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AN ARBITRARY INTERSECTION OF L_p-SPACES

F. ABTAHI ✉, H. G. AMINI, H. A. LOTFI and A. REJALI

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Abstract

For a locally compact group G and an arbitrary subset J of $[1, \infty]$, we introduce $IL_J(G)$ as a subspace of $\bigcap_{p\in J} L^p(G)$ with some norm to make it a Banach space. Then, for some special choice of J, we investigate some topological and algebraic properties of $IL_J(G)$ as a Banach algebra under a convolution product.

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1. Introduction

The set of functions belonging to all L^p-spaces has been studied for many years. Some primary results are due to Arens [4]. He considered the set $\bigcap_{p\in J} L^p[0, 1]$, where $J = [1, \infty]$, and showed that with the smallest topology that contains each relative L^p-norm topology, it is not normable. There are other notable results by Davis *et al.* in [9, 10]. They discuss topological properties of the set $\bigcap_{p\in J} L^p[0, 1]$, where $J = [1, \infty)$. Some other important results on the subject can be found in Bell's work [5]. He pointed out that $\bigcap_{p\in J} L^p[0, 1]$ is not equal to $L^\infty[0, 1]$, for the case where $J = [1, \infty]$. He also used the theory of L^p-spaces for finitely additive set functions, developed by Leader [23], to prove several necessary and sufficient conditions for the normability of a generalisation of $\bigcap_{p\in J} L^p[0, 1]$. For more related results on the subject, see also [12].

Due to the limitation of the set $[0, 1]$, the intersection of L^p-spaces was studied in an extended form, such as $\bigcap_{p\in J} L^p(G)$, where G is a locally compact group and J is a set of two elements of $[1, \infty]$. For some choices of J, $\bigcap_{p\in J} L^p(G)$ was considered as an algebra under convolution, which was of particular importance. Especially for $J = \{1, p\}$, it is known that if $1 \le p < \infty$, then $L^1(G) \cap L^p(G)$ is a Segal algebra, and for $p = \infty$, it is only an abstract Segal algebra with respect to $L^1(G)$ [24]. It is also worthwhile to consider some of Ghahramani's joint work with Lau that introduced some other intersections of Banach spaces, such as $\mathcal{LA}(G) = L^1(G) \cap A(G)$ where $A(G)$ is the Fourier algebra of G; see [16, 17] for more general information.

Motivated by these results, we first consider the set $\bigcap_{p\in J} L^p(G)$, where J is an arbitrary subset of $[1, \infty]$, and notice that for all $a, b \in J$ with $a < b$,

$$\bigcap_{p\in J} L^p(G) \subseteq \bigcap_{p\in[a,b]} L^p(G),$$

as an application of this work. Since our aim is to study $\bigcap_{p\in J} L^p(G)$ as a Banach space, we thus introduce a special subset of $\bigcap_{p\in J} L^p(G)$, denoted by $IL_J(G)$, which is the set of all functions f in $\bigcap_{p\in J} L^p(G)$ such that

$$\|f\|_{IL_J} = \sup_{p\in J} \|f\|_p < \infty.$$

Then we prove that $IL_J(G) = L^{m_J}(G) \cap L^{M_J}(G)$, where

$$m_J = \inf\{p : p \in J\} \quad \text{and} \quad M_J = \sup\{p : p \in J\},$$

which is in fact one of the main results of this paper.

Moreover, we consider $IL_J(G)$ as a Banach algebra under convolution product, for the case where $1 \in J$. This leads us to study some algebraic properties of $IL_J(G)$. In the final section, we first show that the existence of a bounded approximate identity in $IL_J(G)$ is equivalent to discreteness of G. This helps us obtain some results on amenability, weak amenability and also approximate amenability of $IL_J(G)$. Furthermore, we study compact and weakly compact multipliers on $IL_J(G)$ and give some conditions which finally imply that $IL_J(G)$ can be considered as an ideal in its second dual if and only if G is compact. This is also an equivalent condition to Arens regularity of $IL_J(G)$, where G is unimodular.

2. Preliminaries and results

Let G be a locally compact group and λ be a fixed left Haar measure on G. Denote by $C_b(G)$ the space of all bounded continuous complex-valued functions on G with the supremum norm; $C_0(G)$ will be its closed subspace consisting of functions vanishing at infinity and $C_{00}(G)$ the dense subspace of $C_0(G)$ consisting of functions with compact support. For λ-measurable functions f and g on G, convolution multiplication is defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\, d\lambda(y),$$

at each point $x \in G$ for which this makes sense. For $1 \leq p \leq \infty$, the Lebesgue space $L^p(G)$ with respect to λ is as defined in [18]. We denote this space by $\ell^p(G)$, in the case where G is discrete.

For $J = \{p_1, p_2\} \subseteq [1, \infty]$, consider $\bigcap_{p\in J} L^p(G) = L^{p_1}(G) \cap L^{p_2}(G)$ and recall that $\bigcap_{p\in J} L^p(G)$ is a Banach space under the norm

$$\|f\|_J = \max\{\|f\|_{p_1}, \|f\|_{p_2}\} \quad (f \in L^{p_1}(G) \cap L^{p_2}(G)).$$

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Now let J be an arbitrary subset of $[1, \infty]$ and let

$$m_J = \inf\{p : p \in J\} \quad \text{and} \quad M_J = \sup\{p : p \in J\}.$$

In Sections 3 and 4 we shall investigate $\bigcap_{p\in J} L^p(G)$, the set of all complex-valued functions on G that belong to $L^p(G)$ for all $p \in J$, and in fact all available work will be extended to the case where J is not necessarily finite. Our interest in the properties of $\bigcap_{p\in J} L^p(G)$ stemmed from our study of the usual L^p-spaces, and also some results on finite intersections of L^p-spaces.

At the beginning of our discussion, we use a partial case of the Riesz convexity theorem [19, Theorem 13.19] to obtain some more interesting results in this field. A more precise form of the theorem is as follows.

Theorem 2.1. *Let G be a locally compact group, $p, q \in [0, \infty]$ and $f \in L^p(G) \cap L^q(G)$. Then for each $p < r < q$, $f \in L^r(G)$ and also the function T defined by $T(r) = \log(\|f\|_r^r)$ on $[p, q]$ is convex.*

In view of Theorem 2.1, it is understood that there is an intimate relation between L^p-spaces as presented in the following propositions.

Proposition 2.2. *Let G be a locally compact group and $1 \le p < q \le \infty$. Then*

$$L^p(G) \cap L^q(G) = \bigcap_{r\in[p,q]} L^r(G)$$

and, for each $f \in L^p(G) \cap L^q(G)$,

$$\|f\|_r \le \max\{\|f\|_p, \|f\|_q\} \quad (r \in [p, q]).$$

Proposition 2.3. *Let G be a locally compact group and J be a subset of $[1, \infty]$. Then the following assertions hold.*

(i) *If $m_J, M_J \in J$, then $\bigcap_{p\in[m_J,M_J]} L^p(G) = L^{m_J}(G) \cap L^{M_J}(G)$.*
(ii) *If $m_J \in J$ and $M_J \notin J$, then $\bigcap_{p\in J} L^p(G) = \bigcap_{p\in[m_J,M_J)} L^p(G)$.*
(iii) *If $M_J \in J$ and $m_J \notin J$, then $\bigcap_{p\in J} L^p(G) = \bigcap_{p\in(m_J,M_J]} L^p(G)$.*
(iv) *If $m_J, M_J \notin J$, then $\bigcap_{p\in J} L^p(G) = \bigcap_{p\in(m_J,M_J)} L^p(G)$.*

Note that, for all $a, b \in [1, \infty]$,

$$\bigcap_{p\in[a,b)} L^p(G) \subseteq \bigcap_{p\in[a,b]} L^p(G) \quad \text{and} \quad \bigcap_{p\in(a,b]} L^p(G) \subseteq \bigcap_{p\in[a,b]} L^p(G).$$

We finish this section with an example that shows that these inclusions can be proper.

Example 2.4. (1) Take $J = [1, \infty)$ and define

$$f(x) = \begin{cases} \log x & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

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It is plain to see that $f \in L^1(\mathbb{R})$ with $\|f\|_1 = 1$ but $f \notin L^\infty(\mathbb{R})$. Since for each $1 < p < \infty$,

$$(\log y)^p = \left(\frac{2p}{2p}\log y\right)^p = (2p\log(y^{1/2p}))^p \le (2p)^p (y^{1/2p})^p = (2p)^p y^{1/2},$$

we have

$$\int_0^1 |\log x|^p\,dx = \int_\infty^1 |-\log y|^p \frac{-dy}{y^2} = \int_1^\infty \frac{(\log y)^p}{y^2}\,dy \le \int_1^\infty (2p)^p \frac{y^{1/2}}{y^2}\,dy < \infty.$$

So $f \in \bigcap_{p\in J} L^p(\mathbb{R})$ and consequently

$$\bigcap_{p\in[1,\infty)} L^p(\mathbb{R}) \supsetneq \bigcap_{p\in[1,\infty]} L^p(\mathbb{R}).$$

(2) Let $1 \le a < b < \infty$ and take $J = (a, b]$. Define

$$f(x) = \begin{cases} x^{-1/a} & \text{if } 1 \le x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $f \in \bigcap_{p\in J} L^p(\mathbb{R})$, but $f \notin L^a(\mathbb{R})$, and hence

$$\bigcap_{p\in(a,b]} L^p(\mathbb{R}) \supsetneq \bigcap_{p\in[a,b]} L^p(\mathbb{R}).$$

3. The Banach space $IL_J(G)$

Let G be a locally compact group and $J \subseteq [1, \infty]$. We introduce $IL_J(G)$ as a subset of $\bigcap_{p\in J} L^p(G)$ given by

$$IL_J(G) = \left\{ f \in \bigcap_{p\in J} L^p(G) : \sup_{p\in J} \|f\|_p < \infty \right\}.$$

This space will be denoted by $IL_\infty(G)$ when $M_J = \infty$. The main aim of the present section is to describe the properties of $IL_J(G)$ as a Banach space for different subsets J of $[1, \infty]$, and in fact the section is devoted entirely to this important subspace of $\bigcap_{p\in J} L^p(G)$. We first shall discuss Proposition 2.3 again in a different way and get an important result.

Lemma 3.1. *Let G be a locally compact group and J be a subset of* $[1, \infty]$. *Then $IL_J(G) \subseteq L^{M_J}(G)$ and $IL_J(G) \subseteq L^{m_J}(G)$.*

Proof. First, let $M_J < \infty$ and $f \in IL_J(G)$. For integers $n \in \mathbb{N}$ large enough, define functions g_n by

$$g_n(x) = |f(x)|^{M_J - 1/n} \quad (x \in G).$$

Then

$$\int_G |g_n(x)|\,dx = \int_G |f(x)|^{M_J-1/n}\,dx = (\|f\|_{M_J-1/n})^{M_J-1/n} \le \left(\sup_{p\in J}\|f\|_p\right)^{M_J-1/n}.$$

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Since, for each $x \in G$, $\lim_{n\to\infty} g_n(x) = |f(x)|^{M_J}$, Fatou's lemma implies that

$$\begin{aligned}\int_G |f(x)|^{M_J}\,dx &= \int_G \liminf_{n\to\infty} |g_n(x)|\,dx \\ &\le \liminf_{n\to\infty} \int_G |g_n(x)|\,dx \\ &\le \liminf_{n\to\infty} \Big(\sup_{p\in J} \|f\|_p\Big)^{M_J-1/n} \\ &= \Big(\sup_{p\in J} \|f\|_p\Big)^{M_J}.\end{aligned}$$

It follows that $f \in L^{M_J}(G)$ and $\|f\|_{M_J} \le \sup_{p\in J} \|f\|_p$.

Now assume that $M_J = \infty$ and $f \in IL_\infty(G)$ but $f \notin L^\infty(G)$. Then $\|f\|_\infty = \infty$ and so, for each $n \in \mathbb{N}$,

$$\lambda(\{x \in G : |f(x)| \ge n\}) = \alpha_n > 0.$$

It follows that, for each $p \ge 1$, $\|f\|_p \ge n\alpha_n^{1/p}$ and so $\|f\|_{IL_\infty} \ge n$, where $n \ge 1$. Consequently, $\|f\|_{IL_\infty} = \infty$, which is a contradiction. So $f \in L^\infty(G)$ and consequently $\|f\|_\infty \le \sup_{p\in J} \|f\|_p$. For another inclusion, let

$$g_n(x) = |f(x)|^{m_J+1/n} \quad (x \in G),$$

for integers $n \in \mathbb{N}$ large enough. Using a similar argument, one can show that $IL_J(G) \subseteq L^{m_J}(G)$. □

We are now able to give an important difference between the spaces $\bigcap_{p\in J} L^p(G)$ and $IL_J(G)$, comparing Lemma 3.1 and Example 2.4. This is given in the following theorem and is in fact the main result of the present section.

THEOREM 3.2. *Let G be a locally compact group and J be a subset of $[1, \infty]$. Then*

$$\begin{aligned}IL_J(G) &= IL_{(m_J,M_J)}(G) = IL_{[m_J,M_J)}(G) \\ &= IL_{(m_J,M_J]}(G) = IL_{[m_J,M_J]}(G)\end{aligned}$$

and all are equal to $L^{m_J}(G) \cap L^{M_J}(G)$. Furthermore, $IL_J(G)$ is a Banach space under the norm

$$\|f\|_{IL_J} = \sup_{p\in J} \|f\|_p = \max\{\|f\|_{m_J}, \|f\|_{M_J}\} \quad (f \in IL_J(G)).$$

PROOF. The first set of equations follows immediately from Proposition 2.3 and Lemma 3.1. For the last equation, Proposition 2.2 implies that

$$\|f\|_p \le \max\{\|f\|_{m_J}, \|f\|_{M_J}\} \quad (p \in J)$$

and so

$$\|f\|_{IL_J} = \sup_{p\in J} \|f\|_p \le \max\{\|f\|_{m_J}, \|f\|_{M_J}\}.$$

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It is also obvious that

$$\max\{\|f\|_{m_J}, \|f\|_{M_J}\} \leq \sup_{p\in J} \|f\|_p = \|f\|_{IL_J}.$$

Thus the proof of the claim is complete. □

4. Some aspects of $IL_J(G)$ as a Banach algebra

Saeki [25] proved that for a locally compact group G and $1 < p < \infty$, $L^p(G)$ is a Banach algebra under convolution if and only if G is compact. We have also considered only this property, that $f * g$ exists for all $f, g \in L^p(G)$, and proved that, for $2 < p < \infty$, the compactness of G is equivalent to the existence of $f * g$ for all $f, g \in L^p(G)$; see [1]. Furthermore, see [2, 3] for the more general case of weighted L^p-spaces. It is also interesting to find out when $IL_J(G)$ is closed under convolution, where $J \subseteq [1, \infty]$. In particular, let $2 < m_J < \infty$ and $IL_J(G)$ be a Banach algebra under convolution. We first note that with a proof similar to that in [25], we can easily prove that G is unimodular. If G is not compact, with an argument similar to [1] for a fixed compact symmetric neighbourhood B of the identity element of G and a real number α with $\alpha \leq 1/2$, $\alpha M_J > 1$ and $\alpha m_J > 1$, we may find a sequence (a_n) in G such that

$$a_n \in G \Big\backslash \bigcup_{m=1}^{n-1} a_m B^4 \quad (n \geq 2).$$

Hence the functions

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \chi_{Ba_n^{-1}}(x)$$

and

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \chi_{a_n B^2}(x)$$

are well defined, where for each $A \subseteq G$, χ_A denotes the characteristic function of A on G. It follows from an argument entirely analogous to [1, Theorem 1.1] that $f, g \in IL_J(G) = L^{m_J}(G) \cap L^{M_J}(G)$, whereas $f * g \notin IL_J(G)$. These observations confirm the following proposition.

Proposition 4.1. *Let G be a locally compact group and $J \subseteq [1, \infty]$, with $2 < m_J \leq \infty$. Then $IL_J(G)$ is a Banach algebra under convolution if and only if G is compact.*

We now present a sufficient condition for $IL_J(G)$ to be a Banach algebra under convolution.

Proposition 4.2. *Let G be a locally compact group and $J \subseteq [1, \infty]$ such that $m_J = 1$. Then $IL_J(G)$ is a Banach algebra under convolution.*

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Proof. Let $f, g \in IL_J(G)$. Then

$$\begin{aligned}\|f * g\|_{IL_J} &= \max\{\|f * g\|_1, \|f * g\|_{M_J}\}\\ &\le \max\{\|f\|_1\|g\|_1, \|f\|_1\|g\|_{M_J}\}\\ &= \|f\|_1\|g\|_{IL_J}\\ &\le \|f\|_{IL_J}\|g\|_{IL_J}\end{aligned}$$

and the result is proved. □

Proposition 4.2 leads us to study some algebraic properties of $IL_J(G)$, for the case where $m_J = 1$. The particular objects of study in this section are amenability, compact and also weakly compact multipliers of $IL_J(G)$; regularity of $IL_J(G)$ is investigated in the sequel. Before that, we show that $IL_J(G)$ can be studied as a Segal algebra and also an abstract Segal algebra with respect to $L^1(G)$, which is interesting in its own right. We first turn our attention to the case where $IL_J(G)$ admits a left approximate identity.

Proposition 4.3. *Let G be a locally compact group and $J \subseteq [1, \infty]$ such that $m_J = 1$. Then the following assertions hold.*

(1) *If $M_J < \infty$ then $IL_J(G)$ admits a left approximate identity that is bounded in $IL_J(G)$ only when G is discrete.*
(2) *$IL_\infty(G)$ admits a left approximate identity if and only if G is discrete.*

Proof. (1) If $M_J < \infty$ then $IL_J(G) = L^1(G) \cap L^{M_J}(G)$, by Theorem 3.2 and so it has an approximate identity $(e_\lambda)_{\lambda\in\Lambda}$ [24]. If $(e_\lambda)_{\lambda\in\Lambda}$ is bounded in $IL_J(G)$, then $IL_J(G) = L^1(G)$ [7] and hence $L^1(G) \subseteq L^{M_J}(G)$. It follows that G is discrete.

(2) Assume that $(e_\lambda)_{\lambda\in\Lambda}$ is a left approximate identity of $IL_\infty(G)$. Then for every $f \in IL_\infty(G)$,

$$\|e_\lambda * f - f\|_\infty \le \|e_\lambda * f - f\|_{IL_\infty} \to 0.$$

Since for each $\lambda \in \Lambda$, $e_\lambda * f \in \mathrm{LUC}(G)$, the space of bounded left uniformly continuous functions on G, it follows that $f \in \mathrm{LUC}(G)$ and consequently $IL_\infty(G) \subseteq \mathrm{LUC}(G)$. This implies that G is discrete; indeed, for each measurable set $A \subseteq G$, $\chi_A \in IL_\infty(G)$ if and only if $\lambda(A) < \infty$. So for such a subset A of G, $\chi_A \in \mathrm{LUC}(G)$. Thus

$$A = {\chi_A}^{-1}(0, 2),$$

is open in G. Thus we conclude that G is discrete. □

The following corollary is clearly obtained from Proposition 4.3.

Corollary 4.4. *Let G be a locally compact group and $J \subseteq [1, \infty]$ such that $m_J = 1$. Then the following statements are equivalent.*

(1) *$IL_J(G)$ has a bounded left approximate identity.*
(2) *$IL_J(G) = L^1(G)$.*
(3) *$IL_J(G)$ has an identity.*
(4) *G is a discrete group.*

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4.1. $IL_J(G)$ as a Segal algebra. For the sake of completeness, we first repeat and review the basic definitions of Segal algebras and abstract Segal algebras; see [7, 24] for more details.

Let $(\mathcal{A}, \|.\|_{\mathcal{A}})$ be a Banach algebra. Then $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is an abstract Segal algebra with respect to $(\mathcal{A}, \|.\|_{\mathcal{A}})$ if each of the following is true.

(1) $\mathcal{B}$ is a dense left ideal in $\mathcal{A}$ and $\mathcal{B}$ is a Banach algebra with respect to $\|.\|_{\mathcal{B}}$.
(2) There exists $M > 0$ such that $\|f\|_{\mathcal{A}} \leq M\|f\|_{\mathcal{B}}$, for each $f \in \mathcal{B}$.
(3) There exists $C > 0$ such that $\|fg\|_{\mathcal{B}} \leq C\|f\|_{\mathcal{A}}\|g\|_{\mathcal{B}}$, for each $f, g \in \mathcal{B}$.

For a locally compact group G, a linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra if it satisfies the following conditions.

(1) $S(G)$ is dense in $L^1(G)$.
(2) $S(G)$ is a Banach space under some norm $\|.\|_s$ and $\|f\|_1 \leq \|f\|_s$ for all $f \in S(G)$.
(3) $S(G)$ is left translation invariant and the map $x \mapsto {}_xf$ of G into $S(G)$ is continuous.
(4) $\|{}_xf\|_s = \|f\|_s$ for all $f \in S(G)$ and $x \in G$.

Note that every Segal algebra is an abstract Segal algebra with respect to $L^1(G)$ [21, p. 19, Proposition 1].

Proposition 4.5. *Let G be a compact group and $J \subseteq [1, \infty]$ such that $m_J = 1$. Then the following assertions hold.*

(1) *$IL_J(G)$ is always an abstract Segal algebra with respect to $L^1(G)$.*
(2) *If $M_J < \infty$, then $IL_J(G)$ is a Segal algebra.*
(3) *$IL_\infty(G)$ is a Segal algebra if and only if G is discrete.*

Proof. (1) Using Proposition 4.2, one can easily show that $IL_J(G)$ is always an abstract Segal algebra with respect to $L^1(G)$.

(2) If $M_J < \infty$, then $IL_J(G) = L^1(G) \cap L^{M_J}(G)$ is a Segal algebra [24].
(3) Since every Segal algebra admits a left approximate identity [24, p. 19], Proposition 4.3 implies that G is discrete and so $IL_\infty(G) = \ell^1(G)$, which is clearly a Segal algebra. □

4.2. Amenability of $IL_J(G)$. Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. A derivation D from $\mathcal{A}$ into X is inner if there is $\xi \in X$ such that

$$D(a) = a\xi - \xi a \quad (a \in \mathcal{A}),$$

and is approximately inner if there is a net $(\xi_i) \in X$ such that

$$D(a) = \lim_{i\to\infty} a\xi_i - \xi_i a \quad (a \in \mathcal{A}).$$

The Banach algebra $\mathcal{A}$ is (approximately) amenable if every continuous derivation $D : A \to X^*$ is (approximately) inner for all Banach $\mathcal{A}$-bimodules X; and $\mathcal{A}$ is (approximately) weakly amenable if the above holds for $X = \mathcal{A}$.

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THEOREM 4.6. *Let G be a locally compact group and $J \subseteq [1, \infty]$ such that $m_J = 1$. Consider the following conditions.*

(1) *$IL_J(G)$ is amenable.*
(2) *$IL_J(G)$ is weakly amenable.*
(3) *$IL_J(G)$ is approximately amenable.*
(4) *G is discrete and amenable.*

Then (1) ⇔ (4), (1) ⇒ (2), (1) ⇒ (3), (4) ⇒ (2), (4) ⇒ (3). *Furthermore, the following assertions hold.*

(i) *If $M_J = \infty$, then* (1) ⇔ (3) ⇔ (4).
(ii) *If $M_J < \infty$ and G is abelian, then* (2) *holds.*

PROOF. (1) ⇔ (4). If $IL_J(G)$ is amenable, then it has a bounded approximate identity [20, Proposition 1.6] and so G is discrete by Corollary 4.4. Thus $IL_J(G)$ is identical with $\ell^1(G)$ and consequently $\ell^1(G)$ is amenable also. It follows that the discrete group G is amenable [20, Theorem 2.5]. Conversely, if G is discrete and amenable, then $\ell^1(G)$ is amenable [20, Theorem 2.5], and since $\ell^1(G) = IL_J(G)$, it follows that $IL_J(G)$ is amenable.

(1) ⇒ (2), (3) is clear.

(4) ⇒ (2), (3). If G is discrete and amenable, then, by the first paragraph, $IL_J(G)$ and so $\ell^1(G)$ is amenable. Consequently, $\ell^1(G)$ is approximately amenable and thus $IL_J(G)$ is approximately amenable also. Since $\ell^1(G)$ is always weakly amenable [11] it follows that $IL_J(G)$ is weakly amenable.

(i) Let $M_J = \infty$. It suffices to show that (3) is equivalent to (4). If $IL_J(G)$ is approximately amenable, then it has an approximate identity [20, Proposition 1.6] and hence G is discrete by Proposition 4.3. Since $IL_J(G) = \ell^1(G)$, $\ell^1(G)$ is approximately amenable also. Thus G is amenable. Conversely, if G is discrete and amenable, then $\ell^1(G)$ is amenable [20, Theorem 2.5] and so it is approximately amenable. Since $\ell^1(G) = IL_J(G)$, it follows that $IL_J(G)$ is approximately amenable. Consequently (3) is equivalent to (4).

(ii) Let $M_J < \infty$ and G be abelian. Since $L^1(G)$ is always weakly amenable [11, 21], and $IL_J(G) = L^1(G) \cap L^{M_J}(G)$, [16, Corollary 3.4] implies that $IL_J(G)$ is also weakly amenable. □

4.3. Compact and weakly compact multipliers of $IL_J(G)$. Suppose that $\mathcal{A}$ is a Banach algebra. The second dual space $\mathcal{A}^{**}$ of $\mathcal{A}$ can be equipped with two multiplications which make $\mathcal{A}^{**}$ a Banach algebra. Each is a natural extension of the original multiplication in $\mathcal{A}$ [6]. We say that $\mathcal{A}$ is Arens regular if these two multiplications coincide, which is equivalent to $\mathcal{A}^* = Wap(\mathcal{A})$, the set of all weakly almost periodic functionals on $\mathcal{A}$ [13, Theorem 1]. It is well known that Arens regularity of $L^1(G) \cap L^p(G)$ is equivalent to compactness of G, where G is a unimodular locally compact group and $1 < p \leq \infty$; see [17]. In fact we have the following theorem.

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Theorem 4.7. *Let G be a unimodular locally compact group and $J \subseteq [1, \infty]$ such that $m_J = 1$. Then $IL_J(G)$ is Arens regular if and only if G is compact.*

It has been shown that for a unimodular locally compact group G, the Lebesgue Fourier algebra $\mathcal{L}A(G)$ is Arens regular if and only if G is compact [17, Theorem 5.1]. This result has been extended to an arbitrary locally compact group G [14, Corollary 3.6]. These observations lead us to the following natural questions, the second of which was pointed out to us by the referee.

Question 4.8. Let G be a locally compact group and $J \subseteq [1, \infty]$ with $m_J = 1$ such that $IL_J(G)$ is Arens regular. Is G compact?

Question 4.9. Let G be a locally compact group and $J \subseteq [1, \infty]$ with $m_J = 1$. What is the (left or right) topological centre of $IL_J(G)^{**}$? When is the Banach algebra strongly Arens irregular? (See [8, 22].)

An operator $T : \mathcal{A} \to \mathcal{A}$ is a left multiplier if $T(ab) = T(a)b$, for all $a, b \in A$. Right multipliers are defined similarly. It is plain that for each $a \in \mathcal{A}$, the operator ${}_aT : \mathcal{A} \to \mathcal{A}$ ($T_a : \mathcal{A} \to \mathcal{A}$) defined by $b \mapsto ab$ ($b \mapsto ba$) is a left (right) multiplier on $\mathcal{A}$. Recall that $\mathcal{A}$ is a left (right) ideal in $\mathcal{A}^{**}$ if and only if, for every $a \in \mathcal{A}$, the operator $b \mapsto ba$ ($b \mapsto ab$) from $\mathcal{A}$ to $\mathcal{A}$ is weakly compact [13]. As the final result, we will give a necessary and sufficient condition for $IL_J(G)$ to be an ideal in its second dual. It requires some preparation.

Let G be a locally compact group, $J \subseteq [1, \infty]$ with $m_J = 1$ and $\mu \in M(G)$, the Banach space of all complex regular Borel measures on G. Then ${}_\mu T : IL_J(G) \to IL_J(G)$, defined by

$$ {}_\mu T(f) = \mu * f \quad (f \in IL_J(G)), $$

is a bounded left multiplier on $IL_J(G)$; indeed, for each $f \in IL_J(G)$,

$$ \|{}_\mu T(f)\|_{IL_J} = \|\mu * f\|_{IL_J} = \max\{\|\mu * f\|_1, \|\mu * f\|_{M_J}\} \le \|\mu\| \, \|f\|_{IL_J} $$

and so $\|{}_\mu T\| \le \|\mu\|$.

Proposition 4.10. *Let G be a locally compact group, $J \subseteq [1, \infty]$ with $m_J = 1$ and μ be a nonzero element of $M(G)$. Consider the following assertions.*

(1) ${}_\mu T : IL_J(G) \to IL_J(G)$ *is compact and G is unimodular.*
(2) ${}_\mu T : IL_J(G) \to IL_J(G)$ *is weakly compact and G is unimodular.*
(3) *G is compact.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Let (f_n) be a sequence in $L^1(G)$ bounded by K and $g \in IL_J(G)$. Then

$$ \|g * f_n\|_{IL_J} \le \|g\|_{IL_J} \|f_n\|_1 \le K \|g\|_{IL_J}. $$

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Hence $\{g * f_n\}$ is bounded in $IL_J(G)$. With the weak compactness of ${}_\mu T$, there exist a subsequence $(f_{n_j}) \subseteq (f_n)$ and $h \in IL_J(G)$ such that

$$ {}_\mu T(g * f_{n_j}) = \mu * g * f_{n_j} \to h, $$

in the weak topology of $IL_J(G)$. Since the restriction of every element of $L^\infty(G)$ to $IL_\infty(G)$ belongs to the dual of $IL_\infty(G)$, it follows that

$$ {}_\mu T(g * f_{n_j}) = \mu * g * f_{n_j} \to h, $$

in the weak topology of $L^1(G)$. Hence ${}_{\mu * g}T : L^1(G) \to L^1(G)$ is a weakly compact multiplier and so G is compact [15, Theorem 3.1]. □

Note that $IL_\infty(G)$ is identical to $L^\infty(G)$ if and only if G is compact. So the following results are satisfied automatically by Proposition 4.10.

Corollary 4.11. *Let G be a locally compact group, $J \subseteq [1, \infty]$ such that $m_J = 1$ and $M_J = \infty$ and μ be a nonzero element of $M(G)$. Consider the following assertions.*

(1) *${}_\mu T : IL_\infty(G) \to IL_\infty(G)$ is compact and G is unimodular.*
(2) *${}_\mu T : IL_\infty(G) \to IL_\infty(G)$ is weakly compact and G is unimodular.*
(3) *${}_\mu T : L^\infty(G) \to L^\infty(G)$ is compact and G is compact.*
(4) *${}_\mu T : L^\infty(G) \to L^\infty(G)$ is weakly compact and G is compact.*

Then (1) $\Leftrightarrow$ (3) *and* (2) $\Leftrightarrow$ (4).

Corollary 4.12. *Let G be a compact group and $f \in L^\infty(G)$. Then the following are equivalent.*

(1) *${}_f T : IL_\infty(G) \to IL_\infty(G)$ is compact.*
(2) *${}_f T : IL_\infty(G) \to IL_\infty(G)$ is weakly compact.*

Theorem 4.13. *Let G be a unimodular locally compact group and $J \subseteq [1, \infty]$ such that $m_J = 1$. Then $IL_J(G)$ is a left ideal in its second dual if and only if G is compact.*

Proof. Let $IL_J(G)$ be an ideal in its second dual space. So the multiplier ${}_f T : IL_J(G) \to IL_J(G)$ is weakly compact, for all $f \in IL_J(G)$. Since $IL_J(G) \subseteq M(G)$, Proposition 4.10 implies that G is compact. Conversely, suppose that G is compact. Then

$$ IL_J(G) = L^1(G) \cap L^{M_J}(G) = L^{M_J}(G). $$

If $M_J < \infty$, then $IL_J(G) = L^{M_J}(G)$ is obviously an ideal in its second dual because of reflexivity of $L^{M_J}(G)$. Now let $M_J = \infty$. Then $IL_J(G) = L^\infty(G)$ and since G is compact, it follows that for each function $f \in L^1(G)$, the multiplier ${}_f T$ from $L^1(G)$ to $L^1(G)$ is compact [15]. This also holds for each $f \in L^\infty(G)$, because $L^\infty(G) \subseteq L^1(G)$. By some elementary results in operator theory, the multiplier ${}_f T$ from $L^\infty(G)$ to $L^\infty(G)$ is compact and so it is weakly compact and Corollary 4.12 implies that ${}_f T$ from $IL_J(G)$ to $IL_J(G)$ is weakly compact. This concludes the proof. □

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F. ABTAHI, Department of Mathematics, University of Isfahan, Isfahan, Iran
e-mail: f.abtahi@sci.ui.ac.ir

H. G. AMINI, Department of Mathematics, University of Isfahan, Isfahan, Iran
e-mail: ghaeidamini@mau.ac.ir

H. A. LOTFI, Department of Mathematics, University of Isfahan, Isfahan, Iran
e-mail: hali-lotfi@yahoo.com

A. REJALI, Department of Mathematics, University of Isfahan, Isfahan, Iran
e-mail: rejali@sci.ui.ac.ir

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