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Thompson's semigroup and the first Hochschild cohomology

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Abstract. In this paper, we apply the theory of algebraic cohomology to study the amenability of Thompson's group $\mathcal{F}$. We introduce the notion of unique factorization semigroup which contains Thompson's semigroup $\mathcal{S}$ and the free semigroup $\mathcal{F}_n$ on n (≥ 2) generators. Let $\mathfrak{B}(\mathcal{S})$ and $\mathfrak{B}(\mathcal{F}_n)$ be the Banach algebras generated by the left regular representations of $\mathcal{S}$ and $\mathcal{F}_n$, respectively. We prove that all derivations on $\mathfrak{B}(\mathcal{S})$ and $\mathfrak{B}(\mathcal{F}_n)$ are automatically continuous, and every derivation on $\mathfrak{B}(\mathcal{S})$ is induced by a bounded linear operator in $\mathcal{L}(\mathcal{S})$, the weak-operator closed Banach algebra consisting of all bounded left convolution operators on $l^2(\mathcal{S})$. Moreover, we prove that the first continuous Hochschild cohomology group of $\mathfrak{B}(\mathcal{S})$ with coefficients in $\mathcal{L}(\mathcal{S})$ vanishes. These conclusions provide positive indications for the left amenability of Thompson's semigroup.

1 Introduction

The cohomology theory of associative algebras was initiated by Hochschild [10–12] in 1945 in terms of multilinear maps into a bimodule and coboundary operators. In 1953, after discussing with Singer at a conference, Kaplansky went on from there to write his paper [21] proposing some problems about derivations on C^*-algebras and von Neumann algebras. He showed that every derivation on a von Neumann algebra $\mathcal{M}$ of type I is inner, which may be restated in cohomological terms that the first continuous cohomology group of $\mathcal{M}$ with coefficients in itself vanishes.

Recall that a *derivation* of a Banach algebra $\mathcal{A}$ (over the complex field $\mathbb{C}$) with coefficients in a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is a linear map D from $\mathcal{A}$ into $\mathcal{X}$ satisfying the Leibniz rule, i.e., $D(AB) = AD(B) + D(A)B$ for all A, B in $\mathcal{A}$. We say that D is *inner* if there is an element T in $\mathcal{X}$ such that $D(A) = AT - TA$ for each A in $\mathcal{A}$. Let $\mathcal{Z}^1(\mathcal{A}, \mathcal{X})$ denote the space of all (continuous) derivations from $\mathcal{A}$ into $\mathcal{X}$ and $\mathcal{B}^1(\mathcal{A}, \mathcal{X})$ the space of all inner derivations. It is clear that $\mathcal{B}^1(\mathcal{A}, \mathcal{X})$ is a linear subspace of $\mathcal{Z}^1(\mathcal{A}, \mathcal{X})$. The first (continuous) Hochschild cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}$ is then defined to be the following quotient vector space:

$$H^1(\mathcal{A}, \mathcal{X}) = \frac{\mathcal{Z}^1(\mathcal{A}, \mathcal{X})}{\mathcal{B}^1(\mathcal{A}, \mathcal{X})}.$$

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The study of cohomology groups of operator algebras started with the Kadison–Sakai theorem [18, 24]: Every derivation on a von Neumann algebra $\mathcal{M}$ is inner, i.e., $H^1(\mathcal{M},\mathcal{M}) = 0$. Whether every derivation of a von Neumann algebra $\mathcal{M}$ into $B(\mathcal{H})$ is always inner is still an open problem, which is equivalent to several open problems in operator algebras. One of such problems is Kadison's similarity problem [17], which asks if every bounded representation of a C^*-algebra into $B(\mathcal{H})$ is always similar to a $*$-representation. To study the classification of von Neumann algebras, from 1968 to 1972, Johnson, Kadison, and Ringrose [14, 15, 19, 20] proved a series of technical results of the cohomology groups of von Neumann algebras. In particular, they showed that $H^n(\mathcal{M},\mathcal{M}) = 0$ for all $n \geq 1$ when $\mathcal{M}$ is a hyperfinite von Neumann algebra. Due to this fact, Kadison and Ringrose conjectured that these cohomology groups are zero for all von Neumann algebras. With the aid of the theory of completely bounded cohomology groups, this conjecture can be ultimately reduced to the case when $\mathcal{M}$ is a factor of type II_1 with separable predual. In [26], Sinclair and Smith showed that the conjecture holds for von Neumann algebras with Cartan subalgebras and separable preduals. Later in 2003, Christensen et al. [4] proved that the continuous cohomology groups $H^n(\mathcal{M},\mathcal{M})$ and $H^n(\mathcal{M},B(\mathcal{H}))$ of a factor $\mathcal{M} \subseteq B(\mathcal{H})$ of type II_1 with property Γ are zero for all $n \geq 1$. The latest result was proved by Pop and Smith [22]. They showed that the second cohomology group $H^2(\mathcal{M}\overline{\otimes}\mathcal{N},\mathcal{M}\overline{\otimes}\mathcal{N})$ vanishes for arbitrary type II_1 von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$. Note that the free group factor $\mathcal{L}(\mathcal{F}_2)$ satisfies none of the above cases and the higher order cohomology groups of $\mathcal{L}(\mathcal{F}_2)$ are still unknown.

Cohomology groups of Banach algebras are different from that of von Neumann algebras in two main aspects: the automatic continuity of derivations and the cohomology groups. It was conjectured by Kaplansky in [21] (which was finally proved by Sakai in [23]) that every derivation on a C*-algebra is continuous. While, derivations on a Banach algebra are not necessarily continuous. In [1], Bade and Curtis constructed several examples of Banach algebras on which not all derivations are continuous. On the other hand, Johnson and Sinclair [16] showed that the continuity of derivations still holds for semisimple Banach algebras. Up to now, there are no examples showing that the cohomology groups of a von Neumann algebra are nontrivial. Let $\mathcal{A}(\mathbb{D})$ be the set of all complex-valued functions that are continuous on the closed unit disk and analytic in the interior. Then $\mathcal{A}(\mathbb{D})$ endowed with the supremum norm is a unital Banach algebra. The second cohomology group of $\mathcal{A}(\mathbb{D})$ (with coefficients in itself) is nontrivial [13, Proposition 9.1].

Thompson's group $\mathcal{F} = \langle X_0, X_1, X_2, \dots \mid X_i^{-1}X_jX_i = X_{j+1},\ j > i \rangle$ was introduced by Richard Thompson in 1965 [3]. It was conjectured by Geoghegan around 1979 that: (i) the group $\mathcal{F}$ contains no non-abelian free groups; (ii) $\mathcal{F}$ is not amenable. Statement (i) was obtained by Brin and Squier [2] in 1985 while (ii) still remains unknown. Many research works nowadays are developed to answer this question due to two main reasons: (1) $\mathcal{F}$ is related to many branches of mathematics such as geometric group theory; (2) the amenability problem is one of the most significant research areas in mathematics. Every non-unital element in $\mathcal{F}$ has as a unique normal form: $X_0^{\alpha_0}X_1^{\alpha_1}\dots X_n^{\alpha_n}X_n^{-\beta_n}\dots X_0^{-\beta_0}$, where $\alpha_0,\dots,\alpha_n,\beta_0,\dots,\beta_n$, and n are natural numbers such that (i) exactly one of α_n and β_n is nonzero and (ii) if $\alpha_k > 0$ and $\beta_k > 0$ for some integer k with $0 < k < n$, then $\alpha_{k+1} > 0$ or $\beta_{k+1} > 0$. For example, $X_0X_1X_0^{-1}$ and

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$X_0X_1X_2^{-1}X_0^{-1}$ are two normal forms while $X_0X_2X_3^{-1}X_0^{-1}$ is not. The amenability of Thompson's group $\mathcal{F}$ has been an open problem for more than 40 years. We refer to a nice survey paper [3] for more details.

The notion of amenability of a group is introduced by von Neumann. Let G be a discrete group and $l^\infty(G)$ the space of all bounded complex-valued functions on G. Then $l^\infty(G)$ is a commutative C^*-algebra. The group G is said to be *amenable* if there is a state μ on $l^\infty(G)$ such that $\mu(g\varphi) = \mu(\varphi)$, where $\varphi \in l^\infty(G)$, $g \in G$, and $g\varphi$ is defined by $(g\varphi)(h) = \varphi(g^{-1}h)$ for each h in G. The state μ is then called (by von Neumann) a *left invariant mean* on $l^\infty(G)$. The additive group of integers $(\mathbb{Z}, +)$ is amenable while the free group $\mathcal{F}_2$ on two generators is not. In [13], Johnson characterized the amenable group G through the first Hochschild cohomology groups of $l^1(G)$, the space of all absolute-summable complex-valued functions on G, with coefficients in dual Banach $l^1(G)$-bimodules: Let G be a discrete group. Then G is amenable if and only if $H^1(l^1(G), \mathcal{X}^*) = 0$ for each Banach $l^1(G)$-bimodule $\mathcal{X}$. This statement is also true for locally compact groups by [13].

In this paper, we shall apply the theory of cohomology to study the amenability of $\mathcal{F}$. Let $\mathcal{S}$ be the subset $\{X_0^{i_0} \dots X_n^{i_n} \in \mathcal{F} : i_j \in \mathbb{N},\ 1 \le j \le n\}$ of $\mathcal{F}$. Then $\mathcal{S} = \langle X_0, X_1, X_2, \dots \mid X_jX_i = X_iX_{j+1},\ j > i\ \rangle^+$ is a discrete cancellative semigroup. We call it *Thompson's semigroup*. The structure of $\mathcal{F}$ is inherited by $\mathcal{S}$ as well. It was proved by Grigorchuk in 1990 that Thompson's group $\mathcal{F}$ is amenable if and only if Thompson's semigroup $\mathcal{S}$ is left amenable. It is clear that every non-unital element in $\mathcal{S}$ can be uniquely written as $X_{i_1}^{\alpha_1} \dots X_{i_n}^{\alpha_n}$ $(i_1 < \cdots < i_n)$ for some positive integers α_j $(1 \le j \le n)$. This property is similar to the fundamental theorem of arithmetic which states that every natural number (≥ 2) can be uniquely written as the product of primes up to reorder. Moreover, classical arithmetic functions, such as the Möbius function and divisor function, can be generalized on $\mathcal{S}$ [27]. Ge, Ma, and Qi introduced ζ-functions on Thompson's semigroup and studied the analytic structures [9]. These properties can help us to understand the structure of $\mathcal{S}$ better. Therefore, studying $\mathcal{S}$ may bring more useful tools to study the amenability of $\mathcal{F}$.

Let $\mathfrak{B}(\mathcal{S})$ be the Banach algebra generated by the left regular representation of $\mathcal{S}$, and let $\mathcal{X}$ be a Banach $\mathfrak{B}(\mathcal{S})$-bimodule. The Banach algebras $\mathfrak{B}(\mathcal{S})$ and $\mathcal{L}(\mathcal{S})$ (see Section 2) are two important Banach $\mathfrak{B}(\mathcal{S})$-bimodules. We strongly believe that if $H^n(\mathfrak{B}(\mathcal{S}), \mathcal{X}) \neq 0$ or $H^n(\mathfrak{B}(\mathcal{S}), \mathcal{X}^*) \neq 0$ for some $n \ge 1$ and bimodule $\mathcal{X}$ or its dual $\mathcal{X}^*$, then $\mathcal{S}$ is not left amenable. The main topic of this paper is to study the cohomology groups of $\mathfrak{B}(\mathcal{S})$. The basic idea behind the calculation of cohomology groups is to take an average in a suitable way [25]. In most cases, averages are taken on amenable groups. In this article, we shall extend it to $\mathcal{S}$.

The following sections are organized as follows: In Section 2, we provide some basic definitions related to semigroup algebras and amenable semigroups. In Section 3, we introduce the notion of unique factorization semigroup and give three classical examples: Thompson's semigroup, free semigroups, and the amenable semigroup $\mathcal{T}$ (see Example 3.12). The continuity of derivations (see Proposition 4.1) and the cohomology groups of $\mathfrak{B}(\mathcal{S})$ (see Theorems 5.8 and 5.10) are the main results of this paper and are discussed in Sections 4 and 5. We end this paper with some further discussions and open questions.

2 Preliminaries

Group algebras and group actions on manifolds are two major sources for the construction of operator algebras. In applications, generalizations of groups (group algebras), such as semigroups (semigroup algebras), are also used. A brief description of semigroup algebras follows.

The Hilbert space $\mathcal{H}$ is $l^2(S)$, the space of all square-summable complex valued functions on a cancellative semigroup S. The semigroup S (with unit e) is assumed to be discrete and countable. Hence $\mathcal{H}$ is separable. The family of functions $(\delta_s)_{s\in S}$ forms an orthonormal basis of $\mathcal{H}$, where $\delta_s(s) = 1$ and $\delta_s(t) = 0$ for any $t \in S$, $t \neq s$. For each f and g in $\mathcal{H}$, let L_f be the left convolution operator defined as $L_f g = f * g$, where $f * g(s) = \sum_{uv=s} f(u)g(v)$ for each s in S. Note that the convolution operators may be unbounded. We denote by $\mathcal{L}(S)$ the set of all bounded left convolution operators on $\mathcal{H}$. Then $\mathcal{L}(S)$ is a subalgebra of $B(\mathcal{H})$. In general, $\mathcal{L}(S)$ is not a $*$-algebra. Similarly, we denote by $\mathcal{R}(S)$ the subalgebra of $B(\mathcal{H})$ consisting of all bounded right convolution operators. Then $\mathcal{L}(S)' = \mathcal{R}(S)$ and $\mathcal{R}(S)' = \mathcal{L}(S)$, which implies that $\mathcal{L}(S)$ and $\mathcal{R}(S)$ are both weak-operator closed algebras. For each s in S, the operator L_{δ_s} is an isometry on $\mathcal{H}$ and is denoted by L_s in this paper for convenience. We denote by $\mathfrak{B}(S)$ the Banach algebra generated by $\{L_s : s \in S\}$ in norm topology in $B(\mathcal{H})$. Then $\mathfrak{B}(S)$ is a Banach subalgebra of $\mathcal{L}(S)$.

Specific examples for such Banach algebras result from choosing for S any of the free semigroup $\mathcal{F}_n$ on n (≥ 2) generators, Thompson's semigroup $\mathcal{S}$, or the multiplicative semigroup of natural numbers $(\mathbb{N}, \star)$. The algebraic structure of $\mathfrak{B}(S)$ can reflect the structure of S. In [5], Dong, Huang, and Xue proved that the maximal ideal space of the commutative Banach algebra $\mathfrak{B}(\mathbb{N})$ is homeomorphic to the Cartesian product of unit closed disk indexed by primes (see [5, Theorem 1.1]). They pointed out that this result implies the fundamental theorem of arithmetic. Analogously, studying the cohomology of the Banach algebras $\mathfrak{B}(\mathcal{S})$, $\mathfrak{B}(\mathcal{F}_n)$, and $\mathfrak{B}(\mathcal{T})$ can help us to understand the properties of the corresponding semigroups.

In this paper, we will prove that derivations on $\mathfrak{B}(\mathcal{S})$, $\mathfrak{B}(\mathcal{F}_n)$, and $\mathfrak{B}(\mathcal{T})$ are automatically continuous, and every derivation on $\mathfrak{B}(\mathcal{S})$ is spatial and induced by an operator in $\mathcal{L}(\mathcal{S})$. Comparing with a result of Kadison [18, Theorem 4] that every derivation on a C^*-algebra is spatial, we give a nontrivial example in the case of Banach algebras. Moreover, we prove that the first cohomology group of $\mathfrak{B}(\mathcal{S})$ with coefficients in $\mathcal{L}(\mathcal{S})$ is zero, which gives a positive indication for the left amenability of Thompson's semigroup.

In the following, we recall several concepts and results about amenable semigroups. We say that a discrete cancellative semigroup S is *left (resp. right) amenable* if there exists a left (resp. right) invariant mean on $l^\infty(S)$. For example, the additive semigroup of natural numbers is amenable while the free semigroup on n (≥ 2) generators is not. A *left (resp. right) Følner net* of S is a net of non-empty finite subsets $\{F_\alpha\}$ in S such that for any $s \in S$,

$$\lim_{\alpha} \frac{|sF_\alpha \Delta F_\alpha|}{|F_\alpha|} = 0 \left(\text{resp. } \lim_{\alpha} \frac{|F_\alpha s \Delta F_\alpha|}{|F_\alpha|} = 0 \right).$$

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It was proved for groups by Følner, and then generalized to discrete cancellative semigroups by Frey [7] that S is left (resp. right) amenable if and only if S has a left (resp. right) Følner net. In [6], Følner proved that every subgroup of an amenable group is still amenable. For semigroups, it is not always true. In [7], Frey gave an example of a left amenable semigroup which contains a non-amenable semigroup.

3 Unique factorization semigroup

Definition 3.1 Let S be a discrete semigroup. We say that S is a unique factorization semigroup if there exists a subset $\{X_1, X_2, X_3, \dots\}$ of S such that:

(i) every non-unital element in S can be uniquely written as $X_{i_1}^{\alpha_1} \dots X_{i_n}^{\alpha_n}$, where α_i $(1 \le i \le n)$ are positive integers and $i_1 < \cdots < i_n$.

(ii) If $e = X_1^{\beta_1} \dots X_n^{\beta_n}$ for some nonnegative integers β_i $(1 \le i \le n)$, then $\beta_i = 0$. The subset $\{X_1, X_2, X_3, \dots\}$ is called a basis of S.

For example, the multiplicative semigroup of natural numbers is a unique factorization semigroup and the set of all primes is the unique basis up to reorder. It is also clear that Thompson's semigroup $\mathcal{S} = \langle X_0, X_1, \dots \mid X_j X_i = X_i X_{j+1},\ i < j \rangle^+$ is a unique factorization semigroup with the basis $\{X_n \in \mathcal{S} :\ n \in \mathbb{N}\}$.

Next, we introduce some properties of $\mathcal{S}$ that will be frequently used in Sections 4 and 5.

Definition 3.2 Let $X = X_0^{\alpha_0} X_1^{\alpha_1} \dots X_n^{\alpha_n}$ be an element in $\mathcal{S}$, where α_i $(0 \le i \le n)$ are nonnegative integers. We define the index of X at the ith position as $\mathrm{ind}_i(X) := \alpha_i$ and the index of X as $\mathrm{ind}(X) := \sum_{i=0}^{n} \mathrm{ind}_i(X)$. The index of the unit element e is zero.

It is clear that $\mathrm{ind}_0(uv) = \mathrm{ind}_0(u) + \mathrm{ind}_0(v) = \mathrm{ind}_0(vu)$ and $\mathrm{ind}(uv) = \mathrm{ind}(u) + \mathrm{ind}(v) = \mathrm{ind}(vu)$ for all u and v in $\mathcal{S}$. In general, $\mathrm{ind}_i(uv) \ne \mathrm{ind}_i(u) + \mathrm{ind}_i(v)$ for $i \ge 1$. For example, $\mathrm{ind}_2(X_0X_2X_1) = \mathrm{ind}_2(X_0X_1X_3) = 0 \ne 1 = \mathrm{ind}_2(X_0X_2) + \mathrm{ind}_2(X_1)$.

Definition 3.3 [27] Let $u, v, w \in \mathcal{S}$. We call u a divisor of v if $v = uw$ and we denote by $u|v$.

For example, $X_1|X_0X_2 = X_1X_0$ while $X_2 \nmid X_0X_2$. It is clear that if $u|v$ then $\mathrm{ind}(u) \le \mathrm{ind}(v)$.

Lemma 3.4 [27] *The relation "|" is a partial order on* $\mathcal{S}$.

Proof Let $u, v, w \in \mathcal{S}$. We verify the following three axioms of the partial order.

(1) (Reflexivity.) $u = ue$ implies $u|u$.
(2) (Antisymmetry.) If $u|v$ and $v|u$, then $u = vw_1$ and $v = uw_2$ for some w_1, w_2 in $\mathcal{S}$. Then we have $w_2w_1 = e$, which implies $w_1 = w_2 = e$. Thus $u = v$.
(3) (Transitivity.) Suppose that $u|v$ and $v|w$, then $v = uw_1$ and $w = vw_2$ for some w_1, w_2 in $\mathcal{S}$. We have $w = uw_1w_2$. Thus $u|w$.

As a result, we conclude that "|" is a partial order. ∎

Lemma 3.5 *Let X be an element in $\mathcal{S}$ such that $X_0|X$. Then for each $n \in \mathbb{N}$, we have:*

(i) $X_1^{-n} X X_1^n \in \mathcal{S}$ *if and only if* $X_1^n|X$;

(ii) $X_1^n X X_1^{-n} \in \mathcal{S}$ *if and only if* $X = YX_1^n$ *for some* $Y \in \mathcal{S}$.

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Proof (i) If $X_1^n|X$, then it is clear that $X_1^{-n}XX_1^n \in \mathcal{S}$. Conversely, we have $X_1^n|XX_1^n$. If X_1^n is not a divisor of X, since $X_1^{-n}X_0 = X_0X_2^{-n}$, we have that the normal form of $X_1^{-n}XX_1^n$ in Thompson's group is ZX_j^{-m} for some $Z \in \mathcal{S}$ and $j \geq 2$, $m \geq 1$. This leads to a contradiction. Thus $X_1^n|X$. (ii) Assume that $W = X_1^nXX_1^{-n} \in \mathcal{S}$, then we have $X_0|W$ and $X = X_1^{-n}WX_1^n \in \mathcal{S}$. From ($i$), we have $X_1^n|W$. Let $Y = X_1^{-n}W$, then $X = YX_1^n$. The other direction is obvious. We complete the proof. ∎

With the aid of the index function we introduce a total order on $\mathcal{S}$ which plays an important role in Sections 4 and 5.

Definition 3.6 Let $u, v \in \mathcal{S}$. We say $u \prec v$ if one of the following conditions holds:

(i) $\mathrm{ind}(u) < \mathrm{ind}(v)$;

(ii) $\mathrm{ind}(u) = \mathrm{ind}(v)$ and $\mathrm{ind}_0(u) > \mathrm{ind}_0(v)$;

(iii) $\mathrm{ind}(u) = \mathrm{ind}(v)$, $\mathrm{ind}_0(u) = \mathrm{ind}_0(v)$, and there exists a positive integer i such that $\mathrm{ind}_i(u) > \mathrm{ind}_i(v)$ and $\mathrm{ind}_j(u) = \mathrm{ind}_j(v)$ whenever $j < i$.

We use $u \preceq v$ to denote $u \prec v$ or $u = v$.

For example, $X_0 \prec X_1 \prec X_0X_1 \prec X_0X_2 \prec X_1X_2$. The relation "$\preceq$" is a *total order* on $\mathcal{S}$ with the well-ordered properties.

Lemma 3.7 *We have the following statements:*

(i) *There exists a unique minimal element in every non-empty subset of* $\mathcal{S}$ *under the total order.*

(ii) *Let* u_i *and* v_i $(1 \leq i \leq n)$ *be* $2n$ *elements in* $\mathcal{S}$. *If* $u_i \preceq v_i$ *for each* $1 \leq i \leq n$, *then* $\Pi_{i=1}^n u_i \preceq \Pi_{i=1}^n v_i$. *The equality holds if and only if* $u_i = v_i$ *for each* $1 \leq i \leq n$.

Proof It is clear that (i) holds. We now give the proof of (ii). First, we consider the case when $n = 2$. If $\mathrm{ind}(u_1) < \mathrm{ind}(v_1)$ or $\mathrm{ind}(u_2) < \mathrm{ind}(v_2)$, then $\mathrm{ind}(u_1u_2) = \mathrm{ind}(u_1) + \mathrm{ind}(u_2) < \mathrm{ind}(v_1) + \mathrm{ind}(v_2) = \mathrm{ind}(v_1v_2)$. This implies $u_1u_2 \prec v_1v_2$. In the case that $\mathrm{ind}(u_1) = \mathrm{ind}(v_1)$ and $\mathrm{ind}(u_2) = \mathrm{ind}(v_2)$, if $\mathrm{ind}_0(u_1) > \mathrm{ind}_0(v_1)$ or $\mathrm{ind}_0(u_2) > \mathrm{ind}_0(v_2)$, then $\mathrm{ind}_0(u_1u_2) = \mathrm{ind}_0(u_1) + \mathrm{ind}_0(u_2) > \mathrm{ind}_0(v_1) + \mathrm{ind}_0(v_2) = \mathrm{ind}_0(v_1v_2)$. This also implies $u_1u_2 \prec v_1v_2$. Hence, we assume that $\mathrm{ind}_0(u_1) = \mathrm{ind}_0(v_1)$ and $\mathrm{ind}_0(u_2) = \mathrm{ind}_0(v_2)$. If either $\mathrm{ind}(u_1)$ or $\mathrm{ind}(u_2)$ is zero, then it is trivial. Thus we may further assume that $\mathrm{ind}(u_1) = \mathrm{ind}(v_1) \geq 1$ and $\mathrm{ind}(u_2) = \mathrm{ind}(v_2) \geq 1$.

Case I: $u_2 = v_2$. If $u_1 = v_1$, then it is obvious. Otherwise, let $u_1 = X_0^{\alpha_0} \ldots X_n^{\alpha_n}$ and $v_1 = X_0^{\beta_0} \ldots X_m^{\beta_m}$. By the definition of the total order, there exists an integer $i > 0$ such that $\alpha_j = \beta_j$ whenever $j < i$ and $\alpha_i > \beta_i$. Since $\{X_l \in \mathcal{S} : l \in \mathbb{N}\}$ is a basis of $\mathcal{S}$, it only needs to prove $u_1X_l \prec v_1X_l$ for each $X_l \in \mathcal{S}$. In fact, we have

$$(1) \qquad u_1X_l = \begin{cases} X_0^{\alpha_0} \ldots X_{l-1}^{\alpha_{l-1}} X_l^{\alpha_l+1} X_{l+2}^{\alpha_{l+1}} \ldots X_{i+1}^{\alpha_i} \ldots X_{n+1}^{\alpha_n}, & l < i, \\ X_0^{\alpha_0} \ldots X_{i-1}^{\alpha_{i-1}} X_i^{\alpha_i+1} X_{i+2}^{\alpha_{i+1}} \ldots X_{n+1}^{\alpha_n}, & l = i, \\ X_0^{\alpha_0} \ldots X_{i-1}^{\alpha_{i-1}} X_i^{\alpha_i} X_{i_1}^{\alpha'_{i_1}} \ldots X_{i_t}^{\alpha'_{i_t}}, & l > i, \end{cases}$$

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and

$$(2) \qquad v_1X_l = \begin{cases} X_0^{\beta_0}\dots X_{l-1}^{\beta_{l-1}}X_l^{\beta_l+1}X_{l+2}^{\beta_{l+1}}\dots X_{i+1}^{\beta_i}\dots X_{m+1}^{\beta_m}, & l < i,\\ X_0^{\beta_0}\dots X_{i-1}^{\beta_{i-1}}X_i^{\beta_i+1}X_{i+2}^{\beta_{i+1}}\dots X_{m+1}^{\beta_m}, & l = i,\\ X_0^{\beta_0}\dots X_{i-1}^{\beta_{i-1}}X_i^{\beta_i}X_{j_1}^{\beta'_{j_1}}\dots X_{j_s}^{\beta'_{j_s}}, & l > i,\end{cases}$$

where $i < i_1 < \cdots < i_t$ and $i < j_1 < \cdots < j_s$. By comparing equation (1) with (2), we have $u_1X_l \prec u_2X_l$.

Case II: $u_2 \prec v_2$. Let $u_2 = X_0^{\gamma_0}\dots X_k^{\gamma_k}$ and $v_2 = X_0^{\omega_0}\dots X_l^{\omega_l}$, then there exists an integer $h > 0$ such that $\gamma_j = \omega_j$ whenever $j < h$ and $\gamma_h > \omega_h$. By the first case, it can be reduced to the case when $u_2 = X_h^{\gamma_h}\dots X_k^{\gamma_k}$ and $v_2 = X_h^{\omega_h}\dots X_l^{\omega_l}$. If $u_1 = v_1 = X_0^{\alpha_0}\dots X_n^{\alpha_n}$, then

$$(3) \qquad u_1u_2 = \begin{cases} X_0^{\alpha_0}\dots X_n^{\alpha_n}X_h^{\gamma_h}\dots X_l^{\gamma_l}, & h > n,\\ X_0^{\alpha_0}\dots X_{h-1}^{\alpha_{h-1}}X_h^{\alpha_h+\gamma_h}X_{h_1}^{\alpha'_{h_1}}\dots X_{h_t}^{\alpha'_{h_t}}, & h \le n,\end{cases}$$

and

$$(4) \qquad v_1v_2 = \begin{cases} X_0^{\alpha_0}\dots X_n^{\alpha_n}X_h^{\omega_h}\dots X_l^{\omega_l}, & h > n,\\ X_0^{\alpha_0}\dots X_{h-1}^{\alpha_{h-1}}X_h^{\alpha_h+\omega_h}X_{h_1}^{\alpha''_{h_1}}\dots X_{h_s}^{\alpha''_{h_s}}, & h \le n,\end{cases}$$

where $h < h_1 < \cdots < h_{\max\{t,s\}}$. Comparing equation (3) with (4), we have $u_1u_2 \prec v_1v_2$. The proof of the case that $u_1 \prec v_1$ is similar, we omit it here. Moreover, we can obtain from the above process directly that $u_1u_2 = v_1v_2$ if and only if $u_1 = v_1$ and $u_2 = v_2$.

The general case when $n > 2$ can be obtained by induction. ∎

We now turn to the free semigroup $\mathcal{F}_n$ on n (≥ 2) generators a_i $(1 \le i \le n)$. The following two definitions on $\mathcal{F}_n$ are parallel to Definitions 3.2 and 3.6, and Lemma 3.10 is parallel to Lemma 3.7.

Definition 3.8 Let $g = \prod_{j=1}^m \prod_{k=1}^n a_k^{i_{jk}}$ be an element in $\mathcal{F}_n$, where i_{jk} $(1 \le j \le m, 1 \le k \le n)$ are nonnegative integers. The index of g is defined as $\mathrm{ind}(g) := \sum_{j=1}^m \sum_{k=1}^n i_{jk}$.

It is clear that $\mathrm{ind}(gh) = \mathrm{ind}(g) + \mathrm{ind}(h)$ for all g and h in $\mathcal{F}_n$.

Definition 3.9 Let $g = \prod_{j=1}^m \prod_{k=1}^n a_k^{i_{jk}}$ and $h = \prod_{j=1}^m \prod_{k=1}^n a_k^{i'_{jk}}$ be two elements in $\mathcal{F}_n$. We say $g \prec h$ if one of the following conditions holds:

(i) $\mathrm{ind}(g) < \mathrm{ind}(h)$;

(ii) $\mathrm{ind}(g) = \mathrm{ind}(h)$ and there exist some j_0 $(1 \le j_0 \le m)$ and k_0 $(1 \le k_0 \le n)$ such that $i_{j_0k_0} > i'_{j_0k_0}$ and $i_{jk} = i'_{jk}$ whenever $j < j_0$ or $j = j_0$ and $k < k_0$.

We use $g \preceq h$ to denote $g \prec h$ or $g = h$.

The relation "$\preceq$" is a total order on $\mathcal{F}_n$.

Lemma 3.10 *We have the following statements:*

(i) *There exists a unique minimal element in each subset of $\mathcal{F}_n$ under the total order.*

(ii) *Let g_i and h_i $(1 \le i \le m)$ be $2m$ elements in $\mathcal{F}_n$. If $g_i \preceq h_i$ for each i $(1 \le i \le m)$, then $\Pi_{i=1}^m g_i \preceq \Pi_{i=1}^m h_i$. Moreover, the equality holds if and only if $g_i = h_i$ for each i $(1 \le i \le m)$.*

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Proposition 3.11 *For $n \geq 2$, the free semigroup $\mathcal{F}_n$ on n generators a_i $(1 \leq i \leq n)$ is a unique factorization semigroup.*[1]

Proof Let $X_i = a_i$ $(1 \leq i \leq n)$, then $X_1 \prec X_2 \prec \cdots \prec X_n$. Let X_{n+1} be the minimal element of $\mathcal{F}$ such that X_{n+1} cannot be represented as $X_1^{i_1} \dots X_n^{i_n}$ for any nonnegative integers i_j $(1 \leq j \leq n)$. Then $X_n \prec X_{n+1}$. Let X_{n+2} be the minimal element of $\mathcal{F}$ such that X_{n+2} cannot be represented as $X_1^{i_1} \dots X_{n+1}^{i_{n+1}}$ for any nonnegative integers i_j $(1 \leq j \leq n+1)$. Then $X_{n+1} \prec X_{n+2}$. Continuing this process, we can obtain a subset $\{X_1, X_2, \dots\}$ of $\mathcal{F}_n$ such that $X_i \prec X_{i+1}$ for each i (≥ 1). Then every element can be written as the product $X_1^{i_1} \dots X_n^{i_n}$ for some nonnegative integers i_j $(1 \leq j \leq n)$. We next show the uniqueness by induction on the index. It is clear that the uniqueness holds for index ≤ 1. Assume that the uniqueness holds for index $\leq k$ $(k \geq 1)$. Let $g \in \mathcal{F}_n$ and $\mathrm{ind}(g) = k+1$. Suppose that g has two different forms:

$$g = X_{i_1}^{\alpha_1} \dots X_{i_n}^{\alpha_n} = X_{j_1}^{\beta_1} \dots X_{j_m}^{\beta_m},$$

where α_i $(1 \leq i \leq n)$ and β_j $(1 \leq j \leq m)$ are positive integers, $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_m$. If $i_1 = j_1$, then

$$X_{i_1}^{\alpha_1 - 1} \dots X_{i_n}^{\alpha_n} = X_{j_1}^{\beta_1 - 1} \dots X_{j_m}^{\beta_m},$$

which implies that $n = m$, $i_t = j_t$ and $\alpha_t = \beta_t$ $(1 \leq t \leq n)$. This leads to a contradiction.

We assume that $i_1 < j_1$. Then there exists some l $(1 \leq l \leq n-1)$ such that $(X_{i_1}^{\alpha_1} \dots X_{i_l}^{\alpha_l})^{-1} X_{j_1}$ belongs to $\mathcal{F}_n$ and $1 \leq \mathrm{ind}((X_{i_1}^{\alpha_1} \dots X_{i_l}^{\alpha_l})^{-1} X_{j_1}) < \mathrm{ind}(X_{i_{l+1}})$, or $(X_{i_1}^{\alpha_1} \dots X_{i_{l-1}}^{\alpha_{l-1}} \; X_{i_l}^{\alpha_l'})^{-1} X_{j_1}$ $(1 \leq l \leq n)$ belongs to $\mathcal{F}_n$ for some α_l' $(1 \leq \alpha_l' < \alpha_l)$ and $1 \leq \mathrm{ind}((X_{i_1}^{\alpha_1} \dots X_{i_{l-1}}^{\alpha_{l-1}} X_{i_l}^{\alpha_l'})^{-1} \; X_{j_1}) < \mathrm{ind}(X_{i_l})$. In the first case, we have $(X_{i_1}^{\alpha_1} \dots X_{i_l}^{\alpha_l})^{-1} X_{j_1} = X_{s_1}^{\gamma_1} \dots X_{s_t}^{\gamma_t}$ for some positive integers $\gamma_1, \dots, \gamma_t$ and $s_1 < \cdots < s_t < \min\{i_{l+1}, j_1\}$. Then

$$X_{i_{l+1}}^{\alpha_{l+1}} \dots X_{i_n}^{\alpha_n} = X_{s_1}^{\gamma_1} \dots X_{s_t}^{\gamma_t} X_{j_1}^{\beta_1 - 1} \dots X_{j_m}^{\beta_m},$$

which leads to a contradiction. In the second case, we have $(X_{i_1}^{\alpha_1} \dots X_{i_l}^{\alpha_l'})^{-1} X_{j_1} = X_{s_1}^{\gamma_1} \dots X_{s_t}^{\gamma_t}$ for some positive integers $\gamma_1, \dots, \gamma_t$ and $s_1 < \cdots < s_t < \min\{i_l, j_1\}$. Then

$$X_{i_l}^{\alpha_i - \alpha_i'} X_{i_{l+1}}^{\alpha_{l+1}} \dots X_{i_n}^{\alpha_n} = X_{s_1}^{\gamma_1} \dots X_{s_t}^{\gamma_t} X_{j_1}^{\beta_1 - 1} \dots X_{j_m}^{\beta_m},$$

which also leads to a contradiction. Above all, we complete the proof. ∎

Free semigroups $\mathcal{F}_n$ $(n \geq 2)$ are neither left nor right amenable. Thompson's semigroup is not right amenable and whether it is left amenable is still unknown. For completeness, we construct the following both left and right amenable semigroup.

Proposition 3.12 *Let $\mathcal{T}$ be the semigroup generated by*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$$

[1] We thank D.Wu for his proof of Proposition 3.11.

in $GL_2(\mathbb{Z})$. Then $\mathcal{T}$ is a non-commutative unique factorization semigroup. Moreover, $\mathcal{T}$ is both left and right amenable.

Proof It can be verified directly that $AB = BA$, $AC = CB$, $BC = CA$. Hence, $\mathcal{T}$ is non-commutative and every element in $\mathcal{T}$ can be written as $A^{\alpha_1}B^{\alpha_2}C^{\alpha_3}$ for some natural numbers α_1, α_2, and α_3. Let X be a matrix in $\mathcal{T}$ with two different representations: $X = A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} = A^{\beta_1}B^{\beta_2}C^{\beta_3}$, where α_i and β_i $(1 \le i \le 3)$ are natural numbers. Taking the determinant at the both sides, we have $2^{\alpha_1}\dot{2}^{\alpha_2}\dot{(}-6)^{\alpha_3} = 2^{\beta_1}\dot{2}^{\beta_2}\dot{(}-6)^{\beta_3}$. Hence $\alpha_3 = \beta_3$ and $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$. Then we have $A^{\alpha_1-\beta_1} = B^{\beta_2-\alpha_2}$, which implies $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Thus $\mathcal{T}$ is a unique factorization semigroup. For each positive integer N (≥ 2), let $F_N = \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 0 \le \alpha_i \le N,\ i = 1,2,3\}$, we have

$$\begin{aligned} AF_N \cap F_N &= \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \le \alpha_1 \le N,\ 0 \le \alpha_i \le N,\ i = 2,3\},\\ BF_N \cap F_N &= \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \le \alpha_2 \le N,\ 0 \le \alpha_i \le N,\ i = 1,3\},\\ CF_N \cap F_N &= \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \le \alpha_3 \le N,\ 0 \le \alpha_i \le N,\ i = 1,2\} \end{aligned}$$

and $|AF_N \cap F_N| = |BF_N \cap F_N| = |CF_N \cap F_N| = N(N+1)^2$. Then

$$\lim_{N\to\infty} \frac{|AF_N \cap F_N|}{|F_N|} = \lim_{N\to\infty} \frac{|BF_N \cap F_N|}{|F_N|} = \lim_{N\to\infty} \frac{|CF_N \cap F_N|}{|F_N|} = \lim_{N\to\infty} \frac{N(N+1)^2}{(N+1)^3} = 1.$$

Since A, B, and C are generators of $\mathcal{T}$, $(F_N)_{N\in\mathbb{N}}$ is a left Følner sequence of $\mathcal{T}$. Thus $\mathcal{T}$ is left amenable. On the other hand, we have

$$\begin{aligned} F_NA \cap F_N =& \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \le \alpha_1 \le N,\ 0 \le \alpha_i \le N,\ i = 2,3,\ \alpha_3 \text{ is even}\}\\ &\cup\{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \le \alpha_2 \le N,\ 0 \le \alpha_i \le N,\ i = 1,3,\ \alpha_3 \text{ is odd}\},\\ F_NB \cap F_N =& \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \le \alpha_2 \le N,\ 0 \le \alpha_i \le N,\ i = 1,3,\ \alpha_3 \text{ is even}\}\\ &\cup\{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \le \alpha_1 \le N,\ 0 \le \alpha_i \le N,\ i = 2,3,\ \alpha_3 \text{ is odd}\},\\ F_NC \cap F_N =& \{A^{\alpha_1}B^{\alpha_2}C^{\alpha_3} \in \mathcal{T} \mid 1 \le \alpha_3 \le N,\ 0 \le \alpha_i \le N,\ i = 1,2\}. \end{aligned}$$

Similarly, $(F_N)_{N\in\mathbb{N}}$ is also a right Følner sequence of $\mathcal{T}$. Thus $\mathcal{T}$ is both left and right amenable. We complete the proof. ∎

4 The continuity of derivations

In this section, we will prove the following theorem.

Theorem 4.1 *Derivations on the Banach algebras $\mathfrak{B}(\mathcal{S})$ and $\mathfrak{B}(\mathcal{F}_n)$ are continuous.*

Before proving Theorem 4.1, we introduce the following definition.

Definition 4.2 A discrete semigroup S is said to be lower stable if there exists a total order "$\preceq$" on S such that:

(i) There exists a unique minimal element in each non-empty subset of S under the total order.

(ii) Let u_i and v_i $(1 \le i \le n)$ be $2n$ elements in S. If $u_i \preceq v_i$ for each $1 \le i \le n$, then $\Pi_{i=1}^n u_i \preceq \Pi_{i=1}^n v_i$. Moreover, the equality holds if and only if $u_i = v_i$ for each $1 \le i \le n$.

By Lemmas 3.7 and 3.10, Thompson's semigroup $\mathcal{S}$ and free semigroups $\mathcal{F}_n$ $(n \ge 2)$ are both lower stable.

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Definition 4.3 A Banach algebra $\mathcal{A}$ is said to be semisimple if its Jacobson radical $\mathcal{J}$ equals zero, where

$$\mathcal{J} = \bigcap_{\text{maximal left ideals of } \mathcal{A}} \mathcal{I}_l = \bigcap_{\text{maximal right ideals of } \mathcal{A}} \mathcal{I}_r.$$

The following lemma is a characterization of semisimple Banach algebras.

Lemma 4.4 *Let $\mathcal{A}$ be a Banach algebra with the unit I. If $\mathcal{A}$ contains no nonzero quasi-nilpotent operators, then $\mathcal{A}$ is semisimple.*

Proof Let T be a nonzero element in $\mathcal{A}$. Let λ be a nonzero point of the spectrum of T, then at least one of $(\lambda I - T)\mathcal{A}$ and $\mathcal{A}(\lambda I - T)$ is properly contained in $\mathcal{A}$. We assume that the right ideal $(\lambda I - T)\mathcal{A}$ is properly contained in $\mathcal{A}$. Then there exists a maximal right ideal $\mathcal{I}_r$ of $\mathcal{A}$ such that $(\lambda I - T)\mathcal{A} \subseteq \mathcal{I}_r \subset \mathcal{A}$. Thus T does not belong to $\mathcal{I}_r$. This implies T is not in the Jacobson radical $\mathcal{J}$ of $\mathcal{A}$. Consequently, we have $\mathcal{J} = 0$. This completes the proof. ∎

Lemma 4.5 *Let S be a lower stable discrete semigroup and $\mathfrak{B}(S)$ the Banach algebra generated by $\{L_s : s \in S\}$ in $B(l^2(S))$. Then $\mathfrak{B}(S)$ is semisimple.*

Proof Let L_f be a nonzero element in $\mathfrak{B}(S)$. We claim that the spectral radius $r(L_f) > 0$. By the definition of lower stable semigroup, there is a unique minimal element X of the subset $\{X \in S : f(X) \neq 0\}$ under the total order. Then we have

$$\underbrace{f * \cdots * f}_{n}(X^n) = f(X)^n$$

for each $n \geq 1$. Therefore,

$$\begin{aligned} r(L_f) = \lim_{n\to\infty} \|L_f^n\|^{1/n} &\geq \lim_{n\to\infty} \|\underbrace{f * \cdots * f}_{n}\|_2^{1/n} \\ &\geq \lim_{n\to\infty} |\underbrace{f * \cdots * f}_{n}(X^n)|^{1/n} = |f(X)| > 0. \end{aligned}$$

By Lemma 4.4, we obtain that $\mathfrak{B}(S)$ is semisimple. ∎

Corollary 4.6 *The Banach algebras $\mathfrak{B}(\mathcal{S})$ and $\mathfrak{B}(\mathcal{F}_n)$ are semisimple.*

In the semigroup $\mathcal{T}$, if $A \prec B$, then $AC \prec BC = CA \prec CB = AC$. Hence, we cannot have $A \prec B$. Similarly, $B \prec A$ is not allowed. Therefore, the semigroup $\mathcal{T}$ in Example 3.12 is not lower stable. The Banach algebra $\mathfrak{B}(\mathcal{T})$ is also semisimple. The proof is similar to that of Lemma 4.5, we omit it here. Now, we prove Theorem 4.1.

Proof of Theorem 4.1 Theorem 4.1 of [16] states that derivations on a semisimple Banach algebra are continuous, then by Corollary 4.6, we can obtain the conclusion. ∎

5 The first cohomology group of $\mathfrak{B}(\mathcal{S})$

A derivation D on a Banach algebra $\mathfrak{B} \subseteq B(\mathcal{H})$ is said to be *spatial* if there exists a bounded operator T in $B(\mathcal{H})$ such that $D(A) = TA - AT$ for each A in $\mathfrak{B}$. In [18],

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Kadison proved that all derivations on a C^*-algebra are spatial. In this section, we will prove that all derivations on the Banach algebra $\mathfrak{B}(\mathcal{S})$ are spatial and induced by bounded operators in $\mathcal{L}(\mathcal{S})$. Moreover, we prove that the first continuous cohomology group of $\mathfrak{B}(\mathcal{S})$ with coefficients in $\mathcal{L}(\mathcal{S})$ is zero (see Theorem 5.8).

The Hilbert space $\mathcal{H}$ is $l^2(\mathcal{S})$. Recall that $\mathfrak{B}(\mathcal{S})$ is the Banach algebra generated by $\{L_s : s \in \mathcal{S}\}$ in $B(\mathcal{H})$ and $\mathcal{L}(\mathcal{S}) = \{L_f \in B(\mathcal{H}) : f \in \mathcal{H}\}$. We use $\sum_{X\in\mathcal{S}} f(X)X$ and $\sum_{X\in\mathcal{S}} \overline{f(X)}X^*$ to denote L_f and its adjoint operator L_f^* for convenience. It is clear that XX^* is the projection from $\mathcal{H}$ onto the closure of the subspace $\text{span}\{\delta_Y : Y \in \mathcal{S}, X|Y\}$ and $X^*X = I$ (the identity map). Recall that "|" is the partial order of Thompson's semigroup introduced in Section 3 and "$X|Y$" means that there exists an element Z in $\mathcal{S}$ such that $Y = XZ$.

Let $(\mathbb{N}, +)$ be the additive semigroup of natural numbers. The notation $\beta\mathbb{N}$ denotes the maximal ideal space of the commutative C^*-algebra $l^\infty(\mathbb{N})$. The elements in $\beta\mathbb{N}\backslash\mathbb{N}$ are called *free ultrafilters*. Let $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$ be a free ultrafilter. For any n in $\mathbb{N}$ and any f in $l^\infty(\mathbb{N})$, we define $E_n(f) = \frac{1}{n}\sum_{i=0}^{n-1} f(i)$. Then for each f, the function $n \mapsto E_n(f)$ is in $l^\infty(\mathbb{N})$. By Gelfand–Naimark theorem, we have $l^\infty(\mathbb{N}) \cong C(\beta\mathbb{N})$. Thus $E_n(f)$ is a continuous function on $\beta\mathbb{N}$. We use $E_\omega(f)$ or the integral $\int_{\mathbb{N}} f(n)dE_\omega(n)$ to denote the limit of $E_n(f)$ at ω. Then E_ω is an invariant mean on $l^\infty(\mathbb{N})$, i.e.,

$$\int_{\mathbb{N}} f(n)dE_\omega(n) = \int_{\mathbb{N}} f(n+m)dE_\omega(n)$$

for each $m \geq 1$. Moreover, E_ω satisfies that $E_\omega(f) = \lim_{n\to\infty} f(n)$ if the limit exists. The invariant mean E_ω is also called a Banach limit.

Let D be a continuous derivation from the Banach algebra $\mathfrak{B}(\mathcal{S})$ into $\mathcal{L}(\mathcal{S})$. For each $\xi, \eta \in \mathcal{H}$, we define

$$\langle A\xi, \eta\rangle := \int_{\mathbb{N}} \langle (X_0^*)^n D(X_0^{n+1})\xi, \eta\rangle dE_\omega(n). \tag{5}$$

Then A is a bounded linear operator on $\mathcal{H}$. Moreover, we have

$$\begin{aligned}\langle A\xi, \eta\rangle &= \int_{\mathbb{N}} \langle (X_0^*)^{n+1} D(X_0^{n+2})\xi, \eta\rangle dE_\omega(n)\\ &= \int_{\mathbb{N}} \langle D(X_0)\xi, \eta\rangle dE_\omega(n) + \int_{\mathbb{N}} \langle (X_0^*)^{n+1} D(X_0^{n+1})X_0\xi, \eta\rangle dE_\omega(n)\\ &= \langle D(X_0)\xi, \eta\rangle + \langle X_0^* A X_0 \xi, \eta\rangle,\end{aligned}$$

which follows that

$$D(X_0) = A - X_0^* A X_0. \tag{6}$$

Similarly, we define

$$\langle B\xi, \eta\rangle := \int_{\mathbb{N}} \langle (X_1^*)^n D(X_1^{n+1})\xi, \eta\rangle dE_\omega(n). \tag{7}$$

We have $B \in B(\mathcal{H})$ and

$$D(X_1) = B - X_1^* B X_1. \tag{8}$$

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5.1 The case of X_0

We prove the following local inner derivative property for X_0 in this subsection. A similar result for X_1 will be given later.

Lemma 5.1 *There exists some $\widehat{A}$ in $\mathcal{L}(\mathcal{S})$ such that $D(X_0) = \widehat{A}X_0 - X_0\widehat{A}$.*

Proof Let

$$D(X_0) = \sum_{X_0 | X} f(X)X + \sum_{X_0 \nmid X} f(X)X.$$

By the fact that $X_1X_0 = X_0X_2$, we have

$$D(X_1)X_0 + X_1D(X_0) = X_0D(X_2) + D(X_0)X_2.$$

It follows that

$$X_1 \sum_{X_0 \nmid X} f(X)X = \sum_{X_0 \nmid X} f(X)XX_2.$$

Since $X_1X \prec XX_2$ when $X_0 \nmid X$, hence $f(X) = 0$ in this case. Therefore,

$$D(X_0) = \sum_{X_0 | X} f(X)X = X_0L_{f_1}$$

for some L_{f_1} in $\mathcal{L}(\mathcal{S})$. By the Leibniz rule and induction, we can obtain that

(9) $$D(X_0^n) = X_0^n L_{f_n}$$

for some L_{f_n} in $\mathcal{L}(\mathcal{S})$. We define

$$\langle \widehat{A}\xi, \eta \rangle := -\int_{\mathbb{N}} \langle (X_0^*)^{n+1}D(X_0^{n+1})\xi, \eta \rangle dE_\omega(n).$$

Then $\widehat{A} \in B(\mathcal{H})$. For each T in $\mathcal{R}(\mathcal{S})$, we have

$$\begin{aligned}\langle T\widehat{A}\xi, \eta \rangle &= -\int_{\mathbb{N}} \langle (X_0^*)^{n+1}D(X_0^{n+1})\xi, T^*\eta \rangle dE_\omega(n) \\ &= -\int_{\mathbb{N}} \langle (X_0^*)^{n+1}D(X_0^{n+1})T\xi, \eta \rangle dE_\omega(n) \\ &= \langle \widehat{A}T\xi, \eta \rangle.\end{aligned}$$

The second equality is due to equation (9). It follows that $\widehat{A} \in \mathcal{R}(\mathcal{S})' = \mathcal{L}(\mathcal{S})$. By equation (5), we have

$$\begin{aligned}\langle A\xi, \eta \rangle &= \int_{\mathbb{N}} \langle (X_0^*)^{n}D(X_0^{n+1})\xi, \eta \rangle dE_\omega(n) \\ &= \int_{\mathbb{N}} \langle (X_0^*)^{n+1}D(X_0^{n+1})\xi, X_0^*\eta \rangle dE_\omega(n) \\ &= \langle -\widehat{A}\xi, X_0^*\eta \rangle,\end{aligned}$$

which implies $A = -X_0\widehat{A}$. The second equality is due to equation (9). Then by equation (6), we have $D(X_0) = \widehat{A}X_0 - X_0\widehat{A}$. This completes the proof. ∎

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5.2 The case of X_1

The following lemma is the main conclusion of this subsection.

Lemma 5.2 *There exists some $\widehat{B}$ in $\mathcal{L}(\mathcal{S})$ such that $D(X_1) = \widehat{B}X_1 - X_1\widehat{B}$.*

We need Lemmas 5.3 and 5.4 to obtain Lemma 5.2.

Lemma 5.3 *There exists some B in $\mathcal{L}(\mathcal{S})$ such that $D(X_1) = B - X_1^*BX_1$.*

Proof Let $D(X_1) = L_f$ in $\mathcal{L}(\mathcal{S})$. By the continuity of D, we have $f(X_1^n) = 0$ for each $n \in \mathbb{N}$. For each $m \geq 1$, we have

$$(X_1^*)^{m-1}D(X_1^m) = \sum_{i=0}^{m-1}(X_1^*)^i L_f X_1^i$$

and

$$\|(X_1^*)^{m-1}D(X_1^m)\| \leq \|D\|.$$

Let h_1 and h_2 be two elements in $\mathcal{S}$. If $h_2h_1^{-1} = X_1^k$ for some $k \in \mathbb{N}$, then

$$\lim_{n,m\to\infty}\left\langle\left((X_1^*)^{m-1}D(X_1^m) - (X_1^*)^{n-1}D(X_1^n)\right)\delta_{h_1}, \delta_{h_2}\right\rangle = \lim_{n,m\to\infty}\sum_{i=n+1}^{m} f(X_1^i h_2 h_1^{-1} X_1^{-i}) = 0.$$

If $h_2h_1^{-1} \neq X_1^k$ for any $k \in \mathbb{N}$, then $X_1^i h_2 h_1^{-1} X_1^{-i} \notin \mathcal{S}$ when i is sufficiently large. Therefore,

$$\lim_{n,m\to\infty}\left\langle\left((X_1^*)^{m-1}D(X_1^m) - (X_1^*)^{n-1}D(X_1^n)\right)\delta_{h_1}, \delta_{h_2}\right\rangle = \lim_{n,m\to\infty}\sum_{i=n+1}^{m} f(X_1^i h_2 h_1^{-1} X_1^{-i}) = 0.$$

We define

$$\langle T\delta_{h_1}, \delta_{h_2}\rangle := \lim_{m\to\infty}\langle (X_1^*)^{m-1}D(X_1^m)\delta_{h_1}, \delta_{h_2}\rangle. \tag{10}$$

Then T is the weak-operator limit of $(X_1^*)^{m-1}D(X_1^m)$ in $B(\mathcal{H})$.

Claim $T = L_{T\delta_e} \in \mathcal{L}(\mathcal{S})$.

To prove this claim, we distinguish two cases:

Case I: *$h_2h_1^{-1} \in \mathcal{S}$. We have*

$$\langle T\delta_{h_1}, \delta_{h_2}\rangle = \sum_{n=0}^{\infty} f(X_1^n h_2 h_1^{-1} X_1^{-n}) = \langle T\delta_e, \delta_{h_2h_1^{-1}}\rangle = \langle T\delta_e * \delta_{h_1}, \delta_{h_2}\rangle.$$

Case II: *$h_2h_1^{-1} \notin \mathcal{S}$. If $X_1^n h_2 h_1^{-1} X_1^{-n} \notin \mathcal{S}$ for any $n \in \mathbb{N}$, then*

$$\langle T\delta_{h_1}, \delta_{h_2}\rangle = \langle T\delta_e * \delta_{h_1}, \delta_{h_2}\rangle = 0.$$

On the other hand, there exists a natural number n such that $X_1^{n+1}h_2h_1^{-1}X_1^{-n-1} \in \mathcal{S}$ and $X_1^i h_2 h_1^{-1} X_1^{-i} \notin \mathcal{S}$ whenever $i \leq n$. Let $X = X_1^{n+1}h_2h_1^{-1}X_1^{-n-1}$, then $X_0|X$. By Lemma 3.5, we have that $X_1^{-k}XX_1^k \notin \mathcal{S}$ for any $k \geq 1$ and $X_1^kXX_1^{-k} \notin \mathcal{S}$ when $k > \mathrm{ind}(X)$. Let m be an even integer such that m is sufficiently lager than $\mathrm{ind}(X)$. Then we have

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$$\sum_{i=1}^{m}\left|\langle D(X_1^{2m})\delta_e,\delta_{X_1^{2m-i}XX_1^{i-1}}\rangle\right|^2=\sum_{i=1}^{m}\left|\sum_{j=-(i-1)}^{2m-i}f(X_1^jXX_1^{-j})\right|^2=\sum_{i=1}^{m}\left|\sum_{j=0}^{\operatorname{ind}(X)}f(X_1^jXX_1^{-j})\right|^2.$$

Note that $X_1^{2m-i}XX_1^{i-1}\neq X_1^{2m-j}XX_1^{j-1}$ *when* $1\leq i<j\leq m$*, then we have*

$$\sum_{i=1}^{m}\left|\sum_{j=0}^{\operatorname{ind}(X)}f(X_1^jXX_1^{-j})\right|^2\leq\left\|D(X_1^{2m})\delta_e\right\|_2^2\leq\|D\|^2$$

for any sufficiently large even integer m. This implies that

$$\sum_{j=0}^{\operatorname{ind}(X)}f(X_1^jXX_1^{-j})=0.$$

Therefore,

$$\langle T\delta_{h_1},\delta_{h_2}\rangle=\sum_{j=0}^{\infty}f(X_1^jh_2h_1^{-1}X_1^{-j})=\sum_{j=0}^{\operatorname{ind}(X)}f(X_1^jXX_1^{-j})=0.$$

From the above discussion, we have $\langle T\delta_{h_1},\delta_{h_2}\rangle=\langle T\delta_e*\delta_{h_1},\delta_{h_2}\rangle$ *for all* h_1 *and* h_2 *in* $\mathcal{S}$*. Thus the claim holds.* ∎

By equations (7) and (10), we have

$$\begin{aligned}\langle T\delta_{h_1},\delta_{h_2}\rangle&=\lim_{m\to\infty}\left\langle (X_1^*)^{m-1}D(X_1^m)\delta_{h_1},\delta_{h_2}\right\rangle\\&=\int_{\mathbb{N}}\left\langle (X_1^*)^{m-1}D(X_1^m)\delta_{h_1},\delta_{h_2}\right\rangle dE_\omega(m)\\&=\langle B\delta_{h_1},\delta_{h_2}\rangle,\end{aligned}$$

which follows that $B=T$. This completes the proof.

Lemma 5.4 *Let* $D(X_1)=L_f\in\mathcal{L}(\mathcal{S})$*. Then*

$$D(X_1)=\sum_{X_0|X}f(X)X+\sum_{X_0\nmid X,X_1|X}f(X)X.$$

Proof Let

$$D(X_1)=\sum_{X_0|X}f(X)X+\sum_{X_0\nmid X,X_1|X}f(X)X+\sum_{X_0\nmid X,X_1\nmid X}f(X)X.$$

By the fact that $X_1X_3=X_2X_1$, we have

$$D(X_1)X_3+X_1D(X_3)=D(X_2)X_1+X_2D(X_1).$$

Then

$$\sum_{X_0\nmid X,X_1\nmid X}f(X)XX_3=X_2\sum_{X_0\nmid X,X_1\nmid X}f(X)X.$$

Let X be the minimal element of the set $\{X\in\mathcal{S}:\ X_0\nmid X,\ X_1\nmid X,\ f(X)\neq 0\}$. Then $X_2X\prec XX_3$. It follows that $f(X)=0$ when $X_0\nmid X$ and $X_1\nmid X$. This completes the proof. ∎

Now, we are ready to prove Lemma 5.2.

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Proof of Lemma 5.2 By Lemma 5.3, we have $D(X_1) = B - X_1^* B X_1$ for some $B = L_g \in \mathcal{L}(\mathcal{S})$. We assume that $g(e) = 0$, where e is the unit element of $\mathcal{S}$. Let

$$L_g = \sum_{X_1|X} g(X)X + \sum_{X_1\nmid X, X_0|X} g(X)X + \sum_{X_1\nmid X, X_0\nmid X} g(X)X.$$

Then

$$D(X_1) = \sum_{X_1|X} g(X)X - X_1^* \sum_{X_1|X} g(X)XX_1 + \sum_{X_1\nmid X, X_0\nmid X} g(X)X - X_1^* \sum_{X_1\nmid X, X_0\nmid X} g(X)XX_1$$

$$+ \sum_{X_1\nmid X, X_0|X} g(X)X - X_1^* \sum_{X_1\nmid X, X_0|X} g(X)XX_1.$$

Since $D(X_1) \in \mathcal{L}(\mathcal{S})$ and $X_1^{-1}XX_1 \notin \mathcal{S}$ when $X_1 \nmid X$ and $X_0|X$, we have

$$X_1^* \sum_{X_1\nmid X, X_0|X} g(X)XX_1 = 0.$$

For $X \in \mathcal{S}$ with $X_1 \nmid X$ and $X_0|X$, let $X_1^{-1}XX_1 = UV^{-1}$, where $U, V \in \mathcal{S}$. Then

$$g(X) = \left\langle X_1^* \sum_{X_1\nmid X, X_0|X} g(X)XX_1\delta_V, \delta_U \right\rangle = 0.$$

It follows that

$$\sum_{X_1\nmid X, X_0|X} g(X)X = 0.$$

Therefore,

$$D(X_1) = \sum_{X_1|X} g(X)X - X_1^* \sum_{X_1|X} g(X)XX_1 + \sum_{X_1\nmid X, X_0\nmid X} g(X)X - X_1^* \sum_{X_1\nmid X, X_0\nmid X} g(X)XX_1.$$

By Lemma 5.4, we have

$$\sum_{X_1\nmid X, X_0\nmid X} g(X)X - X_1^* \sum_{X_1\nmid X, X_0\nmid X} g(X)XX_1 = 0.$$

Let X be the minimal element of the set $\{X \in \mathcal{S} : X_1 \nmid X,\ X_0 \nmid X,\ g(X) \neq 0\}$. Then $X < X_1^{-1}XX_1$ since $X \neq e$. It follows that $g(X) = 0$ when $X_1 \nmid X$ and $X_0 \nmid X$. Thus $B = \sum_{X_1|X} g(X)X$. Let $\widehat{B} = -X_1^* B$, then $\widehat{B} \in \mathcal{L}(S)$ and

$$D(X_1) = B - X_1^* B X_1 = X_1 X_1^* B - X_1^* B X_1 = \widehat{B} X_1 - X_1 \widehat{B}.$$

We complete the proof. ∎

5.3 Conditional expectation

Definition 5.5 Let $\mathfrak{B}$ be a Banach algebra, and let $\mathcal{A}$ be a Banach subalgebra of $\mathfrak{B}$. Let $E: \mathfrak{B} \to \mathcal{A}$ be a contraction such that:

(i) $E(I) = I$;

(ii) $E(A_1 B A_2) = A_1 E(B) A_2$ whenever $A_1, A_2 \in \mathcal{A}$ and $B \in \mathfrak{B}$.

Then E is described as a conditional expectation from $\mathfrak{B}$ onto $\mathcal{A}$.

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Definition 5.6 We denote by $\mathcal{L}_0(\mathcal{S})$ the subset of $\mathcal{L}(\mathcal{S})$ such that for each L_f in $\mathcal{L}_0(\mathcal{S})$, $f(X) = 0$ if X is not in the semigroup generated by X_0.

It is not difficult to check that $\mathcal{L}_0(\mathcal{S})$ is a Banach subalgebra of $\mathcal{L}(\mathcal{S})$. We have the following theorem.

Theorem 5.7 *Let E be the map: $\mathcal{L}(\mathcal{S}) \to \mathcal{L}_0(\mathcal{S})$, $\sum_{X\in\mathcal{S}} f(X)X \mapsto \sum_{n=0}^{\infty} f(X_0^n)X_0^n$. Then E is a well-defined conditional expectation from $\mathcal{L}(\mathcal{S})$ onto $\mathcal{L}_0(\mathcal{S})$.*

Proof For each $g \in l^2(\mathcal{S})$ such that $g(X) = 0$ when $X \neq X_0^n$, we have

$$\left\| \sum_{n=0}^{\infty} f(X_0^n)X_0^n \sum_{n=0}^{\infty} g(X_0^n)\delta_{X_0^n} \right\|_2^2 \leq \|L_f g\|_2^2 \leq \|L_f\|^2 \|g\|_2^2.$$

This implies that $\sum_{n=0}^{\infty} f(X_0^n)X_0^n$ is bounded on the Hilbert subspace $l^2(\mathcal{S}_0)$, where $\mathcal{S}_0$ is the subsemigroup of $\mathcal{S}$ generated by X_0. The norm of $\sum_{n=0}^{\infty} f(X_0^n)X_0^n$ is bounded by $\|L_f\|$. Let $\mathscr{F}$ be the Fourier transform: $\mathbb{Z} \to \mathbb{S}^1$, $n \mapsto e^{2\pi i n\theta}$, $\theta \in [0,1)$. This induces the following isomorphisms [8]:

$$\begin{array}{ccccccccccccc}
\mathbb{Z} & \subseteq & \mathbb{C}\mathbb{Z} & \subseteq & l^1(\mathbb{Z}) & \subseteq & C^*(\mathbb{Z}) & \subseteq & \mathcal{L}(\mathbb{Z}) & \subseteq & l^2(\mathbb{Z}) & \subseteq & \cdots \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
\mathbb{S}^1 & \subseteq & \mathbb{C}[\mathbb{Z}]_{\mathbb{S}^1} & \subseteq & R(\mathbb{S}^1) & \subseteq & C(\mathbb{S}^1) & \subseteq & L^\infty(\mathbb{S}^1) & \subseteq & l^2(\mathbb{S}^1) & \subseteq & \cdots
\end{array}$$

Restricting the Fourier transform on $\mathbb{N}$, we have

$$\begin{array}{ccccccccccc}
\mathbb{C}\mathbb{N} & \subseteq & l^1(\mathbb{N}) & \subseteq & B(\mathbb{N}) & \subseteq & \mathcal{L}(\mathbb{N}) & \subseteq & l^2(\mathbb{N}) & \subseteq & \cdots \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
\mathbb{C}[\mathbb{Z}]_{\mathbb{D}} & \subseteq & H^1(\mathbb{D}) & \subseteq & H_c(\mathbb{D}) & \subseteq & H^\infty(\mathbb{D}) & \subseteq & H^2(\mathbb{D}) & \subseteq & \cdots
\end{array}$$

Since $\mathcal{S}_0$ is isomorphic to $(\mathbb{N},+)$, $\sum_{n=0}^{\infty} f(X_0^n)\delta_n$ belongs to $\mathcal{L}(\mathbb{N})$ and $\mathscr{F}(\sum_{n=0}^{\infty} f(X_0^n)\delta_n)$ is in $H^\infty(\mathbb{D})$. It follows that $\mathscr{F}(\sum_{n=0}^{\infty} f(X_0^n)\delta_n)$ belongs to $L^\infty(\mathbb{S}^1)$. Thus $\sum_{n=0}^{\infty} f(X_0^n)\delta_n$ is a bounded operator on $l^2(\mathbb{Z})$ and belongs to $\mathcal{L}(\mathbb{Z})$. Since the subgroup $H = \langle X_0 \rangle$ of Thompson's group $\mathfrak{F}$ is isomorphic to $\mathbb{Z}$, we have that $\sum_{n=0}^{\infty} f(X_0^n)X_0^n$ is bounded on $l^2(H)$. Let $\{H_n\}_{n=1}^{\infty}$ be all right cosets of H in Thompson's group $\mathfrak{F}$ such that $\mathfrak{F} = \sqcup H_n$. Then

$$l^2(\mathfrak{F}) = \bigoplus_{n=1}^{\infty} l^2(H_n).$$

Moreover, $l^2(H_n)$ are invariant subspaces of $\sum_{n=0}^{\infty} f(X_0^n)X_0^n$ and the operator norms on each subspace are same. Hence $\sum_{n=0}^{\infty} f(X_0^n)X_0^n$ is bounded on $l^2(\mathfrak{F})$. Thus $\sum_{n=0}^{\infty} f(X_0^n)X_0^n$ is in $\mathcal{L}_0(\mathcal{S})$. Then E is well-defined. It is clear that conditions (i) and (ii) in Definition 5.5 hold for E. We complete the proof. ■

5.4 Proof of main results

In this subsection, we will prove the main results of this paper (see Theorems 5.8 and 5.10). When the $\mathfrak{B}(\mathcal{S})$-bimodule is $\mathcal{L}(\mathcal{S})$, the first continuous cohomology group of $\mathfrak{B}(\mathcal{S})$ vanishes.

Theorem 5.8 *The first continuous cohomology group $H^1(\mathfrak{B}(\mathcal{S}), \mathcal{L}(\mathcal{S})) = 0$.*

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The following result is an immediate corollary of Theorems 4.1 and 5.8.

Corollary 5.9 *Derivations on the Banach algebra $\mathfrak{B}(\mathcal{S})$ are spatial and induced by operators in $\mathcal{L}(\mathcal{S})$.*

Let $\mathcal{M}(\mathcal{S})$ be the set of all $T \in \mathcal{L}(\mathcal{S})$ such that $[T, A] \in \mathfrak{B}(\mathcal{S})$ for each $A \in \mathfrak{B}(\mathcal{S})$, where $[T, A] = TA - AT$. Then $\mathcal{M}(\mathcal{S})$ is a norm closed linear subspace of $\mathcal{L}(\mathcal{S})$ containing $\mathfrak{B}(\mathcal{S})$. The first cohomology group of $\mathfrak{B}(\mathcal{S})$ is characterized as the linear space $\mathcal{M}(\mathcal{S})$ module $\mathfrak{B}(\mathcal{S})$.

Theorem 5.10 *The first continuous cohomology group* $H^1(\mathfrak{B}(\mathcal{S}), \mathfrak{B}(\mathcal{S})) = \dfrac{\mathcal{M}(\mathcal{S})}{\mathfrak{B}(\mathcal{S})}$.

The following two lemmas are crucial to obtain the above results.

Lemma 5.11 *Let D be a continuous derivation from the Banach algebra $\mathfrak{B}(\mathcal{S})$ into $\mathcal{L}(\mathcal{S})$. If $D(X_0) = AX_0 - X_0A$ and $D(X_1) = AX_1 - X_1A$ for some A in $B(l^2(\mathcal{S}))$, then $D(T) = AT - TA$ for each T in $\mathfrak{B}(\mathcal{S})$.*

Proof For each $n \geq 1$, if $D(X_0^n) = AX_0^n - X_0^nA$, then

$$D(X_0^{n+1}) = X_0D(X_0^n) + D(X_0)X_0^n = AX_0^{n+1} - X_0^{n+1}A.$$

Therefore, by induction, we have $D(X_0^n) = AX_0^n - X_0^nA$ for any $n \geq 1$. By the definition of Thompson's semigroup, we have $X_1X_0^m = X_0^mX_{m+1}$ for any $m \geq 1$. Then

$$D(X_1)X_0^m + X_1D(X_0^m) = D(X_0^m)X_{m+1} + X_0^mD(X_{m+1}).$$

We have

$$\begin{aligned} D(X_{m+1}) &= X_0^{*m}(D(X_1)X_0^m + X_1D(X_0^m) - D(X_0^m)X_{m+1}) \\ &= X_0^{*m}(AX_1X_0^m - X_1AX_0^m + X_1AX_0^m - X_1X_0^mA - AX_0^mX_{m+1} + X_0^mAX_{m+1}) \\ &= X_0^{*m}(X_0^mAX_{m+1} - X_1X_0^mA) \\ &= AX_{m+1} - X_{m+1}A. \end{aligned}$$

Similarly, we can prove that $D(X) = AX - XA$ for any X in $\mathcal{S}$. By linearity, we have $D(T) = AT - TA$ for each T in the semigroup algebra $\mathbb{C}[\mathcal{S}]$. Since $\mathbb{C}[\mathcal{S}]$ is a dense subalgebra of $\mathfrak{B}(\mathcal{S})$, we obtain that $D(T) = AT - TA$ for each T in $\mathfrak{B}(\mathcal{S})$ by the continuity of D. ■

The following lemma is a generalization of the above conclusion.

Lemma 5.12 *Let D be a continuous derivation from the Banach algebra $\mathfrak{B}(\mathcal{S})$ into $\mathcal{L}(\mathcal{S})$. If $D(X_0) = AX_0 - X_0A$ and $D(X_1) = BX_1 - X_1B$ for some A and B in $\mathcal{L}(\mathcal{S})$, then there exists an operator C in $\mathcal{L}(\mathcal{S})$ such that $D(T) = CT - TC$ for each T in $\mathfrak{B}(\mathcal{S})$.*

Proof By the definitions of Thompson's semigroup and derivation, we have

$$X_1X_0 = X_0X_2, \; X_2X_0 = X_0X_3, \; X_2X_1 = X_1X_3$$

and

$$\begin{aligned} D(X_1)X_0 + X_1D(X_0) &= D(X_0)X_2 + X_0D(X_2), \\ D(X_2)X_0 + X_2D(X_0) &= D(X_0)X_3 + X_0D(X_3), \\ D(X_2)X_1 + X_2D(X_1) &= D(X_1)X_3 + X_1D(X_3). \end{aligned}$$

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By the above equations, we have

$$\begin{aligned}D(X_2) &= X_0^* X_1(A-B)X_0 + X_0^*(B-A)X_1X_0 + (AX_2 - X_2A),\\ D(X_3) &= (X_0^*)^2 X_1(A-B)X_0^2 + (X_0^*)^2(B-A)X_1X_0^2 + (AX_3 - X_3A),\\ D(X_3) &= X_1^*X_0^*X_1(A-B)X_0X_1 + X_1^*X_0^*(B-A)X_1X_0X_1 + X_1^*(A-B)X_2X_1\\ &\quad + X_1^*X_2(B-A)X_1 + (BX_3 - X_3B).\end{aligned}$$

It follows that

$$\begin{aligned}&(X_0^*)^2 X_1(A-B)X_0^2 + (X_0^*)^2(B-A)X_1X_0^2 + (A-B)X_3 + X_3(B-A)\\ &= X_1^*X_0^*X_1(A-B)X_0X_1 + X_1^*X_0^*(B-A)X_1X_0X_1 + X_1^*(A-B)X_2X_1\\ &\quad + X_1^*X_2(B-A)X_1.\end{aligned}$$

Let $A-B = L_f \in \mathcal{L}(\mathcal{S})$ and $L_g = L_f - f(e)I - \sum_{n\geq 0}\sum_{m\geq 1} f(X_n^m)X_n^m$. Then we have

$$\begin{aligned}&(X_0^*)^2 X_1 L_g X_0^2 - (X_0^*)^2 L_g X_1 X_0^2 + L_g X_3 - X_3 L_g = X_1^*X_0^*X_1L_gX_0X_1 - X_1^*X_0^*L_gX_1X_0X_1\\ &\quad + X_1^*L_gX_2X_1 - X_1^*X_2L_gX_1 + \left(\sum_{n\geq 2}\sum_{m\geq 1} f(X_n^m)X_{n+1}^m\right)X_3 - X_3\left(\sum_{n\geq 2}\sum_{m\geq 1} f(X_n^m)X_{n+1}^m\right)\\ &\quad + X_3\left(\sum_{n\geq 2}\sum_{m\geq 1} f(X_n^m)X_n^m\right) - \left(\sum_{n\geq 2}\sum_{m\geq 1} f(X_n^m)X_n^m\right)X_3.\end{aligned} \tag{11}$$

If $g \neq 0$, then we take the minimal element X of the set $\{X \in \mathcal{S} \mid g(X) \neq 0\}$ under the total order. By the definition of L_g, we have $X = X_{i_1}^{\alpha_{i_1}} \ldots X_{i_t}^{\alpha_{i_t}}$, where $i_1 < i_2 < \cdots < i_t$ and $\alpha_{i_1}, \ldots, \alpha_{i_t} \geq 1$, $t \geq 2$.

Case I: $i_1 \geq 3$. We have $X_3X \prec XX_3$. Taking $\langle \cdot\, \delta_e, \delta_{X_3X}\rangle$ on the both sides of equation (11), we obtain that $g(X) = 0$.

Case II: $i_1 = 1$ or 2. We have $XX_3 \prec X_3X$. Analogously, taking $\langle \cdot\, \delta_e, \delta_{XX_3}\rangle$ on the both sides of equation (11), we also have $g(X) = 0$.

Case III: $i_1 = 0$. If $X_1 | XX_2X_1$, then taking $\langle \cdot\, \delta_e, \delta_{X_1^{-1}XX_2X_1}\rangle$, we can obtain that $g(X) = 0$. If $X_1 \nmid XX_2X_1$, by the normal form (of elements) in Thompson"s group $\mathcal{F}$, there exist Y and X_k in $\mathcal{S}$ such that $X_1^{-1}XX_2X_1 = YX_k^{-1}$, where X_1^{-1} and X_k^{-1} are in $\mathcal{F}$. Then taking $\langle \cdot\, \delta_{X_k}, \delta_Y\rangle$, we can also obtain that $g(X) = 0$.

It follows from the above discussion that $g = 0$. This leads to

$$\begin{aligned}&\left(\sum_{n\geq 2}\sum_{m\geq 1} f(X_n^m)X_{n+1}^m\right)X_3 - X_3\left(\sum_{n\geq 2}\sum_{m\geq 1} f(X_n^m)X_{n+1}^m\right)\\ &\quad + X_3\left(\sum_{n\geq 2}\sum_{m\geq 1} f(X_n^m)X_n^m\right) - \left(\sum_{n\geq 2}\sum_{m\geq 1} f(X_n^m)X_n^m\right)X_3\\ &= 0.\end{aligned}$$

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It is not difficult to verify that $f(X_2^m) = 0$ and $f(X_n^m) = f(X_3^m)$ for all $n \geq 4$ and $m \geq 1$. Since $f \in l^2(\mathcal{S})$, we have $f(X_n^m) = 0$ for any $n \geq 2$ and $m \geq 1$. Therefore,

$$A - B = \sum_{m=1}^{\infty} f(X_0^m)X_0^m + \sum_{m=1}^{\infty} f(X_1^m)X_1^m + f(e)I.$$

Let

$$C = A - \sum_{m=1}^{\infty} f(X_0^m)X_0^m = B + \sum_{m=1}^{\infty} f(X_1^m)X_1^m + f(e)I.$$

Then by Theorem 5.7, we have $C \in \mathcal{L}(\mathcal{S})$ and

$$D(X_0) = CX_0 - X_0C, \quad D(X_1) = CX_1 - X_1C.$$

By Lemma 5.11, we have $D(T) = CT - TC$ for each T in $\mathfrak{B}(\mathcal{S})$. ■

Proof of Theorem 5.8 Let D be a continuous derivation from $\mathfrak{B}(\mathcal{S})$ into $\mathcal{L}(\mathcal{S})$. By Lemma 5.1, Lemma 5.2 and Lemma 5.12, there exits an operator C in $\mathcal{L}(S)$ such that $D(T) = CT - TC$ for each T in $\mathfrak{B}(\mathcal{S})$. Thus D is an inner derivation and then $H^1(\mathfrak{B}(\mathcal{S}), \mathcal{L}(\mathcal{S})) = 0$. We complete the proof. ■

Let us see the following lemma before proving Theorem 5.10.

Lemma 5.13 *The intersection $\mathcal{L}(\mathcal{S}) \cap \mathcal{R}(\mathcal{S}) = \mathbb{C}I$.*

Proof For any $L_f \in \mathcal{L}(\mathcal{S}) \cap \mathcal{R}(\mathcal{S})$, there exists an operator $R_g \in \mathcal{R}(\mathcal{S})$ such that $L_f = R_g$. Hence $f = L_f\delta_e = R_g\delta_e = g$. Moreover,

$$f * \delta_{X_0} = L_f\delta_{X_0} = R_f\delta_{X_0} = \delta_{X_0} * f.$$

It is not hard to check that $f = \sum_{n=0}^{\infty} f(X_0^n)\delta_{X_0^n}$ using Lemma 3.7. Similarly,

$$f * \delta_{X_1} = \delta_{X_1} * f.$$

Hence $f = f(e)\delta_e$. This completes the proof. ■

Proof of Theorem 5.10 Let D be a derivation on the Banach algebra $\mathfrak{B}(\mathcal{S})$ then D is continuous. By Lemmas 5.1, 5.2, and 5.12, there exists an operator C in $\mathcal{L}(\mathcal{S})$ such that $D(T) = CT - TC$ for each T in $\mathfrak{B}(\mathcal{S})$. This induces the following map:

$$\begin{array}{cccc} \pi: & H^1(\mathfrak{B}(\mathcal{S}), \mathfrak{B}(\mathcal{S})) & \longrightarrow & \dfrac{\mathcal{M}(\mathcal{S})}{\mathfrak{B}(\mathcal{S})} \\ & \overline{D} & \longrightarrow & C + \mathfrak{B}(\mathcal{S}). \end{array}$$

We will show that π is well-defined. If there exist two operators C_1 and C_2 in $\mathcal{L}(\mathcal{S})$ such that $D(T) = C_1T - TC_1 = C_2T - TC_2$ for each T in $\mathfrak{B}(\mathcal{S})$, then $C_1 - C_2 \in \mathcal{L}(\mathcal{S}) \cap \mathfrak{B}(\mathcal{S})' = \mathcal{L}(\mathcal{S}) \cap \mathcal{R}(\mathcal{S}) = \mathbb{C}I$ from Lemma 5.13. Thus $C_1 + \mathfrak{B}(\mathcal{S}) = C_2 + \mathfrak{B}(\mathcal{S})$. Now, if $\overline{D_1} = \overline{D_2}$, then $D_1 - D_2$ is an inner derivation of $\mathfrak{B}(\mathcal{S})$. There exists an operator C_3 in $\mathfrak{B}(\mathcal{S})$ such that $(D_1 - D_2)(T) = C_3T - TC_3$ for each T in $\mathfrak{B}(\mathcal{S})$. Assume that $D_1(T) = C_1'T - TC_1'$ and $D_2(T) = C_2'T - TC_2'$, where C_1' and C_2' are in $\mathcal{L}(\mathcal{S})$, then $(C_1' - C_2')T - T(C_1' - C_2') = C_3T - TC_3$. It follows that $C_1' - C_2' - C_3$ belongs to $\mathcal{L}(\mathcal{S}) \cap \mathfrak{B}(\mathcal{S})' = \mathbb{C}I$. Thus $C_1' - C_2'$ belongs to $\mathfrak{B}(\mathcal{S})$, that is $C_1' + \mathfrak{B}(\mathcal{S}) = C_2' + \mathfrak{B}(\mathcal{S})$. The map π is a well-defined group homomorphism. If $\pi(\overline{D}) = 0$, then there exists an

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operator C' in $\mathfrak{B}(\mathcal{S})$ such that $D(T) = C'T - TC'$, which means that D is an inner derivation. The map π is injective. The surjectivity of π is obvious. It follows from the above discussion that π is a group isomorphism. We complete the proof. ∎

6 Further discussions

We now recall the definition of higher order continuous Hochschild cohomology for Banach algebras. Let $\mathcal{M}$ be a Banach algebra, and let $\mathcal{X}$ be a Banach $\mathcal{M}$-bimodule. The space of all n-linear (continuous) maps from n-fold Cartesian product $\mathcal{M}^n = \mathcal{M} \times \cdots \times \mathcal{M}$ into $\mathcal{X}$ is denoted by $L^n(\mathcal{M}, \mathcal{X})$ for $n \geq 1$, while $L^0(\mathcal{M}, \mathcal{X})$ is defined to be $\mathcal{X}$.

The coboundary operator $\partial^n\colon L^n(\mathcal{M}, \mathcal{X}) \to L^{n+1}(\mathcal{M}, \mathcal{X})$ is defined, for $n \geq 1$, by

$$\begin{aligned}\partial^n \phi(a_1, a_2, \ldots, a_{n+1}) = a_1\phi(a_2, \ldots, a_{n+1})& \\ + \sum_{i=1}^{n}(-1)^i \phi(a_1, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_{n+1})& \\ + (-1)^{n+1}\phi(a_1, \ldots, a_n)a_{n+1},&\end{aligned}$$

where $\phi \in L^n(\mathcal{M}, \mathcal{X})$ and $a_1, a_2, \ldots, a_{n+1} \in \mathcal{M}$. When $n = 0$, we define ∂^0 by

$$\partial^0 x(m) = mx - xm \qquad (x \in \mathcal{X},\ m \in \mathcal{M}).$$

It is routine to check that $\partial^n\partial^{n-1}\colon L^{n-1}(\mathcal{M}, \mathcal{X}) \to L^{n+1}(\mathcal{M}, \mathcal{X})$ is zero for all $n \geq 1$, and so $\operatorname{Im}(\partial^{n-1})$ is a linear subspace of $\operatorname{Ker}(\partial^n)$. The nth Hochschild cohomology group $H^n(\mathcal{M}, \mathcal{X})$ is then defined to be the following quotient space:

$$\frac{\operatorname{Ker}(\partial^n : L^n(\mathcal{M}, \mathcal{X}) \to L^{n+1}(\mathcal{M}, \mathcal{X}))}{\operatorname{Im}(\partial^{n-1} : L^{n-1}(\mathcal{M}, \mathcal{X}) \to L^n(\mathcal{M}, \mathcal{X}))}$$

for $n \geq 1$. We end this paper by proposing some problems for future study:

- What are the higher order cohomology groups $H^n(\mathfrak{B}(S), \mathfrak{B}(S))$ for $n \geq 2$?
- When $n \geq 2$, does $H^n(\mathfrak{B}(\mathcal{S}), \mathcal{L}(\mathcal{S})) = 0$? The first step to calculate the high order cohomology groups should be the following. Given a 2-cocycle ϕ, we need to modify it by a 1-coboundary such that ϕ is X_0-multimodular, i.e., $\phi(X_0A, B) = X_0\phi(A, B)$, $\phi(AX_0, B) = \phi(A, X_0B)$, and $\phi(A, BX_0) = \phi(A, B)X_0$ for all $A, B \in \mathfrak{B}(\mathcal{S})$.
- What are the cohomology groups of $\mathfrak{B}(\mathcal{F}_n)$ and $\mathfrak{B}(\mathcal{T})$?

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References

[1] W. G. Bade and P. C. Curtis, Jr., *The continuity of derivations of Banach algebras*. J. Funct. Anal. **16**(1974), 372–387.

[2] M. G. Brin and C. C. Squier, *Groups of piecewise linear homeomorphisms of the real line*. Invent. Math. 79(1985), 485–498.

https://doi.org/10.4153/S0008414X24000154 Published online by Cambridge University Press

[3] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Introductory notes on Richard Thompson's groups*. Enseign. Math. (2) **42**(1996), 215–256.

[4] E. Christensen, F. Pop, A. M. Sinclair, and R. R. Smith, *Hochschild cohomology of factors with property Γ*. Ann. of Math. (2) **158**(2003), 635–659.

[5] A. Dong, L. Huang, and B. Xue, *Operator algebras associated with multiplicative convolutions of arithmetic functions*. Sci. China Math. **61**(2018), 1665–1676.

[6] E. Følner, *On groups with full Banach mean value*. Math. Scand. **3**(1955), 243–254.

[7] A. H. Frey, *Studies on amenable semigroups*. Ph.D. thesis, University of Washington, 1960.

[8] L. Ge, *Numbers and figures* (in Chinese). In: N. Xi, The institute lecture 2010, Science Press, Beijing, 2012, pp. 1–10.

[9] L. Ge, M. Ma, and B. Qi, *Riemann ζ-functions on semigroups (in Chinese)*. Sci. Sin. Math. **51**(2021), 1545–1578.

[10] G. Hochschild, *On the cohomology groups of an associative algebra*, Ann. of Math. (2) **46**(1945), 58–67.

[11] G. Hochschild, *On the cohomology theory for associative algebras*, Ann. of Math. (2) **47**(1946), 568–579.

[12] G. Hochschild, *Cohomology and representations of associative algebras*. Duke Math. J. **14**(1947), 921–948.

[13] B. E. Johnson, *Cohomology in Banach algebras*, Memoirs of the American Mathematical Society, American Mathematical Society, Providence, RI, 1972.

[14] B. E. Johnson, *A class of* II_1 *factors without property P but with zero second cohomology*. Ark. Mat. **12**(1974), 153–159.

[15] B. E. Johnson, R. V. Kadison, and J. R. Ringrose, *Cohomology of operator algebras III. Reduction to normal cohomology*. Bull. Soc. Math. France **100**(1972), 73–96.

[16] B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*. Amer. J. Math. **90**(1968), 1067–1973.

[17] R. V. Kadison, *On the orthogonalization of operator representations*. Amer. J. Math. **77**(1955), 600–620.

[18] R. V. Kadison, *Derivations of operator algebras*. Ann. of Math. **83**(1966), 280–293.

[19] R. V. Kadison and J. R. Ringrose, *Cohomology of operator algebras I: Type I von Neumann algebras*. Acta Math. **126**(1971), 227–243.

[20] R. V. Kadison and J. R. Ringrose, *Cohomology of operator algebras II: Extended cobounding and the hyperfinite case*. Ark. Mat. **9**(1971), 55–63.

[21] I. Kaplansky, *Modules over operator algebras*. Amer. J. Math. **75**(1953), 839–859.

[22] F. Pop and R. R. Smith, *Vanishing of second cohomology for tensor products of type II_1 von Neumann algebras*. J. Funct. Anal. **258**(2010), 2695–2707.

[23] S. Sakai, *On a conjecture of Kaplansky*. Tohoku Math. J. (2) **12**(1960), 31–33.

[24] S. Sakai, *Derivations of W^*-Algebras*. Ann. of Math. (2) **83**(1966), 273–279.

[25] A. M. Sinclair and R. R. Smith, *Hochschild cohomology of von Neumann algebras*, Cambridge University Press, Cambridge, 1995.

[26] A. M. Sinclair and R. R. Smith, *Hochschild cohomology for von Neumann algebras with Cartan subalgebras*. Amer. J. Math. **120**(1998), 1043–1057.

[27] B. Xue, *Non-commutative arithmetics on Thompson's monoid*. Acta Math. Sin. (Engl. Ser.) **37**(2021), 1586–1626.

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