SOME RESULTS ON FINITENESS OF RADICAL ALGEBRAS

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1. Introduction

R denotes always a radical algebra over a field Φ. A left ring ideal of R which is also a subvector space over Φ is called a left algebra ideal of R. R is said to be left algebra noetherian if it satisfies the ascending chain condition for left algebra ideals. If dim $R < \infty$, then

(i) R is finitely generated

(ii) R is left algebra noetherian

(iii) R is algebraic.

Since the radical of an algebraic algebra is nil ([4] P. 19), conditions (i), (ii), (iii) are also sufficient for R to be finite-dimensional.

Amitsur has conjectured that the radical of a finitely generated Φ-algebra is nil (see [1]). This brings finitely generated radical Φ-algebra close to being nilpotent and hence of finite dimension. We are therefore led to restrict our investigation to radical algebras and to various conditions that imply that dim $R < \infty$. It seems probable that the following conjecture is true:

R is finite-dimensional radical algebra if and only if (i) R is finitely generated and (ii) R is left algebra noetherian.

Observe that the Levitzki theorem "in a left noetherian ring any nil left ideal is nilpotent" may be extended to algebras, namely, if R is a left algebra noetherian algebra over Φ, then every nil left algebra ideal is nilpotent (see [3]). Using this result we can easily see that the conditions of the conjecture cannot be weakened. An example is given by Golod (see [2]) of a finitely generated nil algebra over any field Φ which is not of finite dimension.

The purpose of this paper is to prove the conjecture under some additional assumptions on the radical algebra R.

2. Preliminaries

A left algebra ideal L of R is said to be finitely generated if there exists a finite number of elements $u_1, \cdots, u_n$ such that

$$L = \Phi u_1 + Ru_1 + \cdots + \Phi u_n + Ru_n.$$

LEMMA 2.1. *Let R be a radical Φ-algebra and L be a non-zero left algebra ideal generated by a finite number of elements. Then $RL \subsetneqq L$.*

PROOF. See [4] p. 200.

THEOREM 2.2. *Let R be a radical, finitely generated and left algebra noetherian over Φ. If R satisfies a polynomial identity then* dim $R < \infty$.

PROOF. It is known that the radical of a finitely generated algebra satisfying a polynomial identity is nil (see [1]). So R is nilpotent. Since R is also finitely generated, dim $R < \infty$.

COROLLARY 2.3. *If R is a commutative, finitely generated radical algebra and if R is left algebra noetherian then* dim $R < \infty$.

NOTATION. If $S \subset R$ then by $(S)^l$ we mean the left annihilator ideal of S. Similarly $(S)^r$ is defined.

3.

In all the theorems of this section, except 3.1, let R be a radical algebra satisfying the following conditions:

(i) R is finitely generated

(ii) R is left algebra noetherian

If the field Φ is non-denumerable then (i) and (ii) are sufficient for R to be finite-dimensional. Thus the remainder of the paper is only of interest for countable fields.

THEOREM 3.1. *Let*

(1) *R be a radical algebra*

(2) $R = \Phi(x_1, x_2)$

(3) $x_1^2 = x_2^2 = 0$ and $(x_1)^r \cap (x_2)^r \neq 0$.

Then dim $R < \infty$.

PROOF. Let $y = x_1 x_2$. We show that y is nilpotent. Note that $x_1 y = y x_2 = 0$ and every element of R has the form

$$z = P(y)x_1 + x_2 Q(y) + x_2 S(y) x_1 + yT(y)$$

where P, Q, S, T are polynomials with coefficients in Φ and their constant terms need not be zero. It follows that

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$$
\begin{aligned}
x_1 z x_1 x_2 &= x_1 x_2 Q(y) x_1 x_2 = y^2 Q(y) \\
x_1 z x_2 &= x_1 x_2 S(y) x_1 x_2 = y^2 S(y) \\
x_1 x_2 z x_1 x_2 &= x_1 x_2 y T(y) x_1 x_2 = y^3 T y) \\
x_1 x_2 z x_2 &= x_1 x_2 P(y) x_1 x_2 = y^2 P(y)
\end{aligned}
$$

for $z \in R$. If $z \in (x_1)^r \cap (x_2)^r$, $z \neq 0$, then above observation implies that

$$\alpha_1 y^{k_1} + \alpha_2 y^{k_2} + \cdots + \alpha_s y^{k_s} = 0$$

where $2 \leqq k_1 < k_2 < \cdots < k_s$. So if

$$x = -\frac{\alpha_2}{\alpha_1} y^{k_2 - k_1} \cdots - \frac{\alpha_s}{\alpha_1} y^{k_r - k_1}$$

then

$$y^{k_1} = y^{k_1} x.$$

So $y^{k_1} = 0$, because x is a quasi-regular element. So $\dim R < \infty$, because the set $x_i, x_i x_j, x_i x_j x_k, \cdots$ is a finite set.

THEOREM 3.2. *If $x_1, \cdots, x_n$ are generators of R such that*

(1) *x_i is nilpotent for all i*

(2) *$x_i R$ (or $R x_i$) is an ideal for all i then $\dim R < \infty$.*

PROOF. It suffices to show that R is nil. Suppose a is a non-nilpotent element in R. Let $S = \{B;\ B$ is an ideal of R such that $a^k \notin B$ for all $k \geqq 1\}$. Then S has a maximal element, say C. Since C is a prime algebra ideal of R, $\bar{R} = R/C$ is a non-zero prime, finitely generated and left algebra noetherian radical algebra over Φ. Can assume $\bar{R} = \Phi(\bar{x}_1, \cdots, \bar{x}_m)$ where $m \leqq n$. If $\bar{x}_i \bar{R} = 0$ for all i then $\bar{x}_i = 0$ for all i, because $\bar{R}$ is a prime ring. This is a contradiction, because $C \neq R$. Let $T = \bar{x}_i \bar{R} \neq 0$ for some i. If $\bar{x}_i$ is of index k then $\bar{x}^{k-1} T = 0$ which is a contradiction. This completes the proof.

EXAMPLE 3.3. Let R be the ring of all polynomials without constant terms, in two indeterminates x and y, over a field Φ of the form

$$\sum \alpha_i x^i + \sum_{i_1, i_2 \neq 0} \alpha_{i_1 \cdots i_k} x^{i_1} y^{i_2} x^{i_3} \cdots + \sum_{j_1, j_2 \neq 0} \alpha_{j_1 \cdots j_l} y^{j_1} x^{j_2} y^{j_3} \cdots$$

subject to the following conditions:

(1) $x^k y^l = y^l x^k$ for all $l \geqq 2$ and $x \geqq 1$ or $l \geqq 1$ and $k \geqq 2$ and $xyx = x^2 y$

and

(2) $x^m = 0$, $(xy)^m = 0$, and $xy^m = 0$ for all $m \geqq n$ when n is fixed and $n \geqq 3$.

R is an algebra over Φ with $\{x, x^i y^j, y^j x^i, y^i x^j y^k\}_{i,j,k=1,\cdots,n-1}$ as a set of generators. Moreover, if t is a generator then tR is an ideal of R.

REMARK. Conditions (1) and (2) in theorem 3.2. can be replaced by

(1)′ $\alpha_{ij} = x_i x_j - x_j x_i$ is nilpotent for all i and j

(2)′ $\alpha_{ij} R$ (or $R\alpha_{ij}$) is an ideal for all i and j.

PROOF. If we assume that R is not nil then let $\bar{R}$ be as in theorem 3.2. If all $\bar{\alpha}_{ij}\bar{R} = 0$ then $\alpha_{ij} \in C$ for all i and j. Then $\bar{R}$ is a commutative ring. Use Corollary 2.3 to get $\bar{R}$ is of finite dimension which implies that $\bar{R}$ is nilpotent. But this is a contradiction to $C \neq R$. So $T = \bar{\alpha}_{ij}\bar{R} \neq 0$ for some i and j. The same argument as in theorem 3.2. leads to a contradiction. Hence $\dim R < \infty$.

THEOREM 3.4. *Let $R = \Phi(x_1, \cdots, x_n)$ and let $\alpha_{ij} = x_i x_j - x_j x_i$. Assume $R\alpha_{ij}$ (or $\alpha_{ij}R$) is nil for all i and j. Then* $\dim R < \infty$.

PROOF. Let N be the sum of all nilpotent algebra ideals of R. Then N is the same as the sum of all nilpotent ring ideals of R. Since $R\alpha_{ij} \subseteq N$, we get $\bar{\alpha}_{ij} = 0$ in $\bar{R} = R/N$ for all i and j. Hence $\bar{R}$ is commutative. We apply Corollary 2.3 to get $R = N$. Thus R is nilpotent and hence finite-dimensional.

LEMMA 3.5. *If R is a ring, $a \in R$, $a^2 = 0$ and $xa - ax$ is nilpotent for all x in R, then aR is nil.*

PROOF. $a(xa - ax)^k x = (ax)^{k+1}$ *for* $x \in R, k = 1, 2, \cdots$.

LEMMA 3.6. *If R is a ring, $a, b \in R$, $a^2 b = 0$ and $ya - ay$ is nilpotent for all y in bR, then abR is nil.*

PROOF. $a(bxa - abx)^k bx = (abx)^{k+1}$ for any $x \in R$, $k = 1, 2, \cdots$.

LEMMA 3.7. *If R is a ring and $a \in R$ such that a and $xa - ax$ are nilpotent for all x in R, then aR is nil.*

PROOF. Suppose aR is not nil, then there is an integer m such that $a^m R$ is nil, but $a^{m-1}R$ is not. Let $b = a^{m-2}$ and apply lemma 3.6. to get a contradiction.

THEOREM 3.8. *Let $R = \Phi(x_1, \cdots, x_n)$, $\alpha_{ij} = x_i x_j - x_j x_i$ and let α_{ij} and $x\alpha_{ij} - \alpha_{ij}x$ be nilpotent for all i and j and for all $x \in R$. Then* $\dim R < \infty$.

PROOF. Lemma 3.7. implies that $\alpha_{ij}R$ is nil. Now we apply theorem 3.4.

THEOREM 3.9. *Let $R = \Phi(x_1, \cdots. x_n)$. Assume that x_i and $xx_i - x_i x$ are nilpotent for all i and all x in R. Then* $\dim R < \infty$.

PROOF. Lemma 3.7. implies that $x_i R$ is nil for all i. Since $\bar{x}_i\bar{R} = 0$ in $\bar{R} = R/N$, where N is the nilpotent radical of R, we get $\bar{x}_i = 0$ and hence $R = N$.

THEOREM 3.10. *Let R satisfy the following two conditions*:

(1) *α_{ij} is a right zero divisor for all i and j*

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(2) *If I is the intersection of all non-zero left algebra ideals of the form Rx or $(a)^l$ for x in R and a in R then $I \neq 0$.*

Then $\dim R < \infty$.

PROOF. If a is a right zero divisor then $0 \neq (a)^l \supseteq I$. So $a \in (I)^r$. In particular N, the nilpotent radical of R, is contained in $(I)^r$: We show that $(I)^r \subseteq N$.

If $0 \neq t \in I$ and $x \in (I)^r$ then $tx = 0$. The chain $(x)^l \subseteq (x^2)^l \subseteq \cdots$ terminates. So there exists a positive integer m such that $(x^m)^l = (x^s)^l$ for all $s \geqq m$. If $Rx^m \neq 0$ then $t = yx^m$ for some y in R. Since $tx = 0$ we get $yx^{m+1} = 0$ and hence $y \in (x^{m+1})^l = (x^m)^l$. So $t = yx^m = 0$ which is not the case. So $Rx^m = 0$ and hence x is nilpotent. So $N = (I)^r$. This shows that $\bar{R} = R/N$ is a commutative ring. We use corollary 2.3. to get $R = N$. This completes the proof.

COROLLARY 3.11. *If α_{ij} is a right zero divisor for all i and j and if the intersection of all non-zero left algebra ideals is not zero then $\dim R < \infty$.*

LEMMA 3.12. *Let R be a ring satisfying the ascending chain condition for left annihilators. Then for any element $a \in R$ there exists a positive integer $k \geqq 1$ such that $(a)^l \cap Ra^k = 0$.*

PROOF. See [5] p. 297.

THEOREM 3.13. *Let $R = \Phi(x_1, \cdots, x_n)$ satisfy the followings:*

(1) *For some maximal algebra ideal T of R there exists an element $a \in R$ such that $T = Ra$.*

(2) *x_i is nilpotent for all i.*

Then $\dim R < \infty$.

PROOF. As before we show that R is nil. Suppose $z \in R$ is a non-nilpotent element of R. Let C be a prime ideal of R which is maximal with respect to $z^k \notin C$ for all $k \geqq 1$. Lemma 2.1 implies that $0 \neq R^2 \subsetneqq R$. If $R^2 \nsubseteq T$, then $0 \neq (R/T)^2 \subsetneqq (R/T)$, by lemma 2.1,. which is a contradiction to maximality of T. So $R^2 \subseteq T$ and hence $R^2 = T = Ra$. Since $\bar{R} = R/C$ is a prime ring and $\bar{R}^2 = \bar{R}\bar{a}$ we get $(\bar{a})^r = 0$. If $(\bar{a})^l \neq 0$ then $(\bar{a})^l \cap \bar{R}\bar{a}^k = 0$ for some $k \geqq 1$, because of the lemma 3.12. But $\bar{R}\bar{a}^k = \bar{R}^k \neq 0$ and hence $\bar{R}\bar{a}^k$ is essential left ideal of $\bar{R}$. This contradiction shows that $(\bar{a})^l = 0$ and hence $\bar{a}$ is a regular element of $\bar{R}$. Clearly for some i, $\bar{x}_i \notin \bar{R}^2$. So $\Phi\bar{x}_i + \bar{R}^2 = \bar{R}$ and $\Phi\bar{x}_i + \Phi\bar{x}_i\bar{a} + \bar{R}\bar{a}^2 = \bar{R}$. This implies that there exist $\alpha, \beta \in \Phi$ and $\bar{r} \in \bar{R}$ such that $\bar{a}^2 = \alpha\bar{x}_i + \beta\bar{x}_i\bar{a} + \bar{r}\bar{a}^2$. If $\alpha \neq 0$ then $\bar{x}_i \in \bar{R}^2$. So $\alpha = 0$ and $\bar{a} = \beta\bar{x}_i + \bar{r}\bar{a}$, by regularity of $\bar{a}$. If $\bar{x}_i$ is of index k then $\bar{y} = \bar{r}\bar{y}$ where $\bar{y} = \bar{a}\bar{x}_i^{k-1}$. So $\bar{y} = 0$, because $\bar{r}$ is a quasi-regular element. This contradiction completes the proof.

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