A NOTE ON A PAIR OF INTEGRAL OPERATORS INVOLVING WHITTAKER FUNCTIONS

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In recent years various authors have studied integral operators involving confluent hypergeometric functions $M_{\kappa,\mu}$ and $W_{\kappa,\mu}$. Using the method devised by Fox [**2**], Saxena [**5**] obtained the inverse of an integral operator with kernel $(xt)^{\mu-\frac{1}{2}}e^{-\frac{1}{2}xt}W_{\kappa,\mu}(xt)$. Singh [**6**] derived the solution of an integral equation of convolution type with kernel

$$(t-x)^{\mu-\frac{1}{2}}W_{\kappa,\mu}(t-x).$$

In this note we show that the transforms defined by

$$\mathscr{H}_{\kappa,\mu}f(x)=\int_0^\infty (xt)^{\kappa-\frac{1}{2}}e^{-\frac{1}{2}xt}M_{\kappa,\mu}(xt)f(t)\,dt, \tag{1}$$

$$\mathscr{G}_{\kappa,\mu}f(x)=\int_0^\infty (xt)^{\kappa-\frac{1}{2}}e^{-\frac{1}{2}xt}W_{\kappa,\mu}(xt)f(t)\,dt, \tag{2}$$

in which $M_{\kappa,\mu}$ and $W_{\kappa,\mu}$ denote Whittaker's confluent hypergeometric functions can be represented as the composition of two operators, one of which

$$T^\alpha f(x)=x^{\alpha-1}\int_0^\infty t^{\alpha-1}e^{-xt}f(t)\,dt, \qquad (\alpha>0,\, x>0), \tag{3}$$

is a modification of the Laplace transform. The second operator, defined by

$$I_{\eta,\alpha}f(x)=\frac{x^{-\eta-\alpha}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^\eta f(t)\,dt \qquad (\alpha>0,\,\eta>-1,\,x\geqq 0), \tag{4}$$

was introduced by Kober [**3**].

It is easily shown [**4**] that T^α and $I_{\eta,\alpha}$ map $L^2(0,\infty)$ onto itself and that, in terms of the usual inner product in that space, the operator T^α is self-adjoint while the operator $I_{\eta,\alpha}$ has adjoint $K_{\eta,\alpha}$ where

$$K_{\eta,\alpha}f(x)=\frac{x^\eta}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\eta-\alpha}f(t)\,dt, \qquad (\alpha>0,\,\eta>-1,\,x\geqq 0). \tag{5}$$

It follows immediately from these definitions and the result

$$t^{\kappa+\mu}I_{-\frac{1}{2},\frac{1}{2}-\kappa+\mu}[y^{\kappa+\mu}e^{-yt};\,y\to x]=\gamma_{\kappa,\mu}(xt)^{\kappa-\frac{1}{2}}e^{-\frac{1}{2}xt}M_{\kappa,\mu}(xt),$$

where $\gamma_{\kappa,\mu}=\Gamma(\mu+\kappa+\frac{1}{2})/\Gamma(2\mu+1)$ (cf. (14) on p. 187 of [**1**]), that

$$I_{-\frac{1}{2},\frac{1}{2}-\kappa+\mu}T^{\kappa+\mu+1}f(x)=\gamma_{\kappa,\mu}\mathscr{H}_{\kappa,\mu}f(x), \tag{6}$$

where $\mathscr{H}_{\kappa,\mu}$ is defined by equation (1). Similarly the relation

$$K_{\kappa-\mu,\frac{1}{2}-\kappa+\mu}T^{\kappa+\mu+1}f(x)=\mathscr{G}_{\kappa,\mu}f(x) \tag{7}$$

follows immediately from the definitions (3), (5) and formula (13) on p. 202 of [**1**], $\mathscr{G}_{\kappa,\mu}$ being defined by equation (2).

To obtain the inversion theorems for $\mathscr{H}_{\kappa,\mu}$ and $\mathscr{G}_{\kappa,\mu}$ we need the formulae

$$(T^{\alpha})^{-1}f(x)=x^{1-\alpha}\mathscr{L}^{-1}[t^{1-\alpha}f(t);x],$$

$$I^{-1}_{\eta,\alpha}=I_{\eta+\alpha,-\alpha},\qquad K^{-1}_{\eta,\alpha}=K_{\eta+\alpha,-\alpha},$$

where, for $\alpha<0$, $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ are defined by the equations

$$I_{\eta,\alpha}f(x)=x^{-\eta-\alpha}\frac{d^n}{dx^n}x^{n+\alpha+\eta}I_{\eta,\alpha+n}f(x),$$

$$K_{\eta,\alpha}f(x)=(-1)^n x^{\eta}\frac{d^n}{dx^n}x^{n-\alpha}K_{\eta-n,\alpha+n}f(x)$$

with n a positive integer such that $0\leqq\alpha+n<1$.

Now, if $\mathscr{H}_{\kappa,\mu}f=\hat{f}_{\kappa,\mu}$ it follows from (6) that

$$T^{\kappa+\mu+1}f=\gamma_{\kappa,\mu}I_{-\kappa+\mu,\kappa-\mu-\frac{1}{2}}\hat{f}_{\kappa,\mu}$$

and hence that

$$\mathscr{H}^{-1}_{\kappa,\mu}\hat{f}_{\kappa,\mu}(x)=\gamma_{\kappa,\mu}x^{-\kappa-\mu}\mathscr{L}^{-1}[t^{-\kappa-\mu}I_{-\kappa+\mu,\kappa-\mu-\frac{1}{2}}\hat{f}_{\kappa,\mu}(t);x].$$

Similarly the equation $\mathscr{G}_{\kappa,\mu}f=f^*_{\kappa,\mu}$ implies that

$$T^{\kappa+\mu+1}f=K_{\frac{1}{2},\kappa-\mu-\frac{1}{2}}f^*_{\kappa,\mu}$$

and hence that

$$\mathscr{G}^{-1}_{\kappa,\mu}f^*_{\kappa,\mu}(x)=x^{-\kappa-\mu}\mathscr{L}^{-1}[t^{-\kappa-\mu}K_{\frac{1}{2},\kappa-\mu-\frac{1}{2}}f^*_{\kappa,\mu}(t);x]$$

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