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MOMENT SEQUENCES AND BACKWARD EXTENSIONS OF SUBNORMAL WEIGHTED SHIFTS

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Abstract

In this note we examine the relationships between a subnormal shift, the measure its moment sequence generates, and those of a large family of weighted shifts associated with the original shift. We examine the effects on subnormality of adding a new weight or changing a weight. We also obtain formulas for evaluating point mass at the origin for the measure associated with the shift. In addition, we examine the relationship between the measure associated with a subnormal shift and those of a family of shifts substantially different from the original shift.

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1. Introduction

Weighted shifts have been used to provide examples and illustrations of many operator theoretic properties. In several cases major conjectures in operator theory have been reduced to the weighted shift case. The intimate relationship between weighted shifts, subnormality, and moment sequences, first exhibited by C. Berger (as referenced below), has led to a productive and extended area of investigation. In this note we examine the relationships between a subnormal shift, the measure its moment sequence generates, and those of a large family of weighted shifts associated with the original shift.

In the second section, the basic properties of subnormal shifts are stated, and several notational conventions are established. The third section deals with the effects on subnormality of adding a new term to the weight sequence or changing the value of one of the terms. The fourth section is concerned with some technical properties

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of the measures involved. The fifth section examines the relationship between the measure associated with a subnormal shift and those of a family of shifts substantially different from the original shift, but tied to it in a more or less natural way.

2. Preliminaries and notation

Let $\mathscr{H}$ be a separable, complex Hilbert space with orthonormal basis $\{e_0, e_1, \dots\}$. We will denote a sequence α of complex numbers by $\alpha : \alpha_0, \alpha_1, \dots$. The bounded linear operator W_α on $\mathscr{H}$ uniquely determined by the equations $W_\alpha e_n = \alpha_n e_{n+1}$ is called *the weighted shift with weight sequence* α. [5] is a good reference for the general properties of such operators. Throughout this article we will make frequent reference to the *weight product sequence* for W_α; namely $\beta_0 = 1$, $\beta_n = \alpha_0\alpha_1 \cdots \alpha_{n-1}$ $(n \geqslant 1)$. In this note we will only be concerned with the case that α is a strictly positive sequence converging to 1. Berger (as described by Halmos in [3]) showed that W_α is subnormal if and only if there is a probability measure μ on $[0, 1]$ with 1 in the support of μ such that $\{\beta_n^2\}_{n=0}^\infty$ is *the moment sequence* for μ; that is, for each $n \geqslant 0$,

$$\beta_n^2 = \int_0^1 t^n \, d\mu.$$

We will refer to $(W_\alpha, \{\beta_n\}_{n=0}^\infty, \mu)$ as a *subnormal shift system.*

Once one has a subnormal shift, there are many other weighted shifts associated with it such as its restrictions and extensions. Some of these are automatically subnormal, while others may or may not be subnormal. In this note we will be particularly concerned with the relationships between the measure determined by the original subnormal shift and the measures determined by these other shifts. To facilitate this investigation we now establish some notation relating to a subnormal shift W_α with associated measure μ. Write $\mu = a\delta_0 + \omega$, where $0 < a \leq 1$ and $\omega(\{0\}) = 0$. Fix an integer $N \geqslant 1$, and define the sequence $\alpha(N)$ by $\alpha(N) : a_N, \alpha_{N+1}, \dots$. The corresponding shift $W_{\alpha(N)}$ is unitarily equivalent to the restriction of W_α to the subspace $\mathscr{P}_N$ spanned by $\{e_N, e_{N+1}, \dots\}$. Since this is the restriction of a subnormal operator to an invariant subspace, $W_{\alpha(N)}$ is itself a subnormal weighted shift (with norm 1). Let μ_N be its associated probability measure and write $\mu_N = a_N\delta_0 + \omega_N$. The corresponding product sequence is

$$\beta_{N,n} = \alpha_N\alpha_{N+1} \cdots \alpha_{N+n-1} = \frac{\beta_{N+n}}{\beta_N}.$$

LEMMA 2.1. *Let* $N \geqslant 1$. *Then*

$$t\,d\mu_N = \frac{t^{N+1}}{\beta_N^2} d\mu \qquad \textit{and} \qquad d\omega_N = \frac{t^N}{\beta_N^2} d\mu = \frac{t^N}{\beta_N^2} d\omega.$$

PROOF. For each $n > 0$,

$$\int_0^1 t^n \, d\mu_N = \beta_{N,n}^2 = \frac{1}{\beta_N^2} \int_0^1 t^{N+n} \, d\mu,$$

that is,

$$\int_0^1 t^{n-1} t \, d\mu_N = \frac{1}{\beta_N^2} \int_0^1 t^{n-1} t^{N+1} \, d\mu.$$

This shows that $t \, d\mu_N = (t^{N+1}/\beta_N^2) \, d\mu$. The corresponding equation involving ω and ω_N follows because these measures have no point mass at 0. □

As a corollary to Curto's theorem (stated in the next section), we see that in fact for $N \geqslant 1$, $\mu_N = \omega_N$, that is, μ_N has no point mass at 0.

3. Backward extensions and perturbations of shifts

Starting with the subnormal shift system $(W_\alpha, \{\beta_n\}_{n=0}^\infty, \mu)$, we may extend the Hilbert space $\mathscr{H}$ by introducing a unit vector e_{-1} orthogonal to $\mathscr{H}$; that is, form the external direct sum $\{e_{-1}\} \oplus \mathscr{H}$. Then for a given positive scalar x, we may form the weighted shift $W_{\alpha(x)}$ (relative to the orthonormal basis $\{e_{-1}, e_0, e_1, \dots\}$) via the sequence $\alpha(x) : x, \alpha_0, \alpha_1, \dots$. The associated product sequence $\beta_n(x)$ is then given by

$$\beta_0(x) = 1; \quad \beta_n(x) = x\beta_{n-1} \text{ for } n > 0.$$

The question of the subnormality of $W_{\alpha(x)}$ has been completely settled by Curto [1]:

THEOREM 3.1 ([1]). *$W_{\alpha(x)}$ is subnormal if and only if $x^2 \int_0^1 (1/t) \, d\mu \leq 1$. In particular, if μ has a point mass at 0, then $1/t \notin L^1(\mu)$, so $W_{\alpha(x)}$ fails to be subnormal for any choice of x.*

COROLLARY 3.2. *Following the notation of the previous section, for $N \geq 1$, $1/t \in L^1(\mu_N)$. In particular, $\mu_N = \omega_N$.*

PROOF. Fix $N \geq 1$. Since $W_\alpha|_{\mathscr{P}_N}$ can be extended back via α_{N-1} to form the subnormal shift $W_\alpha|_{\mathscr{P}_{N-1}}$, the stated result follows from the preceding theorem. □

We will say that W_α has *subnormal backward extension* if $1/t \in L^1(\mu)$. In this context, the appearance of the symbols $(W_{\alpha(x)}, \{\beta_n(x)\}_{n=0}^\infty, \mu_x)$ is meant to convey the information that W_α is subnormal, $x^2 \int_0^1 (1/t) \, d\mu \leq 1$, and $\mu_x = a(x)\delta_0 + \omega(x)$ is the associated probability measure for $W_{\alpha(x)}$. When this is the case, one may easily verify that $t \, d\mu_x = x^2 \, d\mu$ and $d\omega(x) = (x^2/t) \, d\mu$.

We are now in a position to investigate the effect on subnormality of perturbation of a single weight. We separate the cases $N = 0$ and $N > 0$.

THEOREM 3.3. *Let* $(W_\alpha, \{\beta_n\}_{n=0}^\infty, \mu = a\delta_0 + \omega)$ *be a subnormal system, let* $x > 0$, *and define* $\alpha(0, x)$ *to be the sequence formed by replacing* α_0 *by* x *while leaving the other weights unchanged. Then* $W_{\alpha(0,x)}$ *is subnormal if and only if* $x \leq \alpha_0/\sqrt{\omega([0, 1])}$.

PROOF. Suppose $x \leq \alpha_0/\sqrt{\omega([0, 1])}$. Then we have

$$x^2 \int_0^1 \frac{1}{t}\, d\mu_1 = x^2 \int_0^1 \frac{1}{t}\frac{t}{\alpha_0^2}\, d\mu = x^2 \int_0^1 \frac{1}{\alpha_0^2}\, d\omega = \frac{x^2}{\alpha_0^2}\omega([0, 1]) \leq 1.$$

By Theorem 3.1, $W_{\alpha(0,x)}$ is subnormal.

Conversely, suppose that $W_{\alpha(0,x)}$ is subnormal, with corresponding subnormal shift system $(W_{\alpha(0,x)}, \gamma_n, \mu_x = a(x)\delta_0 + \nu)$. Then $t\, d\nu = (x^2/\alpha_0^2)t\, d\mu = (x^2/\alpha_0^2)t\, d\omega$; hence $x^2/\alpha_0^2 d\omega = d\nu$. In particular, $(x^2/\alpha_0^2)\omega([0, 1]) = \nu([0, 1]) \leq 1$, and the proof is complete. $\square$

In terms of the decomposition $\mu = a\delta_0 + \omega$, the preceding theorem may be restated as follows:

REMARK 3.4. $W_{\alpha(0,x)}$ is subnormal if and only if $0 < x \leq \alpha_0/\sqrt{1 - a}$. Thus if $a = 0$, any increase in the value of α_0 results in the loss of subnormality.

COROLLARY 3.5. *Suppose* W_α *is subnormal, and fix* $N \geq 1$. *Define* $\alpha(N, x)$ *to be the sequence formed from* α *by replacing* α_N *by* x *while leaving the other terms of* α *unchanged. For any positive number* $x \neq \alpha_N$, $W_{\alpha(N,x)}$ *is not subnormal.*

PROOF. Since $W_\alpha|_{\mathcal{P}_N}$ is subnormal and it has a subnormal backward extension, $\mu_N(\{0\}) = 0$. Thus the remark above shows that an increase in the first weight of $W_\alpha|_{\mathcal{P}_N}$ leads to a nonsubnormal shift (of course the first weight of $W_\alpha|_{\mathcal{P}_N}$ is α_N). Since this shift is the restriction of $W_{\alpha(N,x)}$ to an invariant subspace, $W_{\alpha(N,x)}$ must also fail to be subnormal. Now assume that $0 < x < \alpha_N$. Then $W_{\alpha(N,x)}|_{\mathcal{P}_N}$ is subnormal. Let $(W_{\alpha(N,x)}|_{\mathcal{P}_N}, \{\gamma_n\}, \lambda)$ be the corresponding subnormal shift system. Then $\gamma_1 = x$, $\gamma_2 = x\alpha_{N+1}$, etc., so that for $k \geq 1$, $\gamma_k = x\beta_{N+k}/\beta_{N+1}$. Hence

$$\gamma_k^2 = \int_0^1 t^k\, d\lambda = \frac{x^2}{\beta_{N+1}^2}\int_0^1 t^{N+k}\, d\mu.$$

This shows that

$$t\, d\lambda = \frac{x^2}{\beta_{N+1}^2} t^{N+1}\, d\mu.$$

Now write $\lambda = b\delta_0 + \lambda'$, where $\lambda'(\{0\}) = 0$. Then since $N \geq 1$,

$$d\lambda' = \frac{x^2}{\beta_{N+1}^2} t^N \, d\mu$$

and so

$$\lambda'([0, 1]) = \frac{x^2}{\beta_{N+1}^2} \int_0^1 t^N \, d\mu = \frac{x^2}{\beta_{N+1}^2} \beta_N^2 = \frac{x^2}{\alpha_N^2}.$$

But we are working under the assumption that $x < \alpha_N$, so that $b = 1 - \lambda'([0, 1]) > 0$. But then Theorem 3.3 guarantees that $W_{\alpha(N,x)}|_{\mathcal{P}_N}$ fails to have a subnormal backward extension, so, in particular, $W_{\alpha(N,x)}$ must fail to be subnormal. □

4. Evaluation of point mass for a subnormal shift system

Throughout this section we assume that $(W_\alpha, \{\beta_n\}_{n=0}^\infty, \mu = a\delta_0 + \omega)$ is a subnormal shift system. We have seen that the value of a is of significance in determining backward extensions and perturbations of W_α and its related shifts. Even though the moment sequence $\{\beta_n^2\}_{n=0}^\infty$ uniquely determines the measure μ, in practice it might be quite difficult to calculate μ explicitly. But knowing $\{\beta_n^2\}_{n=0}^\infty$ is a moment sequence allows us to approach a. A general method for obtaining a is as follows. Let $\{p_k\}_{k=0}^\infty$ be an arbitrary sequence of nonnegative, continuous functions on $[0, 1]$ such that for each k, $p_k(0) = 0$ and for each $t \in (0, 1]$, $p_k(t) \nearrow 1$. Then, via the Monotone Convergence Theorem,

$$\mu(\{0\}) = 1 - \lim_{k\to\infty} \int_0^1 p_k \, d\mu.$$

We offer two such sequences of functions, and present several examples. First, let $p_k(t) = 1 - (1 - t)^k$. This sequence of polynomials has the desired properties, and

$$\begin{aligned} \int_0^1 p_k \, d\mu &= \int_0^1 \left(1 - \sum_{j=0}^k (-1)^j \binom{k}{j} t^j\right) d\mu = 1 - \sum_{j=0}^k (-1)^j \binom{k}{j} \int_0^1 t^j \, d\mu \\ &= 1 - \sum_{j=0}^k (-1)^j \binom{k}{j} \beta_j^2 = \sum_{j=1}^k (-1)^{j+1} \binom{k}{j} \beta_j^2. \end{aligned}$$

For our second construction, let $f_k(t) = 1 - e^{-kt}$. This sequence of functions also satisfies our criteria. In this case we have

$$\int_0^1 f_k \, d\mu = 1 - \int_0^1 \sum_{j=0}^k \frac{(-k)^j}{j!} t^j \, d\mu = 1 - \sum_{j=0}^\infty \frac{(-k)^j}{j!} \beta_j^2 = \sum_{j=1}^\infty \frac{(-k)^{j+1}}{j!} \beta_j^2.$$

This establishes the following result:

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PROPOSITION 4.1. *Let* $(W_\alpha, \{\beta_n\}, \mu)$ *be a subnormal shift system. Then*

$$\mu(\{0\}) = \lim_{k\to\infty} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \beta_j^2 = \lim_{k\to\infty} \sum_{j=0}^{\infty} \frac{(-k)^j}{j!} \beta_j^2.$$

EXAMPLE 4.2. Let $\alpha : \alpha_n = \sqrt{(n+2)/(n+3)}$ $(n \geq 0)$. The shift W_α is the Bergmann shift. Its corresponding measure is $2t$ times Lebesgue measure. Now let $\alpha(x) : x, \sqrt{2/3}, \sqrt{3/4}, \sqrt{4/5}, \ldots$ define a backward extension of α. Then it follows from Curto's theorem that $W_{\alpha(x)}$ is subnormal if and only if $0 < x \leq \sqrt{1/2}$. Now, let us consider

$$\alpha' = \alpha\left(\sqrt{1/2}\right) : \alpha'_n = \sqrt{\frac{n+1}{n+2}} \ (n \geq 0) \quad \text{and} \quad \alpha'(x) : x, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$$

as a backward extension of α'. Let μ and μ' be the associated probability measure for W_α and $W_{\alpha'}$, respectively (μ' is Lebesgue measure and $\mu = 2\mu'$). Then we may calculate directly

$$\mu(\{0\}) = 1 - \lim_{n\to\infty} \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} \beta_i^2 = 1 - \lim_{n\to\infty} \frac{n(3+n)n!}{\Gamma(n+3)} = 0.$$

Similarly, we have

$$\mu_x(\{0\}) = 1 - \lim_{n\to\infty} \frac{n!}{(n+1)\Gamma(n)} = 0.$$

However $W_{\alpha'(x)}$ is not subnormal for any x with $0 < x \leq \sqrt{1/2}$, because $1/t \notin L^1(\mu')$.

5. Moment vectors of subnormal operators

Suppose that T is an injective subnormal operator on $\mathscr{H}$ with $\|T\| = 1$. For a nonzero vector $x \in \mathscr{H}$, consider a weight sequence

$$\alpha_n(x) = \frac{\|T^{n+1}x\|}{\|T^n x\|}, \quad n = 0, 1, 2, \ldots,$$

and let $W_x = W_{\alpha(x)}$ be the weighted shift on ℓ^2_+ with n-th weight $\alpha_n(x)$. Note that the use of the notation $\alpha(x)$ is not that of previous sections. The product weight sequence for $W_{\alpha(x)}$ is given by

$$\beta_n(x) = \alpha_0(x)\alpha_1(x)\cdots\alpha_{n-1}(x) = \frac{\|T^n x\|}{\|x\|}.$$

We will make use of the following version of a result of Embry [2].

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THEOREM 5.1 ([4]). *Suppose that T is an injective operator on $\mathscr{H}$. Then T is subnormal if and only if W_x is subnormal for every nonzero $x \in \mathscr{H}$.*

So if T is an injective subnormal operator on $\mathscr{H}$ with $\|T\| = 1$, for a nonzero vector $x \in \mathscr{H}$, there exists a probability measure μ_x on $[0, 1]$ such that for all $n \geq 0$,

$$\left(\frac{\|T^n x\|}{\|x\|}\right)^2 = \int_0^1 t^n \, d\mu_x(t).$$

Note that $\mu_x([0, 1]) = 1$ but $\operatorname{supp} \mu_x = [0, \|W_x\|]$, so it is possible that $\|W_x\| < 1$ for some $x \in \mathscr{H}$. We will concentrate our attention to the family of shifts W_x for nonzero x in $\mathscr{H}$ when T is itself a weighted shift. We have already looked at some special members of this collection.

EXAMPLE 5.2. Starting with a shift W_α and its product sequence $\{\beta_n\}_{n=0}^\infty$, fix a nonnegative integer N. The shift W_{e_N} is determined as follows:

$$\left\|\frac{W_\alpha^{n+1} e_N}{W_\alpha^n e_N}\right\| = \left\|\frac{\beta_{N+n+1}/\beta_N}{\beta_{N+n}/\beta_N}\right\| = \left\|\frac{\beta_{N+n+1}}{\beta_{N+n}}\right\| = \alpha_{N+n}.$$

Thus W_{e_N} is (unitarily equivalent to) $W_\alpha|_{\mathscr{P}_N}$.

Of course the situation is considerably more complicated when W_x is considered for more general vectors from $\mathscr{H}$. However, in the presence of subnormality, the dominance of μ *vis a vis* absolute continuity remains:

THEOREM 5.3. *Let $(W_\alpha, \{\beta_n\}_{n=0}^\infty, \mu)$ be a subnormal shift system. For each nonzero vector $x := \sum_{n=0}^\infty x_n e_n$ in ℓ_+^2 relative to the orthonormal basis $\{e_n\}_{n=0}^\infty$ in ℓ_+^2,*

$$d\mu_x = \frac{1}{\|x\|^2}\left(\sum_{i=0}^\infty \frac{|x_i|^2}{|\beta_i|^2} t^i\right) d\mu.$$

PROOF. Let x be a nonzero vector from ℓ_+^2 with $x = \sum_{n=0}^\infty x_n e_n$. Then

$$\|W_\alpha^n x\|^2 = \left\|\sum_{i=0}^\infty x_i W_\alpha^n e_i\right\|^2 = \left\|\sum_{i=0}^\infty x_i \frac{\beta_{n+i}}{\beta_i} e_{n+i}\right\|^2 = \sum_{i=0}^\infty \left|x_i \frac{\beta_{n+i}}{\beta_i}\right|^2.$$

We then have

$$\begin{aligned}\int_0^1 t^n \, d\mu_x &= \frac{1}{\|x\|^2} \sum_{i=0}^\infty \left|x_i \frac{\beta_{n+i}}{\beta_i}\right|^2 = \frac{1}{\|x\|^2} \sum_{i=0}^\infty \left|\frac{x_i}{\beta_i}\right|^2 \int_0^1 t^{n+i} \, d\mu \\ &= \int_0^1 t^n \left(\frac{1}{\|x\|^2} \sum_{i=0}^\infty \left|\frac{x_i}{\beta_i}\right|^2 t^i\right) d\mu.\end{aligned}$$

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Since this is valid for all $n \geq 0$,

$$d\mu_x = \left(\frac{1}{\|x\|^2} \sum_{i=0}^{\infty} \left| \frac{x_i}{\beta_i} \right|^2 t^i \right) d\mu,$$

which proves the theorem. □

COROLLARY 5.4. *Let $(W_\alpha, \{\beta_n\}_{n=0}^{\infty}, \mu)$ be a subnormal shift system.*

(a) *If W_α admits a subnormal backward extension, then for every nonzero x in ℓ_+^2, W_x has a subnormal backward extension.*

(b) *If W_α does not admit a subnormal backward extension, then W_x has a subnormal backward extension if and only if $x \perp e_0$, if and only if $x \in$ Ran W_α.*

PROOF. First note that since the weight sequence α is increasing, W_α is bounded below. Hence it has closed range, namely $\mathscr{P}_1 = \{e_0\}^\perp$. Let x be a nonzero vector in ℓ_+^2. Then we have

$$d\mu_x = \left(\frac{1}{\|x\|^2} \sum_{i=0}^{\infty} \left| \frac{x_i}{\beta_i} \right|^2 t^i \right) d\mu.$$

But then

$$\begin{aligned} \int_0^1 \frac{1}{t}\, d\mu_x &= \frac{|x_0|^2}{\|x\|^2} \int_0^1 \frac{1}{t}\, d\mu + \int_0^1 \left(\frac{1}{\|x\|^2} \sum_{i=1}^{\infty} \left| \frac{x_i}{\beta_i} \right|^2 t^{i-1} \right) d\mu \\ &= \frac{|x_0|^2}{\|x\|^2} \int_0^1 \frac{1}{t}\, d\mu + \frac{1}{\|x\|^2} \sum_{i=1}^{\infty} \left| \frac{x_i}{\beta_i} \right|^2 \beta_{i-1}^2 \\ &= \frac{|x_0|^2}{\|x\|^2} \int_0^1 \frac{1}{t}\, d\mu + \frac{1}{\|x\|^2} \sum_{i=1}^{\infty} \left| \frac{x_i}{\alpha_{i-1}} \right|^2. \end{aligned}$$

Since α is an increasing sequence, $\sum_{i=1}^{\infty} |x_i/\alpha_{i-1}|^2 < \infty$, so that W_x has a subnormal backward extension if and only if $|x_0|^2 \int_0^1 (1/t)\, d\mu < \infty$. This observation establishes both parts of the statement of the theorem. □

EXAMPLE 5.5. Let α be the constant sequence 1. Then W_α is the isometric unilateral shift and the corresponding measure is δ_1. Then for any nonzero vector x,

$$d\mu_x = \frac{1}{\|x\|^2} \left(\sum_{i=0}^{\infty} |x_i|^2 t^i \right) d\delta_1 = \frac{1}{\|x\|^2} \left(\sum_{i=0}^{\infty} |x_i|^2 \right) d\delta_1 = d\delta_1.$$

EXAMPLE 5.6. Let $\alpha_n = \sqrt{(n+2)/(n+3)}$ $(n \geq 0)$, so that W_α is the Bergmann shift, and the corresponding measure is given by $d\mu = 2dt$. Then we have

$$\begin{aligned} d\mu_x &= \frac{1}{\|x\|^2}\left(\sum_{i=0}^{\infty}\frac{|x_i|^2}{|\beta_i|^2}t^i\right)d\mu = \frac{1}{\|x\|^2}\left(\sum_{i=0}^{\infty}\frac{|x_i|^2}{\left|\sqrt{2/(i+2)}\right|^2}t^i\right)2\,dt \\ &= \frac{1}{\|x\|^2}\left(\sum_{i=0}^{\infty}(i+2)|x_i|^2t^i\right)dt = \left[\frac{1}{\|x\|^2}\frac{1}{t}\frac{d}{dt}\left(t^2\sum_{i=0}^{\infty}|x_i|^2t^i\right)\right]dt. \end{aligned}$$

We see that in this case there are a great many different measures involved, and admission to this collection may be stated in terms of a level of analyticity of the Radon-Nikodym derivative $d\mu_x/dt$ for nonzero vector x in ℓ^2_+.

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References

[1] R. Curto, 'Quadratically hyponormal weighted shifts', *Integral Equations Operator Theory* **13** (1990), 49–66.
[2] M. Embry, 'A generalization of the Halmos-Bram criterion for subnormality', *Acta Sci. Math. (Szeged)* **35** (1973), 61–64.
[3] P. Halmos, 'Ten problems in Hilbert space', *Bull. Amer. Math. Soc.* **76** (1970), 887–933.
[4] A. Lambert, 'Subnormality and weighted shifts', *J. London Math. Soc.* **14** (1976), 476–480.
[5] A. Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory, Math. Surveys 13 (Amer. Math. Soc., Providence, RI, 1974).

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