* $W_1 = V_1 \cap O_2(X)$, $W = W_1 W_1^g$ *and* $J = V_1 \cap V_1^g$. *Then* $J \leqslant Z(X)$ *and* $W \trianglelefteq X$. *Set* $\tilde{X} = X/J$. *Then* $\tilde{W} = \tilde{W}_1 \oplus \tilde{W}_1^g$ *is elementary abelian of rank* $2k$, *where* $k = m(\tilde{W}_1) \geqslant m$, *and* $\tilde{W}_1 = [\tilde{W}, V_1] = C_{\tilde{W}}(v)$ *for each* $v \in V_1 - W_1$.

(vi) *Set* $e = m_2(V_1/W_1)$. *Then either* $e = 1$ *and* $X/W \cong D_{2n}$ *for some odd integer* n, *or* $e > 1$ *and* $X/W \cong L_2(2^e)$ *or* $\mathrm{Sz}(2^e)$.

(vii) *If* $e \geqslant k$, *then we have that* $e = k$, $\hat{q}(H_1/C_{H_1}(V_1), V_1) \leqslant 2$, $X/W \cong L_2(2^e)$ *and* $q(X/C_X(\tilde{W}), \tilde{W}) \geqslant 1$.

(viii) *If* $e < k$, *then* $\hat{q}(H_1/C_{H_1}(V_1), V_1) < 2$.

Proof. As $V_1 \nleqslant R_2$, (i) follows from the Baer–Suzuki Theorem (cf. [**1**, 39.6]). Pick $X \in \Gamma_*$. Applying the Baer–Suzuki Theorem again, there exists $g \in X$ with $\langle V_1, V_1^g \rangle$ not a 2-group and, by minimality of X, $X = \langle V_1, V_1^g \rangle$.

As V_1 is abelian, $I \leqslant Z(X)$. As R_2 and V_1 are normal in S, $[V_1, R_2] \leqslant U_1$. Then, as $R_2 \trianglelefteq H_2$, also $[V_1^g, R_2] \leqslant U_1^g$. Hence, $U \trianglelefteq X$, completing the proof of (iii), modulo (ii), which shows that, for each $g \in X - M$, $\langle V_1, V_1^g \rangle = X$.

If $U_1 = I$, then $X \leqslant C_{H_2}(R_2/U_2) \cap C_{H_2}(U_2) \leqslant R_2$: a contradiction. Therefore, $U_1 \neq I$, so $m = m(\bar{U}_1)$ is positive. Now (iv) follows.

By minimality of X,

(a) for each $x \in X$, either $X = \langle V_1, V_1^x \rangle$ or $\langle V_1, V_1^x \rangle$ is a 2-group.

Then it follows from (a) and Baer–Suzuki that

(b) if $V_1 \leqslant Y < X$, then $V_1 \leqslant O_2(Y)$.

For $V_1 \leqslant Y < X$, set $P(Y) = \langle V_1^X \cap Y \rangle$. It follows from (b) that

(c) $P(Y)$ is a normal 2-subgroup of Y.

Let $V_1 \leqslant T \in \mathrm{Syl}_2(X)$ and $M = N_X(P(T))$. We claim that

(d) M is the unique maximal overgroup of V_1 in X.

If not, choose $V_1 \leqslant Y < X$ with $Y \nleqslant M$ and $P = P(Y \cap M)$ maximal subject to this constraint. If $P(T) = P$, then also $P = P(Y)$, so $Y \leqslant N_X(P(Y)) \leqslant M$: a contradiction. Thus, $P < P(N_{P(T)}(P))$ and as $V_1 \leqslant P$ but $V_1 \nleqslant O_2(X)$, $N_X(P) < X$, so by maximality of P, $N_X(P) \leqslant M$. Then $N_{P(Y)}(P) \leqslant M$, so $N_{P(Y)}(P) = P$, and hence $P(Y) = P$. But then $Y \leqslant N_X(P) \leqslant M$, for our final contradiction establishing (d).

Next,

(e) for each involution $t \in X - M$, $C_{\bar{U}_1}(t) = 1$, so $m_2(C_{\bar{U}}(t)) = m$.

Namely, by (d), $X = \langle V_1, t \rangle$, so X centralizes $C_{\bar{U}_1}(t)$, and hence its preimage U_2 in X is normal in X. Thus, $U_2 \leqslant U_1 \cap U_1^g = I$, so $U_2 = I$. Then, as $m_2(\bar{U}) = 2m$ and $m_2(C_{\bar{U}}(t)) \geqslant m_2(\bar{U})/2$, (e) follows.

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(f) For each $v \in V_1 - O_2(X)$, $C_{\bar{U}}(v) = \bar{U}_1$.

Again by the Baer–Suzuki Theorem there exists $x \in X$ with $\langle v, v^x \rangle$ not a 2-group, so, as $V_1 \leqslant P(T) \leqslant O_2(M)$, $v^x \notin M$. Hence, by (e), $m_2(C_{\bar{U}}(v)) = m_2(C_{\bar{U}}(v^x)) = m = m_2(\bar{U}_1)$, so, as v centralizes U_1, (f) follows.

(g) $M = N_X(V_1)$ and V_1 is weakly closed in M with respect to X.

If V_1 is weakly closed in M, then $P(T) = V_1$, so $M = N_X(V_1)$. Thus, we may assume that V_1 is not weakly closed in M, so, as $\langle V_1^X \cap M \rangle = P(T)$, V_1 is not weakly closed in $P(T)$. Then, by Alperin's Fusion Theorem (cf. [**8**, A.10]), there is an overgroup Q of V_1 in $P(T)$ such that $V_1 \trianglelefteq Q$, and $h \in N_X(Q)$ such that $V_1 \neq V_1^h$. Suppose $1 \neq [V_1, V_1^h]$. Then, replacing h by h^{-1} if necessary, we may assume $m(V_1^h/C_{V_1^h}(V)) \geqslant m(V_1/C_V(V_1^h))$, contradicting $q(H_1/C_{H_1}(V_1), V_1) > 1$.

Therefore, $[V_1, V_1^h] = 1$. But, by (f), $C_{\bar{U}}(V_1^h) = \bar{U}_1^h$, so $U_1 = U_1^h$. As $U_1 \neq I$, U_1 is not normal in H_2, so, as $U_1 \trianglelefteq S$ and, by hypothesis, $N_{H_2}(V_1)$ is the unique maximal overgroup of S in H_2, it follows that $N_{H_2}(U_1) \leqslant N_{H_2}(V_1)$. Thus, $V_1 = V_1^h$, contrary to the choice of h. This completes the proof of (g).

Note that (d) and (g) prove (ii), and hence complete the proof of (iii).

As $M = N_X(V_1)$, $W_1 = V_1 \cap O_2(X) \trianglelefteq M$, so $M = N_X(W_1)$ or X by (d). However, if $W_1 \trianglelefteq X$, then as $U_1 \leqslant W_1$; also, $U \leqslant W_1 \leqslant V_1$, so $U \leqslant V_1 \cap R_2 = U_1$, contrary to (iv). Thus, $M = N_X(W_1)$. Then we can apply various arguments above to W_1 in place of U_1, to establish (v).

Set $X^* = X/W$. If $W_1 < U_2 \leqslant V_1$ and $x \in X$ with $U_2^x \leqslant M$, then, as $C_{\tilde{W}}(U_2) = \tilde{W}_1$ by (v), and as $C_{\tilde{W}_1}(U_2^x) \neq 1$, it follows that $x \in M$. Therefore, V_1^* is a strongly closed elementary abelian TI-subgroup of M^*. Then, as $X^* = \langle V_1^*, V_1^{*g} \rangle$, (vi) follows (cf. [**5**, I.8.3]).

Let $A = W_1^g$. Then $A \leqslant M = N_X(V_1)$ and, by (v), $[A, V_1] = W_1$ and $[A, W_1] \leqslant J \leqslant C_{V_1}(A)$, so A is cubic on V_1. Furthermore, by (v), $m(A/C_A(V_1)) = m(A/J) = k$ and $m(V_1/C_{V_1}(A)) \leqslant m(V_1/J) = k + e$, so if $k > e$, then $\hat{q}(H_1/C_{H_1}(V_1), V_1) < 2$. So (viii) holds. On the other hand, if $k = e$, then $\hat{q}(H_1/C_{H_1}(V_1), V_1) \leqslant 2$.

Suppose $e \geqslant k$. As X^* is faithful on $\tilde{W}$ of rank $2k$, it follows from (v) and the representation theory of X^* (cf. [**5**, B.4.2]) that $e = k$, $X^* \cong L_2(2^e)$, and $\tilde{W}$ is the natural module for X^*. In particular, (vii) holds. □

Lemma 5.6. *Assume Hypothesis 5.1 holds, and in addition assume that*

(i) $M \in \mathcal{M}(S)$, $R = C_S(V(M))$, $N_M(R)$ *is a uniqueness group and*

(ii) $q(M/C_M(V(M)), V(M)) > 1$.

Then $\mathrm{Baum}(S) = \mathrm{Baum}(R)$ *and we have the following.*

(1) $\{M\} = \mathcal{M}(G(\mathrm{Baum}(S)))$.

(2) $G(S) \leqslant M$.

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(3) *Suppose $H \in \mathfrak{H}$ with $H/O_2(H)$ S_3-free. Then one of the following holds:*

(a) $H \leqslant M$;

(b) $\hat{q}(M/C_M(V(M)), V(M)) < 2$;

(c) $V(M)$ *is a dual FF-module for* $M/C_M(V(M))$.

Proof. Let $V = V(M)$ and $B = \mathrm{Baum}(M)$. By (ii), $J(S)$ centralizes V, so $B \leqslant R$ by Lemma 4.6. Set $Y = N_M(R)$. Then $Y \leqslant G(B)$ and $\mathcal{M}(Y) = \{M\}$ by (i), so (1) follows. Then, as $G(S) \leqslant G(B)$, (1) implies (2).

Assume the hypothesis of (3) and assume $H \not\leqslant M$. Then by (2) and Lemma 5.3 (i), there exists $H' \in \mathcal{X}^*_M(H)$ and, replacing H by H', we may assume $H \in \mathcal{X}^*_M(H)$. Hence, by (1) and Lemma 5.3 (ii), the pair Y, H satisfies Hypothesis 5.2, and by Lemma 5.3 (iii) $N_H(V)$ is the unique maximal overgroup of S in H.

Suppose $V \not\leqslant O_2(H)$. Then Y, H satisfies the hypothesis of Lemma 5.5, so by Lemma 5.5 (i) we can pick $X \in \Gamma_*(H, V_1)$. As $H/O_2(H)$ is S_3-free, X/W is not isomorphic to $L_2(2^f)$ for any f, so, by Lemma 5.5 (vii), $e < k$. Hence, (b) holds in this case by Lemma 5.5 (viii).

Thus, we may assume $V \leqslant O_2(H)$. Hence, by the qrc-Lemma 5.4, one of the cases (ii)–(v) of that result hold. As $H/O_2(H)$ is S_3-free, case (v) does not hold, and in case (iv), $q < 2$, so, as $\hat{q} = \hat{q}(M/C_M(V), V) \leqslant q$, conclusion (b) of the lemma holds.

Thus, we may assume case (ii) or (iii) of Lemma 5.4 holds. But, by (ii), $q > 1$, so case (3) holds. Hence, (c) holds in this case, completing the proof. □

6. The parameter $\hat{q}$

In this section we assume the following.

Hypothesis 6.1. *G is a finite group with $O_2(G) = 1$ and V is a faithful $\mathbb{F}_2G$-module.*

In addition, we adopt the notation from [**5**, § D.2]. In particular, set $\hat{q} = \hat{q}(G, V)$ and $\hat{\mathcal{Q}}_* = \hat{\mathcal{Q}}_*(G, V)$. The parameter $\hat{q}$ is defined in [**5**, Definition B.4.1]. It is the minimum of $m(V/C_V(A))/m(A)$ as A ranges over elementary abelian 2-subgroups of G such that A is *cubic* on V, i.e. such that $[V, A, A, A] = 0$. From [**5**, Definition D.2.1], $\hat{\mathcal{Q}}_*$ consists of those non-trivial elementary abelian 2-subgroups A of G such that $m(V/C_V(A))/m(A) = \hat{q}$, A is cubic on V and A is minimal subject to these constraints.

For $X = O^2(X) \leqslant G$ and $Y \leqslant N_G(X)$, see [**5**, Definition A.1.40] for the definition of $\mathrm{Irr}_+(X, V)$ and $\mathrm{Irr}_+(X, V, Y)$. Namely, $\mathrm{Irr}_+(X, V)$ consists of the X-submodules I of V such that $I = [I, X]$ and X is irreducible on $\tilde{I} = I/C_I(X)$. Furthermore, $\mathrm{Irr}_+(X, V, Y)$ consists of those $I \in \mathrm{Irr}_+(X, V)$ such that $\tilde{I}$ is an X-homogeneous component of $\langle \tilde{I}^Y \rangle$.

The parameter $\hat{q}$ is a variant of the parameter q appearing in the qrc-Lemma 5.4. The definition of q is the same as that of $\hat{q}$, except that it is defined with respect to *quadratic* subgroups A: those with $[V, A, A] = 0$. In particular, $\hat{q} \leqslant q$. The parameter $\hat{q}$ is important to us because of its appearance in Lemma 5.5 and Lemma 5.6. To apply Lemma 5.6 to an S_3-free fusion system, we need the lower bound $\hat{q} \geqslant 2$ on $\mathbb{F}_2G$-modules V for S_3-free groups G, which is established later in Theorem 6.5. This bound is obtained by a reduction to the case G simple. To make that reduction we next define certain parameters μ and η.

Definition 6.2. Given a non-abelian finite quasisimple group L and a non-trivial irreducible $\mathbb{F}_2L$-module U, define

$$\mu(L,U) = \frac{(\dim_F(U)-1)\dim_{\mathbb{F}_2}(U)}{4(m_2(\mathrm{Aut}(L))+1)},$$

where $F = \mathrm{End}_{\mathbb{F}_2L}(U)$. Define $\eta(L,U) = \hat{q}(N_{GL(U)}(\mathrm{Aut}_L(U)), U)$.

Given a non-abelian finite simple group L, define $\mu(L)$ and $\eta(L)$ to be the minima of the $\mu(\hat{L},U)$, $\eta(\hat{L},U)$, respectively, as U varies over all non-trivial irreducible $\mathbb{F}_2\hat{L}$-modules, where $\hat{L}$ is the universal covering group of L.

Our next two lemmas show that bounds on $\mu(L)$ and $\eta(L)$ for components L of G suffice to establish the bound $\hat{q}(G,V) \geqslant 2$ in Theorem 6.5 on $\mathbb{F}_2G$-modules V for S_3-free groups G. The bounds on μ and η can be obtained from the work of Guralnick and Malle in [**12**].

Lemma 6.3. *Assume that $\hat{q} \leqslant 2$, $A \in \hat{\mathcal{Q}}_*$ and L is a component of G with $[A,L] \neq 1$. Set $H = \langle L^A\rangle$, $\tilde{V} = V/C_V(H)$ and pick $I \in \mathrm{Irr}_+(H,V)$ with $[I,L] \neq 0$. Then, we have the following.*

(i) *Replacing I by a suitable $I_1 \in \mathrm{Irr}_+(H,V)$ with $\tilde{I} \cong \tilde{I}_1$ as an $\mathbb{F}_2H$-module, we have $\hat{q}(\mathrm{Aut}_{HA}(\tilde{I}),\tilde{I}) \leqslant \hat{q}$.*

(ii) *Suppose V is an irreducible $\mathbb{F}_2H$-module and let $U \in \mathrm{Irr}_+(L,V)$. If $\mu(L,U) > 1$, then $H = L$.*

Proof. Set $K = HA$, let $A \leqslant S \in \mathrm{Syl}_2(K)$ and set $I_S = \langle I^S\rangle$. Then $S = (S\cap H)A$, so

(a) $I^S = I^{(H\cap S)A} = I^A$.

Next, by [**5**, A.1.42.2], replacing I by a suitable member I_1 of $\mathrm{Irr}_+(H,I_S)$ with $\tilde{I}_1 \cong \tilde{I}$, we may assume $I \in \mathrm{Irr}_+(H,I_S,S)$. Then by [**5**, A.1.42.3],

(b) $\tilde{I}_S$ is the direct sum of the members of $\tilde{I}^A$.

Next, by [**5**, D.2.7], $\mathrm{Aut}_A(I_S) \in \hat{Q}_r(\mathrm{Aut}_K(I_S),I_S)$ for some $r \leqslant \hat{q}$. In particular, $\hat{q}(\mathrm{Aut}_K(I_S),I_S) \leqslant \hat{q}$. Next, trivially, $\hat{q}(\mathrm{Aut}_K(\tilde{I}_S),\tilde{I}_S) \leqslant \hat{q}(\mathrm{Aut}_K(I_S),I_S)$. Let $A_1 \in \hat{Q}_*(\mathrm{Aut}_K(\tilde{I}_S),\tilde{I}_S)$. Applying [**5**, D.2.9.1] to the direct sum decomposition of $\tilde{I}_S$ in (b), we conclude that $|A_1 \colon N_{A_1}(\tilde{I})| \leqslant 2$. Set $\tilde{W} = \langle \tilde{I}^{A_1}\rangle$. By another application of [**5**, D.2.7], $\hat{q}(\mathrm{Aut}_K(\tilde{W}),\tilde{W}) \leqslant \hat{q}(\mathrm{Aut}_K(\tilde{I}_S),\tilde{I}_S)$. Pick $A_2 \in \hat{\mathcal{Q}}_*(\mathrm{Aut}_K(\tilde{W}),\tilde{W})$. If $\tilde{W} \neq \tilde{I}$, then, applying [**5**, D.2.9.1] to the direct sum decomposition $\tilde{W} = \tilde{I} \oplus \tilde{I}^t$, $t \in A_1 - N_{A_1}(\tilde{I})$, we conclude that A_2 acts on $\tilde{I}$. Then, by yet another application of [**5**, D.2.7], $\hat{q}(\mathrm{Aut}_K(\tilde{I}),\tilde{I}) \leqslant \hat{q}(\mathrm{Aut}_K(\tilde{W}),\tilde{W})$. Thus, in any event we have established (i).

So assume the hypothesis of (ii) with $r = |L^A| > 1$. Let $L^A = \{L_1,\dots,L_r\}$ with $L = L_1$ and set $r = 2^f$. As H is irreducible on V, U is an irreducible $\mathbb{F}_2L$-module

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and V is a homogeneous semisimple $\mathbb{F}_2L$-module. Set $F = \mathrm{End}_{\mathbb{F}_2}(U) \cong \mathbb{F}_{2^e}$, write U_F for U regarded as an FL-module, and set $d = \dim_F(U_F)$. We argue as in the proof of [**5**, D.3.7]. In particular, by [**1**, 27.14], we may regard V as an FH-module V_F, and $V_F \cong U_{1,F} \otimes \cdots \otimes U_{r,F}$, where $U_1 = U$ for $i > 1$, $U_i = U^{a_i}$ for $a_i \in A$ with $L^{a_i} = L_i$, regarded as an $\mathbb{F}_2L_i$-module, and $U_{i,F} = U_F^{a_i}$ is an FL_i-module.

Let $t \in A - N_A(L)$ and choose notation so that $H = J \times J^t$, where $J = L_1 \cdots L_s$ and $s = r/2$. Thus, $V_F = W_F \otimes W_F^t$ with W_F, W_F^t F-modules for J and J^t, respectively. Moreover, $W_F = U_{1,F} \otimes \cdots \otimes U_{s,F}$ as an FJ-module, so $\dim_F(W_F) = d^s$. Now, an argument in the proof of [**5**, D.3.7] shows

(c) $\dim_F(V_F/C_{V_F}(t)) = d^s(d^s - 1)/2$, so $\dim_{\mathbb{F}_2}(V/C_V(t)) = d^s(d^s - 1)e/2$.

Next observe that

(d) $m_2(A) \leqslant f + k$, where $k = m_2(\mathrm{Aut}(L))$.

Namely, let B be a complement to $N_A(L)$ in A. Then $m_2(B) = f$, and (modulo $Z(H)$) $C_L(B)$ is a full diagonal subgroup of H isomorphic to L with $N_A(L)$ faithful on $C_L(B)$, so $m_2(A) = f + m_2(N_A(L)) \leqslant f + k$.

Now, by (c) and (d),

$$2 \geqslant \hat{q} = \frac{m(V/C_V(A))}{m(A)} \geqslant \frac{m(V/C_V(t))}{f + k} = \frac{d^s(d^s - 1)e}{2(f + k)},$$

and hence, as $m(U) = de$, we have

(e) $1 \geqslant d^{s-1}(d^s - 1)m(U)/4(f + k)$.

Furthermore, as $f \geqslant 1$,

$$\frac{d^{s-1}(d^s - 1)}{f + k} \geqslant \frac{d - 1}{k + 1},$$

so it follows from (e) that

(f) $1 \geqslant \mu(L, U)$.

Now (ii) follows from (f), completing the proof of (ii) and the lemma. □

Lemma 6.4. *Assume $\hat{q} \leqslant 2$ and L is a component of G which is not centralized by $\hat{\mathcal{Q}}_*(G, V)$. Assume further that $\mu(L/Z(L)) > 1$. Then $\hat{q}(G, V) \geqslant \eta(L/Z(L))$, so in particular $\eta(L/Z(L)) \leqslant 2$.*

Proof. By hypothesis there exists $A \in \hat{\mathcal{Q}}_*$ with $[A, L] \neq 1$. Set $H = \langle L^A \rangle$. By Lemma 6.3 (i), we may assume V is an irreducible $\mathbb{F}_2HA$-module. Then as $\mu(L/Z(L)) >$ 1, we conclude from Lemma 6.3 2 that $H = L$, so V is an irreducible $\mathbb{F}_2L$-module. Hence, by definition of $\eta(L/Z(L))$, $\hat{q} \geqslant \eta(L/Z(L))$, completing the proof of the lemma. □

The following result is probably known, but we do not have a reference, so instead we supply a proof.

Theorem 6.5. *Let G be a finite S_3-free $\mathcal{K}$-group with $O_2(G) = 1$ and let V be a faithful $\mathbb{F}_2G$-module. Then $\hat{q}(G, V) \geqslant 2$.*

Proof. Assume that the theorem fails and choose a counter-example of minimal order, and, subject to that constraint, with $m(V)$ minimal. Thus, $\hat{q} < 2$ and we can choose $A \in \hat{Q}_*$. By parts (1) and (2) of [**5**, D.2.13], A centralizes $F(G)$. Therefore, there exists a component L of G with $[L, A] \neq 1$. Thus, to obtain a counter-example and complete the proof, it suffices by Lemma 6.4 to show that $\mu(L/Z(L)) > 1$ and $\eta(L/Z(L)) \geqslant 2$.

As G is an S_3-free $\mathcal{K}$-group, it follows that $L/Z(L) \cong \mathrm{Sz}(2^k)$ or $L_2(3^k)$ for some $k \geqslant 3$ odd. In particular, $m_2(\mathrm{Aut}(L/Z(L))) = k, 2$, in the respective case. Observe from [**11**] that the Schur multiplier of $L/Z(L)$ is a 2-group, so L is simple. Pick U and F as in Definition 6.2, and set $d = \dim_F(U)$.

Suppose first that $L \cong \mathrm{Sz}(2^k)$. Then, by [**12**], $\eta(L) = 2$. Furthermore, $d \geqslant 4$ and $\dim(U) \geqslant 4k$, so indeed $\mu(L) > 1$.

Suppose next that $L \cong L_3(3^k)$. By [**12**], $\eta(L) > 2$. Further, $m(U) \geqslant m([B, U])$, where B is a Borel subgroup of L, and, as B is a Frobenius group that is the split extension of E_{3^k} by a cyclic group of order $(3^k - 1)/2$, $m([B, U]) \geqslant 3^k - 1 \geqslant 26$. Thus,

$$\mu(L) \geqslant \frac{m(U)}{12} \geqslant \frac{26}{12} > 1,$$

completing the proof. □

7. S_3-free fusion systems

We begin this section with a proof of Theorem 1. We prove the theorem via a series of reductions. Assume the theorem is false, and choose a minimal counter-example $\mathcal{F}$.

Lemma 7.1. $O_2(\mathcal{F}) = 1$.

Proof. Assume $Q = O_2(\mathcal{F}) \neq 1$. Suppose first that $Q \leqslant Z(\mathcal{F})$. Then by Lemma 1.2 (i), $\mathcal{F}/Q$ is a saturated fusion system on S/Q with $O_2(\mathcal{F}/Q) = 1$. But, by the definition of $\mathcal{F}/Q$ in [**2**, § 8], for $Q \leqslant P \leqslant S$, $\mathrm{Aut}_{\mathcal{F}/Q}(P/Q) = \mathrm{Aut}_{\mathcal{F}}(P/Q)$, so $\mathcal{F}/Q$ is S_3-free. Therefore, by minimality of $\mathcal{F}$, $\mathcal{F}/Q$ is constrained, so $1 \neq O_2(\mathcal{F}/Q)$: a contradiction.

Therefore, $Q \nleqslant Z(\mathcal{F})$, so $\mathcal{C} = C_{\mathcal{F}}(Q) \neq \mathcal{F}$. However, by Lemma 1.2 (ii), $\mathcal{C}$ is a saturated fusion system on $C_S(Q)$. As $\mathcal{C} \leqslant \mathcal{F}$ and $\mathcal{F}$ is S_3-free, $\mathcal{C}$ is S_3-free. Hence, as $\mathcal{C} \neq \mathcal{F}$, $\mathcal{C}$ is constrained by minimality of $\mathcal{F}$. Then $\mathcal{F}$ is constrained by Lemma 1.2 (ii), contrary to the choice of $\mathcal{F}$. □

Lemma 7.2.

(i) *$\mathcal{F}$ is of characteristic 2-type.*

(ii) *$\mathcal{F}$ is a local CK-system.*

Proof. Let $U \in \mathcal{F}^f$ and $\mathcal{N} = N_{\mathcal{F}}(U)$. By Lemma 7.1, $\mathcal{N} < \mathcal{F}$, while as $\mathcal{F}$ is S_3-free, so is $\mathcal{N}$. Hence, by minimality of $\mathcal{F}$, $\mathcal{N}$ is constrained, establishing (i). Further, applying Corollary 3 in an inductive context, all composition factors of $\mathrm{Aut}_{\mathcal{N}}(U)$ are in $\mathcal{K}$, so (ii) also holds. □

Lemma 7.3. $\mathcal{F} = \langle N_{\mathcal{F}}(U) \colon U \in \mathcal{U} \rangle$.

Proof. By Lemma 7.2, $\mathcal{F}$ satisfies the hypotheses of [**3**, Theorem 1], so by that theorem, either the lemma holds or $\mathcal{F}$ is an obstruction to pushing up at the prime 2, and we may assume the latter. However, by inspection of the list of such obstructions in [**3**], none are S_3-free. □

Remark 7.4. The full strength of [**3**, Theorem 1] is not required in the proof of Lemma 7.3; see the discussion in [**3**, Example 8.6], which shows that the weaker [**3**, Theorem 7.13] suffices when $\mathcal{F}$ is S_3-free. Moreover, when $\mathcal{F}$ is S_3-free, the proof of [**3**, Theorem 7.13] is fairly easy; namely in that case, Theorem 7.13 follows from [**3**, 6.6.3].

We are now in a position to complete the proof of Theorem 1. By choice of $\mathcal{F}$, $\mathcal{F}$ is a saturated fusion system on a finite 2-group S, which is not constrained. By Lemma 7.2 (i), $\mathcal{F}$ is of characteristic 2-type. Thus, Hypothesis 2.1 is satisfied. Then, by Lemma 7.3, Hypothesis 4.1, and hence also Hypothesis 5.1, is satisfied.

Choose M to be maximal in $\mathcal{M}(S)$ with respect to $\lesssim$ and set $V = V(M)$. As $\mathcal{F}$ is S_3-free, so is $M/C_M(V)$, and by Lemma 7.2 (ii), $M/C_M(V)$ is a $\mathcal{K}$-group. Therefore, by Theorem 6.5,

$$\hat{q}(M/C_M(V), V) \geqslant 2. \tag{$*$}$$

Thus, hypothesis (ii) of Lemma 5.6 is satisfied, while hypothesis (i) of that lemma holds by Lemma 4.5 (i). Therefore, $G(S) \leqslant M$ by Lemma 5.6 (2). Next, by Lemma 5.3 (i), there exists $H \in \mathfrak{H}_M$. As $\mathcal{F}$ is S_3-free, so is $H/O_2(H)$, so the hypotheses of Lemma 5.6 (3) are satisfied, and therefore one of conclusions (a)–(c) of that lemma hold. As $H \in \mathfrak{H}_M$, conclusion (a) fails. Further conclusion (b) fails by ($*$). Finally, as $M/C_M(V)$ is S_3-free, condition (c) fails. This contradiction completes the proof of Theorem 1.

We next prove Corollary 2. Assume G is an S_4-free finite simple group. Let $S \in \mathrm{Syl}_2(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$. As G is S_4-free, $\mathcal{F}$ is S_3-free. Therefore, $\mathcal{F}$ is constrained by Theorem 1. Hence, there exists a non-trivial abelian subgroup of S that is strongly closed in S with respect to G. Now, G is Goldschmidt group by a theorem of Goldschmidt in [**10**], establishing Corollary 2.

Next, we observe that Corollary 3 follows from Corollary 2. Namely, assume G is a finite non-abelian finite simple group that is S_3-free. By Corollary 2, G is a Goldschmidt group. Hence, Corollary 3 follows, as G is S_3-free.

Finally, observe that Corollary 4 follows from Corollary 3 and the fact that 3 divides the order of $L_2(3^n)$.

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