



Reconstruction of topological graphs and their Hilbert bimodules

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We show that the Hilbert bimodule associated with a compact topological graph can be recovered from the C^* -algebraic triple consisting of the Toeplitz algebra of the graph, its gauge action and the commutative subalgebra of functions on the vertex space of the graph. We discuss connections with work of Davidson–Katsoulis and of Davidson–Roydor on local conjugacy of topological graphs and isomorphism of their tensor algebras. In particular, we give a direct proof that a compact topological graph can be recovered up to local conjugacy from its Hilbert bimodule, and present an example of nonisomorphic locally conjugate compact topological graphs with isomorphic Hilbert bimodules. We also give an elementary proof that for compact topological graphs with totally disconnected vertex space the notions of local conjugacy, Hilbert bimodule isomorphism, isomorphism of C^* -algebraic triples, and isomorphism all coincide.

Keywords: Topological graph; Hilbert bimodule; C^* -correspondence; graph C^* -algebra; vector bundle; KMS state

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1. Introduction

Given a class of mathematical objects, if for each object X there is an associated structure $\text{Struc}(X)$, what kind of relationship does an isomorphism $\text{Struc}(X) \cong \text{Struc}(Y)$ impose between X and Y ? The Gelfand–Naimark theorem gives us a classical example of this question: for X and Y locally compact Hausdorff spaces, an isomorphism $C_0(X) \cong C_0(Y)$ induces an isomorphism $X \cong Y$. Another similar result of Gelfand and Kolmogorov [14] deals with algebra homomorphisms and real-valued functions. A compilation of results of this kind can be found in [13].

This question has recently been of significant interest in the context of graph algebras due to the work of Eilers and his collaborators (see e.g. [9]). In particular, in [4, 5, 9], it was shown that a directed graph E can be completely recovered from its Toeplitz algebra, its canonical gauge action, and its abelian coefficient subalgebra. In [4] this was achieved for finite graphs using KMS theory; in [5] this was extended to arbitrary discrete graphs using ground states, while in [9] they used nonselfadjoint operator-algebra theory.

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The main purpose of this paper is to generalize the results of [4] to totally disconnected compact topological graphs, but our approach yields interesting results, and questions, for more general compact topological graphs along the way. Drawing inspiration from [4], given a compact topological graph E , we use the KMS structure on its Toeplitz algebra, together with its coefficient subalgebra, to recover the corresponding graph bimodule. At first sight it may seem that our result can be recovered from [9, Proposition 4.6 or Corollary 4.7], but their results require that the coefficient algebra A be a subalgebra of compact operators in a Hilbert space and hence must be of the form $\bigoplus_{i \in I} \mathcal{K}(H_i)$ for Hilbert spaces $(H_i)_{i \in I}$ [1, Theorem 1.4.5]. If A is also commutative, then each H_i must be one-dimensional and if it is unital as well, then $|I| < \infty$. Hence the intersection between our hypothesis and those of [9] yields the class of finite discrete graphs as in [4].

In the final two sections of the paper, we consider the extent to which a topological graph E can be recovered from its bimodule, and hence from the triple consisting of its Toeplitz C^* -algebra, gauge action and coefficient algebra. After our paper appeared on the arXiv, we discovered (we thank both Adam Dor-On and the anonymous referee for drawing our attention to the fact) that these questions were considered earlier by Davidson–Katsoulis [6] for local homeomorphisms, and by Davidson–Roydor [7] for topological graphs. We present alternative proofs and examples for some of these results, but we emphasize that the definition of local conjugacy and our results relating isomorphism of Hilbert modules of topological graphs to local conjugacy of the graphs themselves both go back to the work of Davidson–Roydor, and before that to the work of Davidson–Katsoulis. Indeed, there is a substantial body of work on the relationship between isomorphism of graphs and of associated tensor algebras [21, 25–27, 32], and on the relationship between isomorphism, or local conjugacy, of multivariable dynamical systems and isomorphism of the associated Hilbert bimodules [17, 18, 20, 22].

In the preliminaries we recall some details about our main tool for studying the Toeplitz algebra of a compact topological graph – the KMS-states for its gauge action. We also introduce Hilbert modules and the relationship between Hilbert bimodules over commutative C^* -algebras and Hilbert bundles.

In §3, we prove our first main theorem, theorem 3.2: let E and F be compact topological graphs, γ^E and γ^F the gauge actions on the corresponding Toeplitz algebras $\mathcal{TC}^*(E)$ and $\mathcal{TC}^*(F)$, and M_E and M_F the coefficient algebras. We say that the triples $(\mathcal{TC}^*(E), \gamma^E, M_E)$ and $(\mathcal{TC}^*(F), \gamma^F, M_F)$ are isomorphic if there is a $*$ -isomorphism $\theta : \mathcal{TC}^*(E) \rightarrow \mathcal{TC}^*(F)$ that intertwines γ^E and γ^F and $\theta(M_E) = M_F$. Our theorem says that $(\mathcal{TC}^*(E), \gamma^E, M_E)$ and $(\mathcal{TC}^*(F), \gamma^F, M_F)$ are isomorphic if and only if the underlying graph bimodules $X(E)$ and $X(F)$ are isomorphic. We do this by proving that we can explicitly reconstruct $X(E)$ from $(\mathcal{TC}^*(E), \gamma^E, M_E)$, and make explicit the sense in which an isomorphism of Toeplitz triples induces an isomorphism of bimodules.

In §4, we show that we can do more than just reconstruct $X(E)$. We recall the notion of local conjugacy of topological graphs [7, Definition 4.3]. We deduce that, for compact topological graphs, isomorphism of graph bimodules implies local conjugacy of graphs (this can be deduced from [7, Theorem 4.5], but we give a direct proof). Next, we prove corollary 4.7: for compact topological graphs with

totally disconnected vertex spaces, $(\mathcal{TC}^*(E), \gamma^E, M_E)$ and $(\mathcal{TC}^*(F), \gamma^F, M_F)$ are isomorphic if and only if E and F are isomorphic topological graphs (again, this follows from [7, Theorem 5.5], but we give an elementary proof for zero-dimensional graphs).

In §5, we give an example that shows that in general, we cannot recover a compact topological graph from $(\mathcal{TC}^*(E), \gamma^E, M_E)$: we exhibit nonisomorphic topological graphs with isomorphic graph correspondences. Examples of nonisomorphic but locally conjugate topological graphs whose vertex spaces have converging dimension 1 also appear in [6, Example 3.18], and it follows from [7, Theorem 5.5] that their Hilbert modules are also isomorphic. However, our example is explicit and we describe a concrete isomorphism of the resulting Hilbert modules.

We finish off in §6 by giving a characterization, in terms of cohomological data, of the vector-bundle structure associated with the right-Hilbert modules of compact topological graphs. Specifically, we demonstrate that they are precisely the vector bundles admitting local trivialisations whose transition functions take values in the permutation matrices. This is closely related to the characterization of Kaliszewski *et al.* [19] in terms of continuous choices of orthonormal basis. We indicate in a closing remark how this relates to the question of which pairs of locally conjugate topological graphs have isomorphic Hilbert modules.

2. Preliminaries

2.1. KMS-states

We first recall the Kubo–Martin–Schwinger states (KMS-states for short) of a C^* -dynamical system. See [2, 3] for details. By a C^* -dynamical system, we mean a strongly continuous action τ of \mathbb{R} on a C^* -algebra A by automorphisms; we call τ a *dynamics* on A .

Let τ be a dynamics on a C^* -algebra A . An element $a \in A$ is *analytic* for τ if the function $t \mapsto \tau_t(a)$ extends to an analytic function $z \mapsto \tau_z(a)$ from \mathbb{C} to A . If it exists, this extension is unique.

Let (A, τ) be C^* -dynamical system, φ a state on A and $\beta \in \mathbb{R}$. We say φ is a τ -KMS $_\beta$ -state if

$$\varphi(a\tau_{i\beta}(b)) = \varphi(ba)$$

for all analytic elements $a, b \in A$. When τ is implicit, we just say that φ is a KMS $_\beta$ -state. The set of KMS $_\beta$ -states is convex and weak*-compact [3, Theorem 5.3.30]. An extremal point of this set is called an *extremal* KMS $_\beta$ -state.

By a KMS $_\infty$ state, we will mean a weak*-limit of a sequence $(\phi_n)_{n=1}^\infty$ of states such that each ϕ_n is a KMS $_{\beta_n}$ -state for a sequence $(\beta_n)_{n=1}^\infty$ satisfying $\beta_n \rightarrow \infty$.

2.2. Hilbert modules

There are plenty of references that explore the properties of Hilbert modules, for example [24]. We rely mainly on studies [31] for a slightly more recent approach. Let A be a C^* -algebra. A *right Hilbert A -module* is a right A -module V equipped with a map $\langle \cdot, \cdot \rangle_A : V \times V \rightarrow A$, which is linear in the second variable and such that for $x, y \in V, a \in A$,

- $\langle x, x \rangle_A \geq 0$, with equality only when $x = 0$;
- $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$;
- $\langle x, y \rangle_A = \langle y, x \rangle_A^*$; and
- X is complete in the norm defined by $\|x\|_A^2 = \|\langle x, x \rangle_A\|$.

A map $T : V \rightarrow V$ is an *adjointable operator* if there exists $T^* : V \rightarrow V$ called the *adjoint* such that $\langle T^*y, x \rangle_A = \langle y, Tx \rangle_A$, for all $x, y \in V$. The adjoint T^* is unique, and T is automatically a bounded, linear A -module homomorphism. The set $\mathcal{L}(V)$ of adjointable operators on V is a C^* -algebra.

A C^* -correspondence over A (or right-Hilbert A -bimodule) is a right Hilbert A -module X together with a left action of A by adjointable operators on X which is given by a $*$ -homomorphism $\phi : A \rightarrow \mathcal{L}(X)$, in the sense that the left action $a \cdot x$ is given by $\phi(a)x$. This implies the familiar-looking formula

$$\langle a \cdot y, x \rangle_A = \langle y, a^* \cdot x \rangle_A \quad \text{for all } a \in A, x, y \in X. \tag{2.1}$$

Every C^* -algebra A can be regarded as a C^* -correspondence over itself with actions given by multiplication, and inner product $\langle a, b \rangle_A = a^*b$. We denote this C^* -correspondence by ${}_A A_A$.

The internal tensor product $X \otimes_A Y$ of C^* -correspondences over A is the completion of the quotient of the algebraic tensor product $X \odot Y$ by the submodule generated by differences of the form $a \odot y - x \odot a \cdot y$, with respect to the inner product satisfying $\langle x \odot y, x' \odot y' \rangle_A = \langle y, \langle x, x' \rangle_A \cdot y' \rangle_A$ (see [30, Chapter 8]). For $n \geq 1$, we write

$$X^{\otimes n} = \underbrace{X \otimes_A X \otimes_A \cdots \otimes_A X}_{n \text{ terms}}.$$

By convention, $X^{\otimes 0} = {}_A A_A$.

Let A be a C^* -algebra and let X be a C^* -correspondence over A . A *Toeplitz representation* of X in a C^* -algebra B is a pair (ψ, π) such that $\psi : X \rightarrow {}_B B_B$ and $\pi : A \rightarrow B$ together constitute a Hilbert bimodule homomorphism.

Given a Toeplitz representation (ψ, π) of X in B , for each $n \in \mathbb{N}$ there exists [12, Proposition 1.8] a Toeplitz representation $(\psi^{\otimes n}, \pi)$ of $X^{\otimes n}$ in B such that for all $x_1, \dots, x_n \in X$,

$$\psi^{\otimes n}(x_1 \otimes x_2 \otimes \dots \otimes x_n) = \psi(x_1)\psi(x_2)\dots\psi(x_n).$$

Given a C^* -correspondence X over A , there exists a Toeplitz representation (ι_X, ι_A) of X in a C^* -algebra \mathcal{T}_X that is *universal* for Toeplitz representations [12, Proposition 1.3]; [29, Theorem 3.4]. This means that for another Toeplitz representation (ψ, π) of X in a C^* -algebra B , there exists a homomorphism denoted $\psi \times \pi : \mathcal{T}_X \rightarrow B$, such that $(\psi \times \pi) \circ \iota_X = \psi$ and $(\psi \times \pi) \circ \iota_A = \pi$. This \mathcal{T}_X is called the *Toeplitz algebra* of the correspondence X . By [30, Proposition 8.9],

$$\mathcal{T}_X = \overline{\text{span}}\{\iota_X^{\otimes m}(\xi)\iota_X^{\otimes n}(\eta)^* \mid m, n \geq 0, \xi \in X^{\otimes m}, \eta \in X^{\otimes n}\}.$$

The C^* -algebra $\iota_A(A)$ (or A) is called the *coefficient algebra* of \mathcal{T}_X . By [12, Proposition 1.3] there exists a strongly continuous action $\gamma^X : \mathbb{T} \rightarrow \text{Aut}(\mathcal{T}_X)$ such that

for $z \in \mathbb{T}$, $\gamma_z^X(\iota_A(a)) = \iota_A(a)$ and $\gamma_z^X(\iota_X(x)) = z\iota_X(x)$ for $a \in A$ and $x \in X$, called the *gauge action*. The dynamics that we will consider is the action $t \mapsto \gamma_{e^{it}}^X$.

For $n \in \mathbb{Z}$, the n^{th} spectral subspace $(\mathcal{T}_X)_n$ of \mathcal{T}_X for the action γ is the space $\{a \in \mathcal{T}_X : \gamma_z(a) = z^n a \text{ for all } z \in \mathbb{T}\}$. One can check that

$$(\mathcal{T}_X)_n = \overline{\text{span}}\{\iota_X^{\otimes p}(\xi)\iota_X^{\otimes q}(\eta)^* \mid p, q \geq 0, p - q = n, \xi \in X^{\otimes p}, \eta \in X^{\otimes q}\}.$$

We will make frequent use of the first spectral subspace $(\mathcal{T}_X)_1$ later.

Let A, B be C^* -algebras and X, Y be C^* -correspondences over A and B , respectively. Then, a map $\theta : \mathcal{T}_X \rightarrow \mathcal{T}_Y$ will be called an *isomorphism of triples* if it is an isomorphism of C^* -algebras such that $\theta \circ \gamma_z^X = \gamma_z^Y \circ \theta$ for all $z \in \mathbb{T}$ and carries $\iota_A(A)$ onto $\iota_B(B)$.

2.3. Topological graphs and modules

A *topological graph* is a quadruple $E = (E^0, E^1, r, s)$ where the *vertex set* E^0 and the *edge set* E^1 are locally compact Hausdorff spaces, the *range map* $r : E^1 \rightarrow E^0$ is a continuous function and the *source map* $s : E^1 \rightarrow E^0$ is a local homeomorphism. A *path* in E is a finite sequence of edges $\mu = \mu_1\mu_2 \dots \mu_n$ such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \leq i \leq n - 1$ or a vertex $v \in E^0$. For such a path, $|\mu| = n$ is called the *length* of μ and by convention $|v| = 0$ for all $v \in E^0$. For $n \geq 1$, the set of paths of length n is denoted E^n and the vertex set E^0 is considered the set of paths of length 0. Define $E^* := \bigcup_{n \in \mathbb{N}} E^n$ to be the set of all paths. For $\mu \in E^n$, let $r(\mu) := r(\mu_1)$ and $s(\mu) := s(\mu_n)$. For $v \in E^0$, we define $E^*v = \{\mu \in E^* : s(\mu) = v\}$ and $vE^* = \{\mu \in E^* : r(\mu) = v\}$, with analogous notation we can define vE^n and $E^n v$. More generally, for a subset $U \subseteq E^0$ we write E^1U for $s^{-1}(U)$.

LEMMA 2.1. *Let E be a compact topological graph. Then the function $v \mapsto |E^1v|$ is locally constant. For each $v \in E^0$, there exist an open neighbourhood W of v and disjoint open s -sections $(Z_e)_{e \in E^1v}$ such that $s(Z_e) = W$ for all $e \in E^1v$ and $E^1W = \bigsqcup_{e \in E^1v} Z_e$.*

Proof. Fix $v \in E^0$. Since s is local homeomorphism, E^1v is a discrete subspace of the compact space E^1 and thus finite. For each $e \in E^1v$ there exists a neighbourhood V_e of e on which s is a homeomorphism. Since E^1 is Hausdorff and E^1v is finite, by shrinking the V_e , we may assume that $V_e, e \in E^1v$ are pairwise disjoint. Since s is a local homeomorphism, it is an open map, and so $W_0 := \bigcap_{e \in E^1v} s(V_e)$ is open. For $w \in W_0$ we have $|E^1w| \geq |\bigcup_{e \in E^1v} V_e \cap E^1w| = |E^1v|$. For the first statement, it suffices to show that there is an open subset $W \subseteq W_0$ containing v such that $|E^1w| \leq |E^1v|$ for all $w \in W$. We suppose otherwise and derive a contradiction. Then there is a sequence (w_i) in W_0 converging to v such that $|E^1w_i| > |E^1v|$ for all i . It follows that for each i there exists $e_i \in E^1w_i \setminus \bigcup_{e \in E^1v} V_e$. Since E^1 is compact, by passing to a subsequence, we may assume that the sequence $(e_i)_i$ is convergent, say to $e_\infty \in E^1$. Continuity of the source map forces $s(e_\infty) = \lim_i s(e_i) = \lim_i w_i = v$ so that $e_\infty \in E^1v$. So by definition we have $e_\infty \in V_{e_\infty}$ but $e_i \notin V_{e_\infty}$ for all i , contradicting that $e_i \rightarrow e_\infty$.

For the second statement, for each $e \in E^1v$ take V_e and W as before, and let $Z_e = V_e \cap E^1W$. Then $s : Z_e \rightarrow W$ is a homeomorphism for each $e \in E^1v$ and

$E^1W = \bigsqcup_{e \in E^1v} Z_e$, since, otherwise there would be an edge $e' \in E^1W$ such that $e' \notin V_e$ for all $e \in E^1v$. This would lead to $|E^1s(e')| > |E^1v|$ and $s(e') \in W$, a contradiction. \square

REMARK 2.2. Lemma 2.1 implies that the map $v \mapsto |E^1v|$ is continuous and in particular, $\max_{v \in E^0} |E^1v|$ exists and is finite as E^0 is compact. By induction, $\max_{v \in E^0} |E^n v|$ exists and is finite for all n .

What follows comes from [23]. Let E be a topological graph. There are a right action of $C_0(E^0)$ on $C_c(E^1)$ and a $C_0(E^0)$ -valued inner product on $C_c(E^1)$ such that

$$(x \cdot \alpha)(e) = x(e)\alpha(s(e)) \quad \text{and} \quad \langle x, y \rangle_{C_0(E^0)}(v) = \sum_{s(e)=v} \overline{x(e)}y(e)$$

for $x, y \in C_c(E^1)$ and $\alpha \in C_0(E^0)$. If $E^1v = \emptyset$, our convention is that the sum is equal to 0. The completion $X(E)$ of $C_c(E^1)$ in the norm $\|x\|^2 = \|\langle x, x \rangle\|_{C_0(E^0)}$ is a Hilbert A -module and is equal to

$$\{x \in C(E^1) : \langle x, x \rangle \in C_0(E^0)\}. \tag{2.2}$$

The formula

$$(\alpha \cdot x)(e) := \alpha(r(e))x(e)$$

defines an action of $C_0(E^0)$ by adjointable operators on $X(E)$, so that $X(E)$ becomes a C^* -correspondence over $C_0(E^0)$. We call this the *graph correspondence* associated with E . The Toeplitz algebra of the topological graph E is then $\mathcal{TC}^*(E) := \mathcal{T}_{X(E)}$.

There is a categorical equivalence between Hilbert $C(X)$ -modules, for X a compact Hausdorff space and Hilbert bundles over X [8, 10, 33]. Following [10], by a Hilbert bundle over X we mean a triple (p, E, X) where $p : E \rightarrow X$ is an open surjection, E and X are topological spaces, together with operations and inner products making each fibre $E_x = p^{-1}(x)$ a Hilbert space that satisfies certain compatibility conditions [11, Definition 13.4]. We describe briefly how to construct the *canonical Hilbert bundle* \mathcal{E}_V of a Hilbert $C(X)$ -module V . For each $x \in X$, let

$$I_x := \{f \in C(X) : f(x) = 0\} \quad \text{and} \quad J_x := VI_x = \{v \cdot f : v \in V, f \in I_x\}.$$

In the rest of this section we will drop the subscript on the inner product. By the Hewitt–Cohen factorization theorem [31, Proposition 2.31] J_x is a closed submodule of V and by [31, Lemma 3.32] we can write $J_x = \{v \in V : \langle v, v \rangle \in I_x\}$. Since J_x is a closed submodule, we may take the quotient V/J_x , which is a Hilbert $C(X)/I_x$ -module. As $C(X)/I_x \cong \mathbb{C}$, the quotient V/J_x is a Hilbert space. Let $\pi_x : V \rightarrow V/J_x$ be the quotient map and let

$$E := \bigsqcup_{x \in X} V/J_x.$$

Define $p : E \rightarrow X$ by $p(v + J_x) = x$ for each $x \in X$. By [31, Proposition 3.25], for each $v \in V$ and $x \in X$ we have $\|\pi_x(v)\|^2 = \langle v, v \rangle(x)$ and hence, the map $x \mapsto \|\pi_x(v)\|$ is continuous. For $v \in V$, define $\hat{v} : X \rightarrow E$ by $\hat{v}(x) = \pi_x(v)$. Let

$\Gamma := \{\hat{v} : v \in V\}$. Then Γ is a complex linear space of cross-sections and for each $x \in X$, the set $\{f(x) : f \in \Gamma\} = \{\pi_x(v) : v \in V\}$ is equal to V/J_x . Hence, by [11, Theorem II.13.18] there is a unique topology on E making $\mathcal{E}_V = (p, E, X)$ into a Hilbert bundle and all elements of Γ continuous cross-sections. The map $v \mapsto \hat{v}$ is an isomorphism between the Hilbert $C(X)$ -modules V and Γ . We will be interested only in compact topological graphs, so the construction above can be applied to the graph correspondences in this paper.

3. Reconstruction of topological–graph bimodules

From now on, E denotes a compact topological graph. That is, E^0 and E^1 are compact, so r and s are proper maps. We use results from Hawkins’ thesis [16], which we reproduce here for the convenience of the reader. In the following, $\mathcal{M}(X)$ denotes the space of finite Borel measures on a compact Hausdorff space X and $\mathcal{M}^1(X) \subset \mathcal{M}(X)$ denotes the space of Borel probability measures on X . By remark 2.2, we can also define

$$\rho(A_E) := \lim_{n \rightarrow \infty} \max_{v \in E^0} |E^n v|^{1/n} < \infty,$$

which we call the *spectral radius* associated with the compact topological graph E (the notation is motivated by the spectral radius of the adjacency matrix in the discrete case).

THEOREM 3.1 [16, Theorem 5.1.10]. *Let E be a compact topological graph, and fix $\beta > \log(\rho(A_E))$. For each $\epsilon \in \mathcal{M}(E^0)$ satisfying*

$$\int_{E^0} \sum_{\mu \in E^*v} e^{-\beta|\mu|} d\epsilon(v) = 1, \tag{3.1}$$

there exists a KMS $_\beta$ -state ϕ_ϵ such that for $x \in X(E)^{\otimes k}$ and $y \in X(E)^{\otimes \ell}$

$$\phi_\epsilon(\iota_{X(E)}^{\otimes k}(x)\iota_{X(E)}^{\otimes \ell}(y)^*) = \delta_{k,\ell} e^{-\beta k} \int_{E^0} \sum_{\mu \in E^*v} e^{-\beta|\mu|} \langle y, x \rangle_{C(E^0)}(r(\mu)) d\epsilon(v). \tag{3.2}$$

For each $v \in E^0$ and each $\beta > \log(\rho(A_E))$, we write

$$N_v^\beta := \sum_{\mu \in E^*v} e^{-\beta|\mu|}.$$

For $v \in E^0$, the measure $\epsilon_v := (N_v^\beta)^{-1} \delta_v$ satisfies (3.1), so theorem 3.1 supplies an associated KMS $_\beta$ -state $\phi_v^\beta := \phi_{\epsilon_v}^\beta$. If β is clear from context, we just write ϕ_v for ϕ_v^β . The proof of theorem 3.1 shows that for $a \in \mathcal{TC}^*(E)$,

$$\phi_v^\beta(a) = \frac{\sum_{\mu \in E^*v} e^{-\beta|\mu|} \langle \psi_v \times \pi_v(a)e_\mu, e_\mu \rangle}{N_v^\beta}. \tag{3.3}$$

THEOREM 3.2 [16, Theorem 5.1.11]. *Let E be a compact topological graph, and fix $\beta > \log(\rho(A_E))$. Then there is an affine isomorphism of $\mathcal{M}^1(E^0)$ onto the set of KMS_β -states of $\mathcal{TC}^*(E)$ that takes a measure Ω to the state φ_Ω given by*

$$\varphi_\Omega(a) = \int_{E^0} \phi_v(a) d\Omega(v) \quad \text{for all } a \in \mathcal{TC}^*(E).$$

REMARK 3.3. The proof of theorem 3.2 shows, among other things, that $v \mapsto \phi_v$ is a homeomorphism of E^0 onto the space of extremal points of the set of KMS_β -states for τ .

Replicating the proof of [16, Lemma 4.1.7] yields the following proposition bar the last statement which we will prove.

PROPOSITION 3.4. *Let E be a topological graph, fix $v \in E^0$ and let $\{e_\mu : \mu \in E^*v\}$ denote the canonical basis for $\ell^2(E^*v)$. Then there exists a linear map $\psi_v : X(E) \rightarrow \mathcal{B}(\ell^2(E^*v))$ and a homomorphism $\pi_v : C_0(E^0) \rightarrow \mathcal{B}(\ell^2(E^*v))$ such that for all $\xi \in X(E)$, $\alpha \in C_0(E^0)$ and $\mu \in E^*v$,*

$$\psi_v(\xi)e_\mu = \sum_{f \in E^1r(\mu)} \xi(f)e_{f\mu} \quad \text{and} \quad \pi_v(\alpha)e_\mu = \alpha(r(\mu))e_\mu.$$

The pair (ψ_v, π_v) is a Toeplitz representation of $\mathcal{TC}^(E)$ on $\ell^2(E^*v)$ and the direct sum $\bigoplus_{v \in E^0} (\psi_v \times \pi_v)$ is faithful.*

Proof. All but the statement that $\Theta := \bigoplus_{v \in E^0} (\psi_v \times \pi_v)$ is faithful follows from essentially replicating the computations in the proof of [16, Lemma 4.1.7], hence we will only deal with the question of faithfulness.

Let U_v be the canonical inclusion of $\ell^2(E^*v)$ into $\ell^2(E^*)$. This U_v is an isometry with adjoint $U_v^* : \ell^2(E^*) \rightarrow \ell^2(E^*v)$ the corresponding orthogonal projection. We define $U := \bigoplus_{v \in E^0} U_v : \bigoplus_{v \in E^0} \ell^2(E^*v) \rightarrow \ell^2(E^*)$. Then U is unitary, and

$$U^*h = (h|_{E^*v})_{v \in E^0} \quad \text{for all } h \in \ell^2(E^*).$$

Let $\{\delta_\mu : \mu \in E^*\}$ be the canonical basis for $\ell^2(E^*)$. By [16, Lemma 4.1.7], there exist $\lambda^0 : C_0(E^0) \rightarrow \mathcal{B}(\ell^2(E^*))$ and $\lambda^1 : X(E) \rightarrow \mathcal{B}(\ell^2(E^*))$ such that for all $\xi \in X(E)$, $\alpha \in C_0(E^0)$ and $\mu \in E^*$,

$$\lambda^1(\xi)\delta_\mu = \sum_{f \in E^1r(\mu)} \xi(f)\delta_{f\mu} \quad \text{and} \quad \lambda^0(\alpha)\delta_\mu = \alpha(r(\mu))\delta_\mu.$$

Furthermore, (λ^1, λ^0) is a Toeplitz representation of $X(E)$ on $\ell^2(E^*)$ and $\lambda^1 \times \lambda^0$ is faithful. We claim that, for $a \in \mathcal{TC}^*(E)$,

$$\Theta(a) = U^* (\lambda^0 \times \lambda^1(a)) U. \tag{3.4}$$

To show this, we first prove that $U_v(\psi_v \times \pi_v(a)) = (\lambda^1 \times \lambda^0(a))U_v$ for all $v \in E^0$. By linearity and continuity, it suffices to fix $x = x_1 \otimes x_2 \otimes \dots \otimes x_m \in X(E)^{\otimes m}$, $y = y_1 \otimes y_2 \otimes \dots \otimes y_n \in X(E)^{\otimes n}$ (with the convention that if $m = 0$, we mean $x = b \in A$, and similarly if $n = 0$ for y), and then consider $a = \iota_{X(E)}^{\otimes m}(x)\iota_{X(E)}^{\otimes n}(y)^*$. With

the convention that $\prod_{i=1}^0 \psi_v(x_i)$ means $\pi_v(b)$ when $x = b \in A$, and similarly for y , we have

$$U_v(\psi_v \times \pi_v(a)) = U_v\left(\prod_{i=1}^m \psi_v(x_i)\right)\left(\prod_{i=0}^{n-1} \psi_v(y_{n-i})^*\right).$$

Direct calculation with basis vectors shows that $U_v\psi_v(x_i) = \lambda^1(x_i)U_v$ and that $U_v\psi_v(y_j)^* = \lambda^1(y_j)^*U_v$. Therefore

$$U_v(\psi_v \times \pi_v(a)) = \left(\prod_{i=1}^m \lambda^1(x_i)\right)\left(\prod_{i=0}^{n-1} \lambda^1(y_{n-i})^*\right)U_v = (\lambda^1 \times \lambda^0(a))U_v$$

as claimed.

Since U_v is an isometry, we deduce that $\psi_v \times \pi_v(a) = U_v^*(\lambda^1 \times \lambda^0(a))U_v$. Since each $\ell^2(E^*v) \subseteq \ell^2(E^*)$ is invariant for $\lambda^1 \times \lambda^0(a)$,

$$\begin{aligned} \Theta(a) &= \bigoplus_{v \in E^0} (\psi_v \times \pi_v(a)) = \bigoplus_{v \in E^0} U_v^*(\lambda^1 \times \lambda^0(a))U_v \\ &= \left(\bigoplus_{v \in E^0} U_v\right)^* (\lambda^1 \times \lambda^0(a)) \left(\bigoplus_{v \in E^0} U_v\right) = U^*(\lambda^1 \times \lambda^0(a))U, \end{aligned}$$

as in equation (3.4). We conclude that Θ is unitarily equivalent to $\lambda^1 \times \lambda^0$ and hence faithful by [16, Lemma 4.1.7]. \square

Next we describe an important set of KMS_∞ -states for the analysis that will follow. For each $v \in E^0$, let φ_v be the vector state of $\mathcal{TC}^*(E)$ given by

$$\varphi_v(a) := \langle \psi_v \times \pi_v(a)e_v, e_v \rangle, \quad \text{for all } a \in \mathcal{TC}^*(E). \tag{3.5}$$

Denote by S^∞ the set $\{\varphi_v : v \in E^0\}$. To prove our main result, we first describe the GNS-representation of each $\varphi_v \in S^\infty$.

LEMMA 3.5. *Fix $v \in E^0$. The GNS-representation of $\varphi_v \in S^\infty$ is equivalent to $\psi_v \times \pi_v$ on $\ell^2(E^*v)$.*

Proof. We show that e_v is a cyclic vector for $\psi_v \times \pi_v$. It is clear that $\psi_v \times \pi_v(1)e_v = e_v$. Fix $f \in E^1v$. Since $|E^1v| < \infty$ there exists a function $x \in X(E)$ such that $x(f) = 1$ and $x(g) = 0$ for $g \in E^1v \setminus \{f\}$, hence

$$\psi_v \times \pi_v(\iota_{X(E)}(x))e_v = \sum_{g \in E^1v} x(g)e_g = e_f.$$

Thus, $\{e_f : f \in E^1v\} \subset \{\psi_v \times \pi_v(a)e_v : a \in \mathcal{TC}^*(E)\}$. An induction on n , using a similar argument, shows that the set $\{e_\mu : \mu \in E^nv\} \subset \{\psi_v \times \pi_v(a)e_v : a \in \mathcal{TC}^*(E)\}$ for each $n \in \mathbb{N}$. Therefore, e_v is cyclic for $\psi_v \times \pi_v$ and this proves the lemma by the uniqueness of cyclic representations [28, Theorem 5.1.4]. \square

REMARK 3.6. If $v \in E^0$, $\xi \in X(E)$ and $\mu, \mu' \in E^*v$, then

$$\langle \psi_v(\xi)^* e_\mu, e_{\mu'} \rangle = \left\langle e_\mu, \sum_{f \in E^1 r(\mu')} \xi(f) e_{f\mu'} \right\rangle.$$

Hence, if $\mu = f\nu$, for $f \in E^1$ and $\nu \in E^*v$, then $\psi_v(\xi)^* e_\mu = \overline{\xi(f)} e_\nu$. Otherwise, if $\mu = v$, then $\psi_v(\xi)^* e_\mu = 0$. In the first case, we also have the identity

$$\psi_v(\xi) \psi_v(\xi)^* e_\mu = \overline{\xi(f)} \sum_{g \in E^1 r(\nu)} \xi(g) e_{g\nu}.$$

LEMMA 3.7. Let E be a topological graph and fix $v \in E^0$. Suppose that $x \in X(E)^{\otimes m}$ and $y \in X(E)^{\otimes n}$ are elementary tensors. If either $m > 0$ or $n > 0$, then $\varphi_v(\iota^{\otimes m}(x)\iota^{\otimes n}(y)^*) = 0$.

Proof. Let $x = x_1 \otimes x_2 \otimes \dots \otimes x_m \in X(E)^{\otimes m}$ and $y = y_1 \otimes y_2 \otimes \dots \otimes y_n \in X(E)^{\otimes n}$. First, suppose $n > 0$. Then,

$$\begin{aligned} \varphi_v(\iota^{\otimes m}(x)\iota^{\otimes n}(y)^*) &= \langle \psi_v^{\otimes m}(x)\psi_v^{\otimes n}(y)^* e_v, e_v \rangle \\ &= \langle \psi_v^{\otimes m}(x)\psi_v(y_n)^* \dots \psi_v(y_1)^* e_v, e_v \rangle = 0, \end{aligned}$$

by remark 3.6. Now, suppose $n = 0$ and $m > 0$. Then,

$$\varphi_v(\iota^{\otimes m}(x)\iota_{C_0(E^0)}(y)) = \langle \psi_v^{\otimes m}(x)\pi_v(y) e_v, e_v \rangle = \langle \psi_v(x_1) \dots \psi_v(x_m \cdot y) e_v, e_v \rangle.$$

By proposition 3.4,

$$\langle \psi_v(x_1) \dots \psi_v(x_m \cdot y) e_v, e_v \rangle = \sum_{\mu = \mu_1 \dots \mu_m \in E^m v} \langle x_1(\mu_1) \dots x_m(\mu_m) y(v) e_\mu, e_v \rangle = 0,$$

which concludes the proof. □

LEMMA 3.8. Let E be a compact topological graph. Let $(v_n)_{n \in \mathbb{N}}$ be a sequence in E^0 . Then φ_{v_n} is weak* convergent if and only if v_n converges in E^0 , in which case, writing $v := \lim_n v_n$, we have $\varphi_{v_n} \xrightarrow{w^*} \varphi_v$.

Proof. First suppose that $v_n \rightarrow v$ in E^0 . We want to show that $\varphi_{v_n}(a) \rightarrow \varphi_v(a)$ for every $a \in \mathcal{TC}^*(E)$. By linearity and continuity it is enough to prove this for $a = \iota^{\otimes m}(x)\iota^{\otimes n}(y)^*$ where $x \in X(E)^{\otimes m}$, $y \in X(E)^{\otimes n}$ are elementary tensors. If $n > 0$ or $m > 0$, then $\varphi_{v_n}(a) = 0 = \varphi_v(a)$ by lemma 3.7, so we just need to check the case when $a = \iota_{C(E^0)}(\alpha)$ for $\alpha \in C(E^0)$. Indeed,

$$\varphi_{v_n}(\iota_{C(E^0)}(\alpha)) = \langle \pi_{v_n}(\alpha) e_{v_n}, e_{v_n} \rangle = \alpha(v_n) \rightarrow \alpha(v) = \varphi_v(\iota_{C(E^0)}(\alpha)).$$

This proves the first direction. Now suppose that v_n does not converge in E^0 . Since E^0 is compact, (v_n) has at least two distinct accumulation points $v, v' \in E^0$. Take $\alpha \in C(E^0)$ such that $\alpha(v) \neq \alpha(v')$. Let $(v_{n_\ell})_\ell$ and $(v_{n_k})_k$ be subsequences of v_n

that converge to v and v' , respectively. If φ_n weak* converges to some ϕ , then by the previous paragraph,

$$\alpha(v) = \lim_{\ell} \varphi_{v_n\ell}(\iota_{C(E^0)}(\alpha)) = \phi(\iota_{C(E^0)}(a)) = \lim_k \varphi_{v_nk}(\iota_{C(E^0)}(\alpha)) = \alpha(v'),$$

a contradiction. □

PROPOSITION 3.9. *Let E be a compact topological graph. The set S^∞ is the set of weak* limit points of sequences $(\phi_n)_{n \in \mathbb{N}}$ such that there exists a sequence $\beta_n \rightarrow \infty$ of real numbers such that each ϕ_n is an extremal KMS_{β_n} -state.*

Proof. First, assume that (β_n) and (ϕ_n) are sequences as in the statement of the proposition such that ϕ_n weak* converges to some ϕ . We need to show that there exists $v \in E^0$ such that $\phi = \varphi_v$. Since $\beta_n \rightarrow \infty$, we may assume that $\beta_n > \log(\rho(A_E))$ for all n . By remark 3.3, for each n there is a vertex $v_n \in E^0$ such that $\phi_n = \phi_{v_n}^{\beta_n}$, the extremal KMS_{β_n} -state described just before theorem 3.1. By equation (3.3), for every $a \in \mathcal{TC}^*(E)$,

$$\begin{aligned} \phi_{v_n}^{\beta_n}(a) &= (N_{\phi_{v_n}^{\beta_n}})^{-1} \sum_{E^*v_n} e^{-\beta_n|\mu|} \langle \psi_{v_n} \times \pi_{v_n}(a)e_\mu, e_\mu \rangle \\ &= (N_{\phi_{v_n}^{\beta_n}})^{-1} \left(\langle \psi_{v_n} \times \pi_{v_n}(a)e_{v_n}, e_{v_n} \rangle + \sum_{E^*v_n \setminus \{v_n\}} e^{-\beta_n|\mu|} \langle \psi_{v_n} \times \pi_{v_n}(a)e_\mu, e_\mu \rangle \right). \end{aligned}$$

We claim that $\lim_{n \rightarrow \infty} N_{\phi_{v_n}^{\beta_n}} = 1$ and that

$$\lim_{n \rightarrow \infty} \sum_{E^*v_n \setminus \{v_n\}} e^{-\beta_n|\mu|} \langle \psi_{v_n} \times \pi_{v_n}(a)e_\mu, e_\mu \rangle = 0.$$

For the first of these equalities, note that by remark 2.2, we have for $n \in \mathbb{N}$,

$$1 \leq N_{\phi_{v_n}^{\beta_n}} \leq 1 + \sum_{i=1}^{\infty} e^{-\beta_n i} \max_{v \in E^0} |E^i v|.$$

Hence $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} e^{-\beta_n i} \max_{v \in E^0} |E^i v| = 0$ by the dominated convergence theorem. Thus $N_{\phi_{v_n}^{\beta_n}} \rightarrow 1$.

The second equality follows similarly once we observe that

$$\sum_{E^*v_n \setminus \{v_n\}} e^{-\beta_n|\mu|} \langle \psi_{v_n} \times \pi_{v_n}(a)e_\mu, e_\mu \rangle \leq \|a\| \sum_{i=1}^{\infty} e^{-\beta_n i} \max_{v \in E^0} |E^i v|.$$

The claim implies that $\phi_{v_n}^{\beta_n} - \varphi_{v_n}$ weak* converges to 0. In particular, φ_{v_n} weak* converges to the same weak* limit as $\phi_n = \phi_{v_n}^{\beta_n}$, namely ϕ . So lemma 3.8 implies that v_n converges to some $v \in E^0$ and $\phi = \varphi_v$.

Now, assume that $v \in E^0$. We have to show that there exist sequences (β_n) and (ϕ_n) as in the statement of the proposition such that ϕ_n weak* converges to φ_v . Pick any sequence (β_n) of real numbers such that $\beta_n > \log(\rho(A_E))$ and $\beta_n \rightarrow \infty$. Take the constant sequence $v_n = v$. By remark 3.3, $\phi_n := \phi_{v_n}^{\beta_n} = \phi_v^{\beta_n}$ is an extremal

KMS $_{\beta_n}$ -state for all n . The same argument as before shows that $\phi_{v_n}^{\beta_n} - \varphi_{v_n} = \phi_n - \varphi_v$ weak* converges to 0 and we are done. \square

LEMMA 3.10. *Let E be a compact topological graph. For each state ϕ of $\mathcal{TC}^*(E)$, denote its GNS-representation by π_ϕ . There exists a unique element $p_E \in \mathcal{TC}^*(E)$ such that for every $\varphi \in S^\infty$,*

- (i) $\pi_\varphi(p_E)$ is a minimal projection in $\pi_\varphi(\mathcal{TC}^*(E))$ and
- (ii) $\varphi(p_E) = 1$.

Proof. We begin with existence. As E^1 is compact and the source map is a local homeomorphism, there exists a finite open cover $(U_i)_{i=1}^n$ of E^1 by s-sections. Let $\{\xi_i\}_{i=1}^n$ be a partition of unity subordinate to the U_i with each $\xi_i \in C_0(U_i, [0, 1])$. Writing $\sqrt{\xi_i}$ for the pointwise square-root of each $\xi_i : E^1 \rightarrow [0, 1]$, we define an element $p_E \in \mathcal{TC}^*(E)$ by

$$p_E := 1 - \sum_{i=1}^n \iota_{X(E)}(\sqrt{\xi_i}) \iota_{X(E)}(\sqrt{\xi_i})^*. \tag{3.6}$$

Let $v \in E^0$ be the vertex such that $\varphi = \varphi_v$, and fix $\mu \in E^*v \setminus \{v\}$. Write $\mu = f\nu$ with $f \in E^1$. By remark 3.6, since each ξ_i is supported in an s-section,

$$\psi_v \times \pi_v(p_E)e_\mu = e_\mu - \sum_{i=1}^n \sum_{g \in E^1r(\nu)} \sqrt{\xi_i(f)} \sqrt{\xi_i(g)} e_{g\nu} = e_\mu - \left(\sum_{i=1}^n \xi_i(f) \right) e_\mu = 0.$$

On the other hand, again by remark 3.6, $\psi_v \times \pi_v(p_E)e_v = e_v$, so $\psi_v \times \pi_v(p_E)$ is the projection onto the basis vector e_v , thus by lemma 3.5, p_E satisfies (i) and (ii).

Now we prove uniqueness. Fix $a \in \mathcal{TC}^*(E)$ such that $\pi_\varphi(a)$ is a minimal projection in $\pi_\varphi(\mathcal{TC}^*(E))$ and $\varphi(a) = 1$ for every $\varphi \in S^\infty$. Fix $v \in E^0$. By lemma 3.5, $\psi_v \times \pi_v(a)$ is a minimal projection in $\psi_v \times \pi_v(\mathcal{TC}^*(E))$. Since $\psi_v \times \pi_v(p_E) \in \psi_v \times \pi_v(\mathcal{TC}^*(E)) \cap \mathcal{K}(\ell^2(E^*v))$, we have $\mathcal{K}(\ell^2(E^*v)) \subset \pi_{\varphi_v}(\mathcal{TC}^*(E))$ [28, Theorem 2.4.9]. Hence, $\psi_v \times \pi_v(a)$ is the rank-one projection $\xi \otimes \xi^*$, for a unit vector $\xi \in \ell^2(E^*v)$ and

$$1 = \varphi_v(a) = \langle \psi_v \times \pi_v(a)e_v, e_v \rangle = \langle \xi \otimes \xi^*(e_v), e_v \rangle = |\langle \xi, e_v \rangle|^2.$$

Since ξ is a unit vector, $\xi = \lambda e_v$, for some $\lambda \in \mathbb{T}$. This implies that $\psi_v \times \pi_v(a) = \xi \otimes \xi^* = \lambda e_v \otimes (\lambda e_v)^* = e_v \otimes e_v^*$. Hence $\psi_v \times \pi_v(a) = \psi_v \times \pi_v(p_E)$ for every $v \in E^0$. Proposition 3.4 implies that $\bigoplus_{v \in E^0} \psi_v \times \pi_v$ is faithful, so $a = p_E$. \square

REMARK 3.11. Direct calculations with basis vectors, similar to those in the proof of lemma 3.10, show that the faithful representation $\lambda^0 \times \lambda^1$ of [16, Lemma 4.1.7] defined in the proof of proposition 3.4 carries p_E to the projection onto $\overline{\text{span}}\{e_v : v \in E^0\} \subset \ell^2(E^*)$. Define $M := \iota_{C(E^0)}(C(E^0))$ and observe that $p_E \in M'$,

the commutant of M . Indeed, for $\alpha \in C(E^0)$ and $\mu \in E^*$,

$$\begin{aligned} \lambda^0 \times \lambda^1(\iota_{C(E^0)}(\alpha)p_E)e_\mu &= \begin{cases} \lambda^0(\alpha)e_\mu & \text{if } |\mu| = 0 \\ 0 & \text{if } |\mu| > 0 \end{cases} \\ &= \lambda^0 \times \lambda^1(p_E\iota_{C(E^0)}(\alpha))e_\mu. \end{aligned}$$

In particular, if $\iota_{C(E^0)}(\alpha)p_E = 0$, then $\lambda^0(\alpha)e_v = 0$ for all $v \in E^0$ so that $\alpha = 0$. Thus the map $\rho_E : M \rightarrow Mp_E$ given by $\rho(m) := mp_E = p_E m$ is an isomorphism of C^* -algebras.

In what follows, we write $\iota_{X(E)}(\xi)$ as $\iota_X(\xi)$ to lighten notation.

For the following lemma, recall that $\mathcal{TC}^*(E)_1$ denotes the first spectral subspace $\{a \in \mathcal{TC}^*(E) : \gamma_z(a) = za \text{ for all } z \in \mathbb{T}\}$.

LEMMA 3.12. *Let E be a compact topological graph. Let $M := \iota_{C(E^0)}(C(E^0)) \subset \mathcal{TC}^*(E)$, let $p_E \in \mathcal{TC}^*(E)$ be as in lemma 3.10 and let $\rho_E : M \rightarrow Mp_E$ be the isomorphism $\rho_E(m) = mp_E$ of remark 3.11. Define*

$$\psi_E : X(E) \rightarrow_{\mathcal{TC}^*(E)} \mathcal{TC}^*(E)_{\mathcal{TC}^*(E)}$$

by $\psi_E(\xi) = \iota_X(\xi)p_E$. Then $(\psi_E, \iota_{C(E^0)})$ is a bimodule morphism, and $\psi_E(X(E)) = \mathcal{TC}^*(E)_1 p_E$. Furthermore, there is an M -valued inner product on $\mathcal{TC}^*(E)_1 p_E$ such that

$$\langle \psi_E(\xi), \psi_E(\eta) \rangle_M = \rho_E^{-1}(\psi_E(\xi)^* \psi_E(\eta)) \text{ for all } \xi, \eta \in X(E), \tag{3.7}$$

and with respect to this inner product, $(\psi_E, \iota_{C(E^0)}) : X(E) \rightarrow \mathcal{TC}^*(E)_1 p_E$ is an isomorphism of C^* -correspondences.

Proof. We start by showing that $\psi_E(X(E)) = \mathcal{TC}^*(E)_1 p_E$. For this, recall that $\mathcal{TC}^*(E) = \overline{\text{span}}\{\iota_X^{\otimes m}(x)\iota_X^{\otimes n}(y)^* : n, m \in \mathbb{N}, x \in X(E)^{\otimes m}, y \in X(E)^{\otimes n}\}$. Note that $\gamma_z(\iota_X^{\otimes m}(x)\iota_X^{\otimes n}(y)^*) = z^{m-n}\iota_X^{\otimes m}(x)\iota_X^{\otimes n}(y)^*$, so

$$\mathcal{TC}^*(E)_1 = \overline{\text{span}}\{\iota_X^{\otimes n+1}(x)\iota_X^{\otimes n}(y)^* : n \geq 0, x \in X(E)^{\otimes n+1}, y \in X(E)^{\otimes n}\}.$$

Hence,

$$\mathcal{TC}^*(E)_1 p_E = \overline{\text{span}}\{\iota_X^{\otimes n+1}(x)\iota_X^{\otimes n}(y)^* p_E : n \geq 0, x \in X(E)^{\otimes n+1}, y \in X(E)^{\otimes n}\}.$$

Suppose that $x = x_1 \otimes x_2 \otimes \dots \otimes x_{n+1}$ and that $y = y_1 \otimes y_2 \otimes \dots \otimes y_n$ if $n > 0$ or $y = a \in A A_A$ if $n = 0$. For $\mu \in E^*$, the representation (λ^0, λ^1) as in remark 3.11

satisfies

$$\lambda^0 \times \lambda^1(\iota_X^{\otimes n+1}(x)\iota_X^{\otimes n}(y)^*p_E)e_\mu = \begin{cases} \lambda^0 \times \lambda^1(\iota_X^{\otimes n+1}(x)\iota_X^{\otimes n}(y)^*)e_\mu, & \text{if } |\mu| = 0, \\ 0, & \text{if } |\mu| > 0. \end{cases}$$

If $|\mu| = 0$, then by the proof of [16, Lemma 4.1.7], if $n > 0$, we have $\lambda^1(y_1)^*e_\mu = 0$. Hence, for $|\mu| = 0$,

$$\lambda^0 \times \lambda^1(\iota_X^{\otimes n+1}(x)\iota_X^{\otimes n}(y)^*)e_\mu = \begin{cases} \lambda^1(x \cdot a^*)e_\mu, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$

Since $\lambda^1 \times \lambda^0$ is faithful representation, we conclude that $\iota_X^{\otimes n+1}(x)\iota_X^{\otimes n}(y)p_E = 0$ if $n > 0$ and

$$\begin{aligned} \mathcal{TC}^*(E)_1p_E &= \overline{\text{span}}\{\iota_X(x)\iota_{C(E^0)}(a)^*p_E : x \in X(E), a \in C(E^0)\} \\ &= \{\psi_E(\xi) : \xi \in X(E)\} = \psi_E(X(E)). \end{aligned}$$

Now we show that $(\psi, \iota_{C(E^0)})$ is a bimodule morphism. Indeed, for $a \in C(E^0)$ and $\xi \in X(E)$,

$$\psi_E(a \cdot \xi) = \iota_X(a \cdot \xi)p_E = \iota_{C(E^0)}(a)\iota_X(\xi)p_E = \iota_{C(E^0)}(a)\psi_E(\xi).$$

Since p_E commutes with M as in remark 3.11, it follows from a similar computation that ψ preserves the right actions of $C(E^0)$. So $(\psi_E, \iota_{C(E^0)})$ preserves the bimodule structure. It remains to show that $(\psi_E, \iota_{C(E^0)})$ is an isomorphism of Hilbert modules. A simple calculation shows that it respects the M -valued inner-product of (3.7):

$$\begin{aligned} \langle \psi_E(\xi), \psi_E(\eta) \rangle_M &= \rho_E^{-1}(\psi_E(\xi)^*\psi_E(\eta)) \\ &= \rho_E^{-1}((\iota_X(\xi)a)^*\iota_X(\eta)p_E) = \iota_X(\xi)^*\iota_X(\eta) = \iota_{C(E^0)}(\langle \xi, \eta \rangle_{C(E^0)}). \end{aligned}$$

Hence $X(E) \cong \mathcal{TC}^*(E)_1p_E$ as C^* -correspondences. □

We can now prove our first main theorem. Recall the notion of an isomorphism of triples from the introduction.

THEOREM 3.13. *Let E and F be compact topological graphs. Let γ^E and γ^F be the gauge actions on $\mathcal{TC}^*(E)$ and $\mathcal{TC}^*(F)$ and let $M_E = \iota_{C(E^0)}(C(E^0))$ and $M_F = \iota_{C(F^0)}(C(F^0))$. Suppose that $\theta : (\mathcal{TC}^*(E), \gamma^E, M_E) \rightarrow (\mathcal{TC}^*(F), \gamma^F, M_F)$ is an isomorphism of triples. Let $\theta_M := \theta|_{M_E}$. Then there exists a unique linear map $\theta_X : X(E) \rightarrow X(F)$ such that*

$$\theta(\iota_X(\xi)p_E) = \iota_X(\theta_X(\xi))p_F \quad \text{for all } \xi \in X(E), \tag{3.8}$$

and (θ_X, θ_M) is an isomorphism of C^* -correspondences.

Proof. We first claim that $\theta(\mathcal{TC}^*(E)_1) = \mathcal{TC}^*(F)_1$ and $\theta(p_E) = p_F$. The first is because θ intertwines the gauge actions γ_E, γ_F . For the second we show that $\theta(p_E) \in \mathcal{TC}^*(F)$ satisfies properties (i) and (ii) of lemma 3.10. Let $\varphi \in S_F^\infty$ (the subscript F

is to make it explicit that we are referring to the states of $\mathcal{TC}^*(F)$). Then writing ξ for the corresponding cyclic vector in the GNS-space \mathcal{H}_φ , for any $b \in \mathcal{TC}^*(E)$,

$$\varphi(\theta(b)) = \langle \pi_\varphi(\theta(b))\xi, \xi \rangle.$$

Since θ is a gauge invariant isomorphism, $\phi \rightarrow \phi \circ \theta$ is an isomorphism of the KMS_β -simplex of $(\mathcal{TC}^*(F), \gamma_F)$ to that of $(\mathcal{TC}^*(E), \gamma_E)$ for each β . Thus, given an extremal KMS_β -state ϕ of $\mathcal{TC}^*(F)$, the composition $\phi \circ \theta$ is an extremal KMS_β -state of $\mathcal{TC}^*(E)$. Since p_E satisfies condition (ii) of lemma 3.10 in $\mathcal{TC}^*(E)$, we have $\phi(\theta(p_E)) = \phi \circ \theta(p_E) = 1$ for each extremal ϕ , and so $\theta(p_E)$ satisfies condition (ii) of lemma 3.10 in $\mathcal{TC}^*(F)$. So, writing η for the cyclic vector in the GNS-space $\mathcal{H}_{\varphi \circ \theta}$, for $b \in \mathcal{TC}^*(E)$ we have

$$\varphi \circ \theta(b) = \langle \pi_{\varphi \circ \theta}(b)\eta, \eta \rangle.$$

Hence, by uniqueness of the GNS construction, $\pi_\varphi \circ \theta$ is unitarily equivalent to $\pi_{\varphi \circ \theta}$. Since $\pi_\varphi(\theta(b))$ is a minimal projection if and only if $\pi_{\varphi \circ \theta}(b)$ is a minimal projection, $\theta(p_E)$ satisfies condition (i) of lemma 3.10. Thus by uniqueness, $\theta(p_E) = p_F$.

Considering the diagram

$$\begin{array}{ccccc} X(E) & \xrightarrow{\psi_E} & \mathcal{TC}^*(E)_1 p_E & \hookrightarrow & \mathcal{TC}^*(E) \\ \theta_X \downarrow & & \downarrow \theta & & \downarrow \theta \\ X(F) & \xrightarrow{\psi_F} & \mathcal{TC}^*(F)_1 p_F & \hookrightarrow & \mathcal{TC}^*(F), \end{array} \tag{*}$$

we show that the map θ_X that makes the diagram \star commute satisfies (3.8). For $\xi \in X(E)$,

$$\begin{aligned} \iota_{X(F)}(\theta_X(\xi))p_F &= \iota_{X(F)}(\psi_F^{-1} \circ \theta \circ \psi_E(\xi))p_F \\ &= \iota_{X(F)}(\psi_F^{-1} \circ \theta(\iota_{X(E)}(\xi)p_E))p_F. \end{aligned} \tag{3.9}$$

We already saw that $\theta(\mathcal{TC}^*(E)_1) = \mathcal{TC}^*(F)_1$ and $\theta(p_E) = \theta(p_F)$. Consequently, $\theta(\iota_{X(E)}(\xi)p_E) = \theta(\iota_{X(E)}(\xi))p_F \in \mathcal{TC}^*(F)_1 p_F$, so there exists $\xi' \in X(F)$ such that

$$\theta(\iota_{X(E)}(\xi)p_E) = \iota_{X(F)}(\xi')p_F = \psi_F(\xi').$$

Hence, we obtain

$$\iota_{X(F)}(\theta_X(\xi)p_F) \stackrel{(3.9)}{=} \iota_{X(F)}(\xi')p_F = \theta(\iota_{X(E)}(\xi)p_E),$$

that is (3.8). Since the maps ψ_E and ψ_F in the diagram are isomorphisms, we deduce that (θ_X, θ_M) is an isomorphism of C^* -correspondences. \square

4. Local reconstruction of topological graphs

In this section, we investigate how to recover a compact topological graph up to local conjugacy in the sense of Davidson–Roydor [7] from its Hilbert bimodule. We arrived at this result independently, but subsequently discovered that it can be recovered from Davidson and Roydor’s results about tensor algebras. We thank

both Adam Dor-On and the anonymous referees for bringing these results to our attention. We include a proof here because we feel that the direct passage from the bimodule to the local conjugacy class of the graph is more elementary than the approach that passes through the tensor algebra.

We first recall Davidson and Roydor’s notion of local conjugacy, which in turn is based on Davidson and Katsoulis’ notion of local conjugacy for local homeomorphisms [6].

DEFINITION 4.1 [7, Definition 4.3]. *Let E and F be topological graphs. We say that E and F are locally conjugate, and write $E \cong_{\text{loc}} F$ if there exists a homeomorphism $\phi^0 : E^0 \rightarrow F^0$ such that for each $v \in E^0$ there is a neighbourhood U of v and a homeomorphism $\phi_U^1 : E^1U \rightarrow F^1\phi^0(U)$ such that*

$$r_F \circ \phi_U^1 = \phi^0 \circ r_E|_{E^1U} \quad \text{and} \quad s_F \circ \phi_U^1 = \phi^0 \circ s_E|_{E^1U}. \tag{4.1}$$

Our main result in this section is that if compact topological graphs have isomorphic graph modules, then they are locally conjugate. For that, we need to collect some technical lemmas first.

LEMMA 4.2. *Let (e_1, \dots, e_k) be the standard basis for \mathbb{C}^k . If $\{x_1, \dots, x_k\} \subset \mathbb{C}^k$ is a basis, there exists a permutation σ of $\{1, \dots, k\}$ such that $\langle x_i, e_{\sigma(i)} \rangle \neq 0$ for all $i = 1, \dots, k$.*

Proof. Let A be the $k \times k$ matrix with i th column x_i , and let S_k be the symmetric group. Since $\{x_1, \dots, x_k\}$ are linearly independent, A is invertible. Therefore,

$$\det(A) = \sum_{\sigma \in S_k} \left(\text{sgn}(\sigma) \prod_{i=1}^k A_{i,\sigma(i)} \right) \neq 0.$$

Hence there exists $\sigma \in S_k$ such that $\prod_{i=1}^k A_{i,\sigma(i)} \neq 0$, and hence

$$\langle x_i, e_{\sigma(i)} \rangle = A_{i,\sigma(i)} \neq 0$$

for all $i = 1, \dots, k$. □

LEMMA 4.3. *Let E be a compact topological graph. Suppose that $h \in C(E^0, [0, 1])$ and $g_1, \dots, g_k \in X(E)$ satisfy*

- (1) $\langle g_i, g_j \rangle_{C(E^0)} = \delta_{i,j} h$;
- (2) $\overline{X(E) \cdot h} \subseteq \overline{\text{span}\{g_i \cdot a : i \leq k, a \in C(E^0)\}}$; and
- (3) For each $i \leq k$, there exists a continuous function $\alpha_i : \text{supp}(h) \rightarrow E^0$ such that for every $a \in C(E^0)$ and for any function $\tilde{a} \in C(E^0)$ such that $\tilde{a}|_{\text{supp}(h)} = a \circ \alpha_i$, we have $a \cdot g_i = g_i \cdot \tilde{a}$.

Then, for each $v \in \text{supp}^\circ(h)$, there exist a neighbourhood $W \subseteq \text{supp}(h)$ of v , s -sections $Z_e, e \in E^1v$ as in lemma 2.1 and a bijection $\sigma : \{1, \dots, k\} \rightarrow E^1v$ such that each $\alpha_i = r \circ (s|_{Z_{\sigma(i)}})^{-1}$ on W .

Proof. Fix $v \in \text{supp}^\circ(h)$. For $i \leq k$, regard $g_i|_{E^1v}$ as an element of $\ell^2(E^1v)$. Then for $i, j \leq k$,

$$\langle g_i|_{E^1v}, g_j|_{E^1v} \rangle = \sum_{e \in E^1v} \overline{g_i|_{E^1v}(e)} g_j|_{E^1v}(e) = \langle g_i, g_j \rangle(v) = \delta_{i,j} h(v).$$

Since $h(v) \neq 0$ it follows that $\{g_i|_{E^1v} : i \leq k\}$ is linearly independent. We claim that $\text{span}\{g_i|_{E^1v} : i \leq k\} = \ell^2(E^1v)$. For this, note that $\text{res} : \xi \mapsto \xi|_{E^1v}$ is a norm-decreasing linear map from $X(E) \cdot h$ to $\ell^2(E^1v)$, and is surjective since $h(v) \neq 0$. So condition (2) gives

$$\begin{aligned} \ell^2(E^1v) &\subseteq \text{res}(\overline{\text{span}\{g_j \cdot a : j \leq k, a \in C(E^0)\}}) \\ &\subseteq \overline{\text{span}\{a(v)g_i|_{E^1v} : a \in C_0(E^0), i \leq k\}} = \text{span}\{g_i|_{E^1v} : i \leq k\}. \end{aligned}$$

Hence $|E^1v| = k$, and so lemma 4.2 yields a bijection $\sigma : \{1, \dots, k\} \rightarrow E^1v$ such that $g_i(\sigma(i)) \neq 0$ for all $i = 1, \dots, k$.

We claim that whenever $g_i(e) \neq 0$, the function α_i of (3) satisfies $\alpha_i(s(e)) = r(e)$. To see this, we prove the contrapositive. So suppose that $\alpha_i(s(e)) \neq r(e)$. By Tietze’s theorem there exists $a \in C(E^0)$ such that $a(r(e)) = 1$ and $a(\alpha_i(s(e))) = 0$. The function $a \circ \alpha_i|_{\text{supp}h}$ is a continuous function on the compact set $\text{supp}(h)$, so by Tietze’s theorem there is a function $\tilde{a} \in C(E^0)$ that extends $a \circ \alpha_i|_{\text{supp}h}$. Since $s(e) \in \text{supp}(h)$ we have $\tilde{a}(s(e)) = a(\alpha_i(s(e)))$, and so condition (3) gives

$$0 = g_i(e)a(\alpha_i(s(e))) = (g_i \cdot \tilde{a})(e) = (a \cdot g_i)(e) = a(r(e))g_i(e) = g_i(e),$$

proving the claim.

Fix a neighbourhood W of v and s -sections $Z_e, e \in E^1v$ as in lemma 2.1. Since the g_i are continuous and each $g_i(\sigma(i)) \neq 0$, for each i there is a neighbourhood $\sigma(i) \in E^1$ on which g_i is everywhere nonzero. Shrinking the $Z_{\sigma(i)}$ and W appropriately, we may therefore assume that $Z_{\sigma(i)} \subseteq \text{supp}g_i$ for each i and $W \subseteq \text{supp}(h)$. Fix $w \in W$. Then $(s|_{Z_{\sigma(i)}})^{-1}(w) \in \text{supp}g_i$ and by the claim above $\alpha_i(w) = \alpha_i(s \circ (s|_{Z_{\sigma(i)}})^{-1}(w)) = r((s|_{Z_{\sigma(i)}})^{-1}(w))$ as needed. \square

REMARK 4.4. The combination of (1) and (2) of lemma 4.3 actually implies the stronger condition (2’) that $\overline{X(E) \cdot h} = \overline{\text{span}\{g_j \cdot a : j \leq k, a \in C(E^0)\}}$. To see this, recall from the proof of [31, Proposition 2.31] that each $x \in X(E)$ can be written as $x = y \cdot \langle y, y \rangle_{C_0(E^0)}$ where y is the upper right-hand entry of the matrix $\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}^{1/3}$. Applying this with $x = g_j$ we obtain

$$\langle y, y \rangle_{C_0(E^0)} = \langle g_j, g_j \rangle_{C_0(E^0)}^{1/3} = h^{1/3} \in \overline{C_0(E^0)h},$$

and so $g_j = y \cdot \langle y, y \rangle_{C_0(E^0)} \in X(E) \cdot \overline{C_0(E^0)h} \subseteq \overline{X(E) \cdot h}$. But since (2) is easier to check than (2’) we have stated the lemma with the formally weaker hypothesis.

THEOREM 4.5. *Let E and F be compact topological graphs. Suppose that $X(E) \cong X(F)$ as Hilbert bimodules. Then $E \cong_{\text{loc}} F$.*

Proof. Let $\theta = (\theta^0, \theta^1)$ be a bimodule isomorphism from $X(F)$ to $X(E)$, so $\theta^1 : X(F) \rightarrow X(E)$ and $\theta^0 : C(F^0) \rightarrow C(E^0)$ preserve the bimodule structure. Let

$\phi^0 = \widehat{\theta}^0 : E^0 \rightarrow F^0$ be the Gelfand transform of θ^0 which is a homeomorphism between E^0 and F^0 . Fix $v \in E^0$. Applying lemma 2.1 to $\phi^0(v)$ we obtain an open neighbourhood $W' \subseteq F^0$ of $\phi^0(v)$ and s-sections Z'_e for $e \in F^1\phi^0(v)$ such that $F^1W' = \bigsqcup_{e \in F^1\phi^0(v)} Z'_e$. Since F^0 is normal, Urysohn's lemma gives a function $h' \in C_c(W', [0, 1])$ (in particular, the compact set $\text{supp}(h)$ is contained in W') such that $h'(\phi^0(v)) = 1$. For each $e \in F^1\phi^0(v)$, define $g'_e \in C_0(Z'_e) \subseteq X(F)$ by $g'_e = \sqrt{h' \circ s|_{Z'_e}}$. Define $\alpha'_e : \text{supp}(h') \rightarrow F^0$ by $\alpha'_e = r_F \circ (s_F|_{Z'_e})^{-1}$. Direct computation shows that h' , the g'_e and the α'_e satisfy condition (1) of lemma 4.3. To see that they also satisfy condition (2), fix $\xi \in X(F) = C(F^1)$ and $e \in F^1\phi^0(v)$. If $f \in Z'_e$. Then

$$\begin{aligned} (\xi \cdot h')(f) &= \xi((s_F|_{Z'_e})^{-1}(s_F(f)))\sqrt{h'(s_F(f))}\sqrt{h'(s_F(f))} \\ &= \left(g'_e(f) \cdot (\xi \circ (s_F|_{Z'_e})^{-1}\sqrt{h'})\right)(f). \end{aligned}$$

For each $e \in F^1\phi^0(v)$, let $a_e := (\xi \circ (s_F|_{Z'_e})^{-1}\sqrt{h'}) \in C_0(W') \subseteq C(F^0)$. Since the Z'_e are disjoint and $\text{supp}(\xi \cdot h') \subseteq F^1W' = \bigsqcup_{e \in F^1\phi^0(v)} Z'_e$, we can write $\xi \cdot h' = \sum_{e \in F^1\phi^0(v)} g'_e \cdot a_e$. This gives condition (2). For condition (3), fix $a' \in C(F^0)$ and suppose that $\tilde{a}' \in C(F^0)$ extends $a' \circ \alpha'_e$. We must show that $(a' \cdot g'_e)(f) = (g'_e \cdot \tilde{a}')(f)$ for all $f \in F^1$. We consider two cases. First suppose that $f \notin \text{supp}(g'_e)$, then $(a' \cdot g'_e)(f) = a'(r(f))g'_e(f) = 0 = g'_e(f)\tilde{a}'(s(f)) = (g'_e \cdot \tilde{a}')(f)$. Now suppose that $f \in \text{supp}(g'_e)$. Then $f \in Z'_e$, which implies that $\alpha'_e(s(f)) = r(f)$; and $s(f) \in \text{supp}(h')$, giving $\tilde{a}'(s(f)) = a'(\alpha'_e(s(f))) = a'(r(f))$. Hence $(a' \cdot g'_e)(f) = a'(r(f))g'_e(f) = \tilde{a}'(s(f))g'_e(f) = (g'_e \cdot \tilde{a}')(f)$.

Let $h := \theta^0(h') = h' \circ \phi^0$ and for each $e \in F^1\phi^0(v)$, let $g_e := \theta^1(g'_e)$ and $\alpha_e = (\phi^0)^{-1} \circ \alpha'_e \circ \phi^0$, defined on $(\phi^0)^{-1}(\text{supp}(h')) = \text{supp}(h) \subseteq E^0$. Now we claim that h , the g_e and the α_e also satisfy (1)–(3) of lemma 4.3. Since (θ^0, θ^1) is a bimodule isomorphism, conditions (1) and (2) follow from straightforward calculations using that h' , the g'_e and the α'_e satisfy (1) and (2). We show that condition (3) holds. For this, fix $a \in C(E^0)$ and let $a' := (\theta^0)^{-1}(a) \in C(F^0)$. Suppose that $\tilde{a} \in C(E^0)$ extends $a \circ \alpha_e$. We claim that $\tilde{a}' := (\theta^0)^{-1}(\tilde{a})$ extends $a' \circ \alpha'_e|_{\text{supp}(h')}$. To see this, note first that, by definition of h , we have $\text{supp}(h) = (\phi^0)^{-1}(\text{supp}(h'))$. Also, by definition of α_e , we have $\alpha_e \circ (\phi^0)^{-1} = (\phi^0)^{-1} \circ \alpha'_e$. So for $w \in \text{supp}(h')$,

$$\begin{aligned} (\theta^0)^{-1}(\tilde{a})(w) &= \tilde{a}((\phi^0)^{-1}(w)) = (a \circ \alpha_e)((\phi^0)^{-1}(w)) \\ &= (a \circ (\phi^0)^{-1}) \circ \alpha'_e(w) = a' \circ \alpha'_e(w). \end{aligned}$$

Consequently, condition (3) for h' , g'_e and α'_e gives $a' \cdot g'_e = g'_e \cdot \tilde{a}'$. Hence

$$a \cdot g_e = \theta^1(a' \cdot g'_e) = \theta^1(g'_e \cdot \tilde{a}') = \theta^1(g'_e) \cdot \theta^0(\tilde{a}') = g_e \cdot \tilde{a}.$$

Thus condition (3) holds for h , the g_e and the α_e .

As $h(v) = h'(\phi^0(v)) = 1$, we have $v \in \text{supp}^\circ(h)$, so by lemma 4.3, there exists an open neighbourhood $W \subseteq (\phi^0)^{-1}(W')$ of v and s-sections $Z_f, f \in E^1v$ as in lemma 2.1, and a bijection $\sigma : F^1\phi^0(v) \rightarrow E^1v$ such that $\alpha_e = r_E \circ (s_E|_{Z_{\sigma(e)}})^{-1}$ on W for each $e \in F^1\phi^0(v)$.

Define $\phi_W^1 : E^1W \rightarrow F^1\phi^0(W)$ by $\phi_W^1|_{Z_{\sigma(e)}} = (s_F|_{Z'_e})^{-1} \circ \phi^0 \circ s_E$ for each $e \in F^1\phi^0(v)$. Since each $\phi_W^1|_{Z_{\sigma(e)}}$ is a homeomorphism and their images are disjoint, ϕ_W^1 is a homeomorphism by the pasting lemma. We claim that ϕ_W^1 satisfies the conditions necessary for local conjugacy, that is, $r_F \circ \phi_W^1 = \phi^0 \circ r_E|_{E^1W}$ and $s_F \circ \phi_W^1 = \phi^0 \circ s_E|_{E^1W}$. That $s_F \circ \phi_W^1 = \phi^0 \circ s_E|_{E^1W}$ is by definition of ϕ_W^1 . Now, fix $\bar{f} \in E^1W$. There exists a unique $f \in E^1v$ such that $\bar{f} \in Z_f$. Since σ is a bijection there is a unique $e \in F^1\phi^0(v)$ such that $\sigma(e) = f$, hence $\bar{f} \in Z_{\sigma(e)}$. Thus, we have

$$\begin{aligned} \phi^0 \circ r_E(\bar{f}) &= \phi^0(r_E((s|_{Z_{\sigma(e)}})^{-1}(s_E(\bar{f})))) = \phi^0(\alpha_e(s_E(\bar{f}))) \\ &= \phi^0((\phi^0)^{-1} \circ \alpha'_e \circ \phi^0(s_E(\bar{f}))) = \alpha'_e(\phi^0(s_E(\bar{f}))) \\ &= r_F((s_F|_{Z'_e})^{-1}(\phi^0(s_E(\bar{f})))) = r_F((s_F|_{Z'_e})^{-1}(s_F(\phi_W^1(\bar{f}))). \end{aligned}$$

Since $\bar{f} \in Z_{\sigma(e)}$, we have $\phi_W^1(\bar{f}) = (s_F|_{Z'_e})^{-1}(\phi^0 \circ s_E(\bar{f})) \in Z'_e$, therefore $\phi^0 \circ r_E(\bar{f}) = r_F(\phi_W^1(\bar{f}))$. Hence E and F are locally conjugate. \square

REMARK 4.6. Theorem 4.5 is a corollary of [7, Theorem 4.5]: if E and F have isomorphic bimodules, then the tensor algebras of these bimodules are also isomorphic, and so Davidson and Roydor’s result implies that E and F are locally conjugate.

Our next corollary says that for topological graphs with totally disconnected vertex spaces, local conjugacy coincides with isomorphism. This is a consequence of the general theorem [7, Theorem 5.5], which says that local conjugacy coincides with isomorphism whenever the covering dimension of the vertex-spaces of the graphs involved is at most 1. We have included a proof since the argument is simpler for zero-dimensional spaces.

COROLLARY 4.7. *Let E and F be compact topological graphs and suppose that E^0 is totally disconnected. Let γ^E and γ^F be the gauge actions on $\mathcal{TC}^*(E)$ and $\mathcal{TC}^*(F)$ and let $M_E = \iota_{C(E^0)}(C(E^0))$ and $M_F = \iota_{C(F^0)}(C(F^0))$. Suppose that $(\mathcal{TC}^*(E), \gamma^E, M_E) \cong (\mathcal{TC}^*(F), \gamma^F, M_F)$. Then E and F are isomorphic as topological graphs.*

Proof. Theorems 3.2 and 4.5 imply that E and F are locally conjugate. Let $\phi^0 : E^0 \rightarrow F^0$ be the map on vertices implementing the local conjugacy. Then for each $v \in E^0$ there exists a neighbourhood U_v of v and a homeomorphism $\phi_{U_v}^1 : E^1U_v \rightarrow F^1\phi^0(U_v)$ such that $r_F \circ \phi_{U_v}^1 = \phi^0 \circ r_E|_{E^1U_v}$ and $s_F \circ \phi_{U_v}^1 = \phi^0 \circ s_E|_{E^1U_v}$. Since E^0 is totally disconnected, we may suppose the (U_v) are compact open. The $(U_v)_v$ cover E^0 , so admit a finite subcover $(U_i)_{i=1}^n$. Put $V_1 = U_1$ and let $V_i = U_i \setminus \bigcup_{j=1}^{i-1} U_j$ for $i \geq 2$. Then the V_i are mutually disjoint compact open sets that cover E^0 and we can define the homeomorphisms $\phi_{V_i}^1 : E^1V_i \rightarrow F^1\phi^0(V_i)$ by the restriction of $\phi_{U_i}^1$ on E^1V_i . Note that $(E^1V_i)_i$ are mutually disjoint open sets that cover E^1 and, similarly, $(F^1\phi^0(V_i))_i$ are mutually disjoint open sets that cover F^1 . We define $\phi^1 : E^1 \rightarrow F^1$ by $\phi^1|_{E^1V_i} = \phi_{V_i}^1$. By the pasting lemma ϕ^1 is continuous and bijective, thus a homeomorphism. It only remains to verify that $r_F \circ \phi^1 = \phi^0 \circ r_E$ and $s_F \circ \phi^1 = \phi^0 \circ s_E$. We show this only for the range map, since it is analogous for the source map. If $f \in E^1$, then $f \in E^1V_i$ for some $i = 1, \dots, n$ and hence $(r_F \circ \phi^1)(f) = r_F(\phi_{V_i}^1(f)) = (\phi^0 \circ r_E)(f)$. \square

COROLLARY 4.8. *Let E and F be compact topological graphs. If $X(E) \cong X(F)$ as Hilbert bimodules, then $E \cong F$ as discrete directed graphs, via an isomorphism that implements a homeomorphism $E^0 \cong F^0$.*

Proof. By theorem 4.5, $E \cong_{\text{loc}} F$. Hence there is a homeomorphism $\phi^0 : E^0 \rightarrow F^0$ as in definition 4.1. Fix $v \in E^0$. There is a neighbourhood U of v and a homeomorphism $\phi_U^1 : E^1U \rightarrow F^1\phi^0(U)$ satisfying equations (4.1). Hence, for $w \in E^0$ we have $\phi_U^1(wE^1v) = \phi^0(w)F^1\phi^0(v)$. Since ϕ_U^1 is a bijection, $|wE^1v| = |\phi^0(w)F^1\phi^0(v)|$. Since v, w were arbitrary, it follows that there is a range-and-source-preserving bijection $\phi^1 : E^1 \rightarrow F^1$ such that (ϕ^0, ϕ^1) is an isomorphism of discrete graphs. \square

5. Example

In this section, we describe two nonisomorphic topological graphs whose graph correspondences are isomorphic (so that, in particular, the graphs are locally conjugate). This proves that a generalization of theorem 4.5 to arbitrary compact topological graphs is not possible.

Examples of locally conjugate local homeomorphisms that are not conjugate (so that the associated topological graphs are not isomorphic) appear in [6, Example 3.18]. Furthermore, since the graphs described below have covering dimension equal to 1, once we have established that they are locally conjugate, we could deduce that their Hilbert modules are isomorphic from [7, Theorem 5.5]. However, as our example is explicit and we are able to describe an explicit isomorphism of the Hilbert modules of the topological graphs involved, we present the details.

Let $E^0 = F^0 = F^1 = \mathbb{T}$, and let $E^1 = \mathbb{T} \times \{0, 1\}$. Define range and source maps in E, F by $r_E(z, j) = s_E(z, j) = z$ and $r_F(z) = s_F(z) = z^2$. Then E and F are not isomorphic as topological graphs: F^1 is connected and E^1 is not. We show that $X(E)$ is isomorphic to $X(F)$. For every $f \in C(E^1)$ and $z \in \mathbb{T}$,

$$\langle f, f \rangle(z) = \sum_{s_E(w,j)=z} |f(w, j)|^2 = |f(z, 1)|^2 + |f(z, 2)|^2,$$

therefore $\langle f, f \rangle \in C(E^0)$. Hence (2.2) gives $X(E) = C(E^1)$ and, analogously, $X(F) = C(\mathbb{T})$. Define $\psi : C(E^1) \rightarrow C(\mathbb{T}, \mathbb{C}^2)$ by

$$\psi(h)(z) = (h(z, 1), h(z, 2)) \quad \text{for all } h \in C(E^1), z \in \mathbb{T}.$$

This map is an isomorphism of vector spaces, with inverse given by

$$\phi(g)(z, j) = \pi_j(g(z)) \quad g \in C(\mathbb{T}, \mathbb{C}^2), (z, j) \in \mathbb{T} \times \{0, 1\}.$$

The map ψ induces a Hilbert $C(\mathbb{T})$ -bimodule structure on $C(\mathbb{T}, \mathbb{C}^2)$ as follows. Writing $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ for the inner product in \mathbb{C}^2 (linear in the second component), for $g_1, g_2 \in C(\mathbb{T}, \mathbb{C}^2)$ and $f \in C(\mathbb{T})$,

$$\begin{aligned} (f \cdot g_1)(z) &:= \psi(f \cdot \phi(g_1))(z) = f(z)g_1(z), \\ (g_1 \cdot f)(z) &:= \psi(\phi(g_1) \cdot f) = g_1(z)f(z), \quad \text{and} \\ \langle g_1, g_2 \rangle(z) &:= \langle \psi^{-1}(g_1), \psi^{-1}(g_2) \rangle(z) = \langle g_1(z), g_2(z) \rangle_{\mathbb{C}}. \end{aligned}$$

Hence ψ is an isomorphism $C(E^1) \rightarrow C(\mathbb{T}, \mathbb{C}^2)$.

For $t \in [0, 2\pi]$, define

$$\mathcal{U}_t = \begin{pmatrix} e^{it/2} \cos(t/4) & -e^{it/2} \sin(t/4) \\ \sin(t/4) & \cos(t/4) \end{pmatrix} \in \mathcal{U}(\mathbb{C}^2).$$

For $f \in C(\mathbb{T})$ define $\tilde{\rho}(f) : [0, 2\pi] \rightarrow \mathbb{C}^2$ by

$$\tilde{\rho}(f)(t) = \mathcal{U}_t \begin{pmatrix} f(e^{it/2}) \\ f(-e^{it/2}) \end{pmatrix}.$$

Then $\tilde{\rho}$ is continuous, and

$$\tilde{\rho}(f)(0) = \begin{pmatrix} f(1) \\ f(-1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(-1) \\ f(1) \end{pmatrix} = \tilde{\rho}(f)(2\pi).$$

Hence $\tilde{\rho}$ determines a continuous function $\rho(f) \in C(\mathbb{T}, \mathbb{C}^2)$ by $\rho(f)(e^{it}) = \tilde{\rho}(f)(t)$. Identifying $X(F)$ with $C(\mathbb{T})$ as above, we obtain maps,

$$X(F) \xrightarrow{\rho} C(\mathbb{T}, \mathbb{C}^2) \xrightarrow{\phi} X(E).$$

We show that ρ is an isomorphism of Hilbert bimodules. We start by proving that it is isometric. For $f_1, f_2 \in C(\mathbb{T})$, we calculate

$$\begin{aligned} \langle \rho(f_1), \rho(f_2) \rangle (e^{it}) &= \langle \rho(f_1)(e^{it}), \rho(f_2)(e^{it}) \rangle_{\mathbb{C}} \\ &= \left\langle \mathcal{U}_t \begin{pmatrix} f_1(e^{it/2}) \\ f_1(-e^{it/2}) \end{pmatrix}, \mathcal{U}_t \begin{pmatrix} f_2(e^{it/2}) \\ f_2(-e^{it/2}) \end{pmatrix} \right\rangle_{\mathbb{C}} \\ &= \left\langle \begin{pmatrix} f_1(e^{it/2}) \\ f_1(-e^{it/2}) \end{pmatrix}, \begin{pmatrix} f_2(e^{it/2}) \\ f_2(-e^{it/2}) \end{pmatrix} \right\rangle_{\mathbb{C}} \\ &= \langle f_1, f_2 \rangle (e^{it}). \end{aligned}$$

To see that ρ preserves the bimodule structure, fix $f \in X(F)$ and $a \in C(\mathbb{T})$. Then

$$\begin{aligned} \rho(f \cdot a)(e^{it}) &= \mathcal{U}_t \begin{pmatrix} f \cdot a(e^{it}) \\ f \cdot a(-e^{it}) \end{pmatrix} = \mathcal{U}_t \begin{pmatrix} f(e^{it})a(s_F(e^{it})) \\ f(-e^{it})a(s_F(-e^{it})) \end{pmatrix} \\ &= \mathcal{U}_t \begin{pmatrix} f(e^{i\pi t}) \\ f(-e^{i\pi t}) \end{pmatrix} a(e^{it}) = \rho(f)(e^{it})a(e^{it}) = (\rho(f) \cdot a)(e^{it}). \end{aligned}$$

Since the left and right actions on each of $X(E)$ and $X(F)$ coincide, $\rho(a \cdot f) = a \cdot \rho(f)$ as well. It remains to prove that ρ is surjective. For that, we use the Stone–Weierstrass theorem for Banach bundles [15, Corollary 4.3]. We must show that $\rho(C(\mathbb{T})) \subset C(\mathbb{T}, \mathbb{C}^2)$, when viewed as sections on the canonical bundle associated with $C(\mathbb{T}, \mathbb{C}^2)$, is fiberwise dense in that bundle. First, we define some sets. For $z \in \mathbb{T}$, let

$$I_z := \{f \in C(\mathbb{T}) : f(z) = 0\},$$

$$C(\mathbb{T}, \mathbb{C}^2)_z := \{h \in C(\mathbb{T}, \mathbb{C}^2) : \langle h, h \rangle \in I_z\} = \{h \in C(\mathbb{T}, \mathbb{C}^2) : h(z) = 0\}.$$

For $h \in C(\mathbb{T}, \mathbb{C}^2)$, the associated section of the bundle $\bigsqcup_{z \in \mathbb{T}} C(\mathbb{T}, \mathbb{C}^2)/C(\mathbb{T}, \mathbb{C}^2)_z$ is given by $\hat{h} : z \mapsto h + C(\mathbb{T}, \mathbb{C}^2)_z$. The map $h \mapsto \hat{h}$ is an isometric isomorphism.

For $z \in \mathbb{T}$ write $\text{ev}_z : C(\mathbb{T}, \mathbb{C}^2) \rightarrow C(\mathbb{T}, \mathbb{C}^2)/C(\mathbb{T}, \mathbb{C}^2)_z$ for the ‘evaluation’ map

$$\text{ev}_z(h) = h + C(\mathbb{T}, \mathbb{C}^2)_z \quad h \in C(\mathbb{T}, \mathbb{C}^2).$$

We must show that $\text{ev}_z(\rho(\widetilde{C}(\mathbb{T})))$ is dense in $C(\mathbb{T}, \mathbb{C}^2)/C(\mathbb{T}, \mathbb{C}^2)_z$ for each $z \in \mathbb{T}$. For this, fix $z = e^{it} \in \mathbb{T}$ and $h + C(\mathbb{T}, \mathbb{C}^2)_z \in C(\mathbb{T}, \mathbb{C}^2)/C(\mathbb{T}, \mathbb{C}^2)_z$. We show that there exists $f \in C(\mathbb{T})$ such that $h - \rho(f) \in C(\mathbb{T}, \mathbb{C}^2)_z$ or, in other words, $h(e^{it}) = \rho(f)(e^{it})$. First, we solve the equation,

$$\mathcal{U}_t \begin{pmatrix} x \\ y \end{pmatrix} = h(e^{it}). \tag{5.1}$$

Since \mathcal{U}_t is unitary, there is a unique solution $(x_0, y_0) \in \mathbb{C}^2$. Let $A = \{-e^{it/2}, e^{it/2}\}$ and define $\tilde{f} : A \rightarrow \mathbb{C}$ by $\tilde{f}(e^{it/2}) = x_0$ and $\tilde{f}(-e^{it/2}) = y_0$. By Tietze’s extension theorem, there exists $f \in C(\mathbb{T})$ such that $f(e^{it/2}) = x_0$ and $f(-e^{it/2}) = y_0$. This implies that $\rho(f)(e^{it}) = h$. Therefore, by Corollary 4.3 of [15] $\rho(C(\mathbb{T}))$ is dense in $C(\mathbb{T}, \mathbb{C}^2)$ and since the former is closed, they are equal.

6. A cohomological obstruction

We finish with a discussion of the relationship between isomorphism of topological graphs, isomorphism of the associated C^* -algebraic triples, isomorphism of their graph bimodules, and local conjugacy of topological graphs. To begin this discussion, we make an observation about the structure of graph bimodules in terms of the description of vector bundles using local trivialisations and transition functions. This is essentially a rephrasing of Kaliszewski *et al.* characterization of graph modules as those admitting a continuous choice of basis [19].

PROPOSITION 6.1. *Let E be a compact topological graph. Then there exists a local trivialization of the canonical Hilbert bundle \mathcal{E} associated with the Hilbert $C(E^0)$ -module $X(E)$ whose transition functions take values in the permutation matrices.*

Proof. By lemma 2.1 there exists a finite open cover $\{U_i\}_{i \in F}$, of the vertex space E^0 , such that, for each $i \in F$ there exists $k(i) \in \mathbb{N}$ and s -sections $Z_j^{(i)}$, $j \leq k(i)$ such that the source map restricts to a homeomorphism $Z_j^{(i)} \cong U_i$ for each j , and $E^1U_i = \bigsqcup_{j=1}^{k(i)} Z_j^{(i)}$.

Fix $i \in F$ and for each $j \leq k(i)$, define $\phi_i^j : Z_j^{(i)} \rightarrow U_i \times \{1, \dots, k(i)\}$ by $\phi_i^j(e) = (s|_{Z_j^{(i)}}(e), j)$. Then each ϕ_i^j is a homeomorphism of $Z_j^{(i)}$ onto $U_i \times \{j\}$. Hence, by the pasting lemma, $\phi_i := \bigsqcup_{j=1}^{k(i)} \phi_i^j : E^1U_i \rightarrow U_i \times \{1, \dots, k(i)\}$ is a homeomorphism.

If $U_i \cap U_j \neq \emptyset$, then we may consider the composition $\phi_j \circ \phi_i^{-1}$ with domain $(U_i \cap U_j) \times \{1, \dots, k(i)\}$. Let $(x, \ell) \in (U_i \cap U_j) \times \{1, \dots, k(i)\}$. Then $\phi_i^{-1}(x, \ell) = (s|_{Z_\ell^{(i)}})^{-1}(x)$, and there exists a $m \in \{1, \dots, k(j)\}$ such that $(s|_{Z_\ell^{(i)}})^{-1}(x) \in Z_m^{(j)}$, since otherwise $(s|_{Z_\ell^{(i)}})^{-1}(x) \notin E^1U_j$, contradicting $x \in U_j$. This m is unique

because the $Z_n^{(j)}$, $n \in \{1, \dots, k(j)\}$ are disjoint. Hence

$$\phi_j \circ \phi_i^{-1}(x, \ell) = \phi_j((s|_{Z_\ell^{(i)}})^{-1}(x)) = (x, m).$$

So, there is a function $\sigma_x^{j,i} : \{1, \dots, k(i)\} \rightarrow \{1, \dots, k(j)\}$ such that $(s|_{Z_\ell^{(i)}})^{-1}(x) \in Z_{\sigma_x^{j,i}(\ell)}^{(j)}$ for all $\ell \in \{1, \dots, k(i)\}$.

We claim that $\sigma_x^{j,i}$ is a bijection. Indeed, if $\sigma_x^{j,i}(\ell) = \sigma_x^{j,i}(\ell')$, then $\phi_j \circ \phi_i^{-1}(x, \ell) = (x, \sigma_x^{j,i}(\ell)) = (x, \sigma_x^{j,i}(\ell')) = \phi_j \circ \phi_i^{-1}(x, \ell')$. Since ϕ_j and ϕ_i are injective, $\ell = \ell'$, thus $\sigma_x^{j,i}$ is injective.

We now claim that $\sigma_x^{j,i}$ is surjective. For this, fix $m \in \{1, \dots, k(j)\}$. Then $\phi_j^{-1}(x, m) = (s|_{Z_m^{(j)}})^{-1}(x) \in E^1(U_i \cap U_j) \subseteq E^1U_i$. Thus, there is a unique $m' \in \{1, \dots, k(i)\}$ such that $\phi_j^{-1}(x, m) \in Z_{m'}^{(i)}$, so $\phi_j^{-1}(x, m) = (s|_{Z_{m'}^{(i)}})^{-1}(x)$. Hence

$$\phi_j \circ \phi_i^{-1}(x, m') = \phi_j((s|_{Z_{m'}^{(i)}})^{-1}(x)) = \phi_j(\phi_j^{-1}(x, m)) = (x, m).$$

This implies that $\sigma_x^{j,i}(m') = m$ and $\sigma_x^{j,i}$ is a bijection. In particular, $k(i) = k(j)$.

Recall the canonical Hilbert bundle \mathcal{E} associated with the bimodule $X(E)$ discussed in the preliminaries. For each i the map ϕ_i induces a right-Hilbert $C_0(U_i)$ -module isomorphism $\phi_i^* : C_0(U_i \times \{1, \dots, k(i)\}) \cong C_0(U_i, \mathbb{C}^{k(i)}) \rightarrow X(E) \cdot C_0(U_i)$ by $\phi_i^*(\xi) = \xi \circ \phi_i$. This induces a vector-bundle isomorphism $\psi_i : U_i \times \mathbb{C}^{k(i)} \cong \mathcal{E}|_{U_i}$ satisfying $\psi_i(x, e_m) = \phi_i^{-1}(x, m)$ for all $x \in U_i$ and $m \leq k(i)$.

The maps ψ_i , $i \in F$ are a local trivialization of \mathcal{E} , and for $x \in U_i \cap U_j$ and $\ell \leq k(i)$, the transition function $\psi_j^{-1} \circ \psi_i$ satisfies $\psi_j^{-1} \circ \psi_i(x, e_\ell) = \psi_j^{-1}(\phi_i^{-1}(x, \ell)) = \psi_j^{-1}(\phi_j^{-1}(\sigma_x^{j,i}(\ell))) = (x, e_{\sigma_x^{j,i}(\ell)})$. That is, the matrix implementing $\psi_j^{-1} \circ \psi_i$ in the fibre over x is precisely the permutation matrix corresponding to $\sigma_x^{j,i}$. \square

COROLLARY 6.2. *Let K be a second-countable compact Hausdorff space, and let X be a right-Hilbert $C(K)$ -module. The following are equivalent.*

- (1) X is isomorphic, as a right-Hilbert module, to the graph module of a compact topological graph E with $E^0 \cong K$.
- (2) X admits a continuous choice of finite orthonormal bases in the sense of Kaliszewski–Quigg–Patani [19].
- (3) The canonical vector bundle \mathcal{E} over K associated with X is finite rank and admits a local trivialization whose transition functions take values in the permutation matrices.

Proof. That (1) and (2) are equivalent follows from [19, Theorem 6.4]. Proposition 6.1 gives (1) \implies (3). So it suffices to show that if \mathcal{E} has a local trivialization whose transition functions take values in permutation matrices, then it admits a global choice of orthonormal bases. For this, fix such a local trivialization, say $\{U_i\}_{i \in F}$ is an open cover of K and for each i , we have a bundle isomorphism $\psi_i : U_i \times \mathbb{C}^{k(i)} \rightarrow \mathcal{E}|_{U_i}$ so that the transition functions $\psi_i^{-1} \circ \psi_j$ are permutation-matrix valued. Fix $x \in K$. If $i, j \in F$ satisfy $x \in U_i \cap U_j$ then since

$\psi_i^{-1} \circ \psi_j$ takes values in permutation matrices, there is a permutation σ such that $\psi_i^{-1} \circ \psi_j(x, e_\ell) = (x, e_{\sigma(\ell)})$ for all $\ell \leq k(i)$. Hence $\{\psi_i(x, e_1), \dots, \psi_i(x, e_{k(i)})\} = \{\psi_j(x, e_{\sigma(1)}), \dots, \psi_j(x, e_{\sigma(k(i))})\} = \{\psi_j(x, e_1), \dots, \psi_j(x, e_{k(i)})\}$. So there is a well-defined set-valued map $\psi : K \rightarrow \mathcal{P}(\mathcal{E})$ such that, for each i , the restriction $\psi|_{U_i}$ is the map $x \mapsto \{\psi_i(x, e_1), \dots, \psi_i(x, e_{k(i)})\}$. The set $B := \bigcup_{x \in K} \psi(x)$ is a subset of \mathcal{E} . Each $B \cap \mathcal{E}_x = \psi(x)$ is an orthonormal basis for \mathcal{E}_x . For each i , the restriction π_i of the bundle map to $B \cap \mathcal{E}|_{U_i}$ satisfies $\pi_i(\psi_i(x, e_\ell)) = x$ and for each x, ℓ the set $\psi_i(U_i \times \{e_\ell\}) \cong U_i \times \{e_\ell\}$ is an open neighbourhood of (x, e_ℓ) on which π_i restricts to a homeomorphism. So the bundle map restricts to a local homeomorphism $B \rightarrow K$. That is B is a continuous choice of basis. \square

REMARK 6.3. Our results give the following string of implications for a pair E and F of compact topological graphs:

$$\begin{aligned} E \cong F &\xrightarrow{(1)} (\mathcal{TC}^*(E), \gamma^E, M_E) \cong (\mathcal{TC}^*(F), \gamma^F, M_F) \\ &\iff X(E) \cong X(F) \\ &\xrightarrow{(2)} E \cong_{\text{loc}} F. \end{aligned}$$

Section 5 details a pair E, F of topological graphs that have isomorphic modules (and so, in particular, are also locally conjugate) but are not isomorphic as topological graphs. Thus implication (1) does not admit a converse.

Though we do not have a counterexample, it also seems unlikely that implication (2) admits a converse due to the following cohomological considerations.

Corollary 6.2 shows that graph modules of compact topological graphs, regarded purely as right modules, can be characterized as the finite-rank vector bundles that admit a local trivialization whose transition functions take values in the permutation matrices. Any (right) graph module can be made into a graph bimodule by making the left action identical to the right action; this amounts to making all of the edges in the graph loops.

Hence, the existence of a nontrivial vector bundle over a compact space whose transition functions are permutation matrices would provide a counterexample to the converse of implication (2): given such a bundle B , say of rank k , corollary 6.2 yields a topological graph E whose graph module is the module of sections of B . By construction, the range and source maps on E coincide, and each vertex of E is the base of k loop-edges. Hence E is locally conjugate to the topological graph F with $F^0 = E^0$, and $F^1 = E^0 \times \{1, \dots, k\}$ as a topological space, and with range and source maps given by $(v, i) \mapsto v$. Since the graph module of F is the module of sections of a trivial bundle, it follows that E and F are locally conjugate topological graphs with non-isomorphic graph modules.

The larger question is how to characterize exactly what additional cohomological data should be paired with a local conjugacy of graphs to determine an isomorphism of graph bimodules. The question seems complicated since one cannot directly apply the standard classification theory for vector bundles – a more refined cohomology is required that takes into account the left action of $C(E^0)$ on $X(E)$, likely via the functions α_i appearing in lemma 4.3(3).

The discussion above is consistent with the results of [7]. As mentioned in §4, their theorem says that if the covering dimension of E^0 is at most 1, then local conjugacy of topological graphs implies isomorphism of their Hilbert modules. Our proposed strategy for constructing locally conjugate graphs with non-isomorphic Hilbert bimodules is outside the scope of that theorem: if the covering dimension of E^0 is at most 1, then there are no nontrivial vector bundles over E^1 —with transition functions taking values in permutation matrices or otherwise.

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