A Proof of the Binomial Theorem when the Exponent is a Positive Integer.

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$$\frac{(1+x)^n - 1}{(1+x) - 1} = (1+x)^{n-1} + (1+x)^{n-2} + (1+x)^{n-3} + \dots + (1+x) + 1;$$

$$\therefore\ (1+x)^n - 1 = x\{(1+x)^{n-1} + (1+x)^{n-2} + \dots + (1+x) + 1\} \qquad \text{(i)}$$

Thus $(1+x)^n - 1$ is divisible by x.

Hence $(1+x)^n = 1 +$ terms in x and its powers.

Then (i) may be written:

$$(1+x)^n = 1 + x\{n + \text{ terms in } x \text{ and its powers}\}$$
$$= 1 + nx + \text{ higher powers of } x. \qquad \text{(ii)}$$

Now $\qquad (a+b)^n = a^n + (n,\ 1)a^{n-1}b + (n,\ 2)a^{n-2}b^2 + \dots + b^n,$

where $(n,\ 1)$, $(n,\ 2)\dots$ are numerical coefficients independent of a and b, and by (ii) we see that $(n,\ 1) = n$.

Again $(1+x+y)^n = (1+x)^n + n(1+x)^{n-1}y + (n,\ 2)(1+x)^{n-2}y^2 + \dots$ (iii)

Also $(1+x+y)^n = x^n + nx^{n-1}(1+y) + \dots + (n,\ r)x^{n-r}(1+y)^r + \dots$ (iv)

The coefficient of $x^{n-r}y$ in (iii) is $n\,.\,(n-1,\ r-1)$;

the coefficient of $x^{n-r}y$ in (iv) is $(n,\ r)\,.\,r$.

$$\text{Hence}\quad (n,\ r) = \frac{n}{r}(n-1,\ r-1)$$
$$= \frac{n}{r}\cdot\frac{n-1}{r-1}(n-2,\ r-2)$$
$$= \frac{n}{r}\cdot\frac{n-1}{r-1}\cdots\cdots\frac{(n-r+1)}{1}.$$

$$\text{Thus}\quad (a+b)^n = a^n + na^{n-1}b + \frac{n.\overline{n-1}}{1\,.\,2}a^{n-2}b^2 + \dots$$
$$+ \frac{n.\overline{n-1}\dots\overline{n-r+1}}{1\,.\,2\ \dots\ r}a^{n-r}b^r + \dots + b^n.$$

https://doi.org/10.1017/S0013091500029084 Published online by Cambridge University Press