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NON-EMBEDDINGS OF THE REAL FLAG MANIFOLDS $\mathbf{R}F(1, 1, n-2)$

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Abstract

This paper gives non-embeddings and non-immersions for the real flag manifolds $\mathbf{R}F(1, 1, n-2)$, $n > 3$ and shows that Lam's immersions for $n = 4$ and 5 and Stong's result for $n = 6$ are the best possible.

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1. Introduction

The real flag manifold

$$\mathbf{R}F(1, 1, n-2) = \frac{0(n)}{0(1) \times 0(1) \times 0(n-2)}, \quad n \geq 3$$

is a smooth connected compact homogeneous manifold of dimension $2n-3$.

In [4, Corollary 5.2], Lam's immersion result on general real flag manifolds gives better results than Whitney's [11, 12] in the case of $\mathbf{R}F(1, 1, n-2)$ only for $n = 4, 5$ and 6.

We shall use dual Stiefel-Whitney classes of $\mathbf{R}F(1, 1, n-2)$ to prove the following theorem:

THEOREM. (a) *For $2^{r-1} + 2 \leq n \leq 2^r - 1$ and $s = 2^r$, we have:*

$$\mathbf{R}F(1, 1, n-2) \not\subset \mathbf{R}^{2s-2}, \quad \mathbf{R}F(1, 1, n-2) \not\subseteq \mathbf{R}^{2s-3};$$

(b) $\mathbf{R}F(1, 1, n-2) \not\subset \mathbf{R}^{2n-2}$, $\mathbf{R}F(1, 1, n-2) \not\subseteq \mathbf{R}^{2n-3}$, *if $n = 2^{r-1}$;*
(c) $\mathbf{R}F(1, 1, n-2) \not\subset \mathbf{R}^{3n-5}$, $\mathbf{R}F(1, 1, n-2) \not\subseteq \mathbf{R}^{3n-6}$, *if $n = 2^{r-1} + 1$, where $X \subset Y$ denotes X embeds in Y, and $X \subseteq Y$ denotes X immerses in Y.*

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2. Proof of theorem

Let $F = \mathbf{R}F(1, 1, n-2)$. Then from [4], the tangent bundle of F is given by

$$\tau(F) = (\gamma_1 \otimes \gamma_2) \oplus (\gamma_1 \otimes \xi) \oplus (\gamma_2 \otimes \xi)$$

where γ_1 and γ_2 are the two canonical line bundles, ξ is the complementary $(n-2)$-plane bundle and $\gamma_1 \oplus \gamma_2 \oplus \xi$ is an n-plane trivial bundle, all over F. By considering $(\gamma_1 \oplus \gamma_2 \oplus \xi) \otimes (\gamma_1 \oplus \gamma_2 \oplus \xi)$ one sees that

$$\tau(F) \oplus (\gamma_1 \otimes \gamma_1) \oplus n\xi \oplus (\gamma_1 \otimes \gamma_2) \oplus (\gamma_2 \otimes \gamma_2)$$

is an n^2-plane trivial bundle, where $n\xi$ stands for the n-fold Whitney sum of ξ.

Taking the total Stiefel–Whitney classes and using the Whitney product formula, we have

$$w(F) = \bar{w}(n\xi)\bar{w}(\gamma_1 \otimes \gamma_2)$$

where $\bar{w}$ is the dual total Stiefel–Whitney class to w. Let $x = w_1(\gamma_1)$, $y = w_1(\gamma_2)$ be the first Stiefel–Whitney classes of γ_1 and γ_2, respectively. Put $\sigma_1 = x + y$ and $\sigma_2 = xy$. Then

$$(1) \qquad w(F) = (1 + \sigma_1 + \sigma_2)^n (1 + \sigma_1)^{-1}.$$

Note that from [1], $H^*(F; \mathbf{Z}_2)$ is generated by x and y subject to the relations $\bar{\sigma}_{n-1} = 0 = \bar{\sigma}_n$ so that $x^n = 0 = y^n$, where $\bar{\sigma}_i = \bar{\sigma}_i(x, y)$ denotes the i-th complete symmetric function in x and y. Also an additive basis for $H^*(F; \mathbf{Z})$ is the set $\{x^i y^j \mid 0 \le i \le n-1, 0 \le j \le n-2\}$, so that $\sigma_1^s \ne 0$, $1 \le s \le n-2$ and $\sigma_2^k \ne 0$, $1 \le k \le n-2$.

We now use the fact that if M^n is a smooth manifold of real dimension n, then $\bar{w}_k(M) \ne 0$ implies $M^n \not\subseteq \mathbf{R}^{n+k-1}$ and $M^n \not\subset \mathbf{R}^{n+k}$, (see [5, p. 120]).

Now if $s = 2^r$, we have $(1+\sigma_1+\sigma_2)^s = 1+\sigma_1^s+\sigma_2^s = 1+x^s+y^s+x^s y^s = 1$ since $s \ge n+1$ for $2^{r-1} \le n \le 2^r - 1$. Hence from (1) above, the total dual Stiefel-Whitney class of F is given by

$$(2) \qquad \bar{w}(F) = (1 + \sigma_1)(1 + \sigma_1 + \sigma_2)^{s-n}.$$

Hence

$$\bar{w}_{2s-2n+1}(F) = \sigma_1 \sigma_2^{s-n} \begin{cases} = 0, & \text{if } n = 2^{r-1}, 2^{r-1} + 1 \\ \ne 0, & \text{if } 2^{r-1} + 2 \le n \le 2^r - 1. \end{cases}$$

It follows that

$$\mathbf{R}F(1, 1, n-2) \not\subset \mathbf{R}^{2s-2}, \quad \mathbf{R}F(1, 1, n-2) \not\subseteq \mathbf{R}^{2s-3}$$

if $2^{r-1}+2 \leq n \leq 2^r - 1$ and $s = 2^r$. This proves part (a) of the theorem.

If $n = 2^{r-1}$, then (2) becomes $\bar{w}(\mathrm{F}) = (1+\sigma_1)(1+\sigma_1+\sigma_2)^n = 1+\sigma_1$, since $(1+\sigma_1+\sigma_2)^n = 1+x^n+y^n+x^ny^n = 1$. Hence $\bar{w}_1(\mathrm{F}) = \sigma_1 \neq 0$. This proves part (b) of the theorem.

If $n = 2^{r-1}+1$, then (2) above becomes $\bar{w}(\mathrm{F}) = (1+\sigma_1)(1+\sigma_1+\sigma_2)^{2^{r-1}-1} = (1+\sigma_1)\sum_{i=0}^{2^{r-1}-1}(1+\sigma_1)^{2^{r-1}-1-i}\sigma_2^i$. This implies that

$$
\begin{aligned}
\bar{w}_{n-2} &= \sum_{i=0}^{2^{r-2}-1}\left[\binom{2^{r-1}-i}{i-1}+\binom{2^{r-1}-i}{i}\right]\sigma_1^{2^{r-1}-2i-1}\sigma_2^i \\
&= \sum_{i=0}^{2^{r-1}-1}\binom{2^{r-1}-i+1}{i}\sigma_1^{2^{r-1}-2i-1}\sigma_2^i \\
&= \sum_{i=0}^{2^{r-3}-1}\binom{2^{r-1}-2i+1}{2i}\sigma_1^{2^{r-1}-4i-1}\sigma_2^{2i}, \quad \text{since } \binom{\text{even}}{\text{odd}} = 0, \quad \text{mod } 2 \\
&= \sum_{i=0}^{2^{r-3}-1}\sigma_1^{2^{r-1}-4i-1}\sigma_2^{2i}, \quad \text{since } \binom{2^{r-1}-2i+1}{2i} = 1, \quad \text{mod } 2.
\end{aligned}
$$

When $r=3$, $\bar{w}_3 = \sigma_1^3 = x^3+x^2y+xy^2+y^3 \neq 0$, since a basis for cohomology is $\{1, x, y, x^2, y^2, xy, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3\}$.

When $r=4$, $\bar{w}_7 = \sigma_1^7+\sigma_1^3\sigma_2^2 = x^7+x^6y+xy^6+y^7 \neq 0$, since a basis for cohomology is $\{x^iy^j : 0 \leq i \leq 8, 0 \leq j \leq 7\}$.

We now prove, by induction for $r > 4$, that

$$\bar{w}_{n-2} = (\sigma_1^{2^{r-1}}+\sigma_2^{2^{r-2}})(\sigma_1^{2^{r-2}}+\sigma_2^{2^{r-3}})\cdots(\sigma_1^8+\sigma_2^4)(\sigma_1^7+\sigma_1^3\sigma_2^2).$$

When $r=5$,

$$\bar{w}_{15} = \sigma_1^{15}+\sigma_1^{11}\sigma_2^2+\sigma_1^7\sigma_2^4+\sigma_1^3\sigma_2^6 = (\sigma_1^8+\sigma_2^4)(\sigma_1^7+\sigma_1^3\sigma_2^2).$$

Assume as an inductive hypothesis, that the formula for $\bar{w}_{n-2}$ when $s = r-1$ is true. Now for $s=r$,

$$
\begin{aligned}
\bar{w}_{n-2} &= (\sigma_1^{2^{r-1}-1}+\sigma_1^{2^{r-1}-5}\sigma_2^2+\sigma_1^{2^{r-1}-9}\sigma_2^4+\cdots+\sigma_1^{2^{r-2}+3}\sigma_2^{2^{r-3}-2}) \\
&\quad + (\sigma_1^{2^{r-2}-1}\sigma_2^{2^{r-3}}+\sigma_1^{2^{r-2}-5}\sigma_2^{2^{r-3}+2}+\cdots+\sigma_1^3\sigma_2^{2^{r-2}-2}) \\
&= \sigma_1^{2^{r-2}}(\sigma_1^{2^{r-2}-1}+\sigma_1^{2^{r-2}-5}\sigma_2^2+\cdots+\sigma_1^3\sigma_2^{2^{r-3}-2}) \\
&\quad + \sigma_2^{2^{r-3}}(\sigma_1^{2^{r-2}-1}+\sigma_1^{2^{r-2}-5}\sigma_2^2+\cdots+\sigma_1^3\sigma_2^{2^{r-3}-2}) \\
&= (\sigma_1^{2^{r-2}}+\sigma_2^{2^{r-1}})(\sigma_1^{2^{r-2}-1}+\sigma_1^{2^{r-2}-5}\sigma_2^2+\cdots+\sigma_1^3\sigma_2^{2^{r-3}-2}) \\
&= (\sigma_1^{2^{r-1}}+\sigma_2^{2^{r-1}})(\sigma_1^{2^{r-2}}+\sigma_2^{2^{r-3}})(\sigma_1^{2^{r-3}}+\sigma_2^{2^{r-4}})\cdots(\sigma_1^8+\sigma_2^4)(\sigma_1^7+\sigma_1^3\sigma_2^2),
\end{aligned}
$$

(by the inductive hypothesis).

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Hence, by the principle of mathematical induction, the formula for $\bar{w}_{n-2}$ is true for all $r > 4$. Now

$$\begin{aligned}\bar{w}_{n-2} &= (x^{2^{r-1}} + y^{2^{r-1}} + x^{2^{r-2}}y^{2^{r-2}})(x^{2^{r-2}} + y^{2^{r-2}} + x^{2^{r-3}}y^{2^{r-3}}) \cdots \\ &\quad \cdots (x^8 + y^8 + x^4y^4)(x^7 + x^6y + xy^6 + y^7) \\ &= x^{2^r-1} + y^{2^r-1} + (\text{lower powers of } x \text{ and } y) \neq 0\end{aligned}$$

since a basis for cohomology is $\{x^iy^j : 0 \leq i \leq 2^r, 0 \leq j \leq 2^r - 1\}$. Thus $\bar{w}_{n-2} \neq 0$ for $r \geq 3$. This proves part (c) of the theorem.

REMARKS. 1. Part (a) of the theorem is strongest if $n = 2^{r-1} + 2$ when $\mathbf{R}F(1, 1, 2^{r-1}) \not\subseteq \mathbf{R}^{2^{r+1}-3}$ and by Whitney's classical result, $\mathbf{R}F(1, 1, 2^{r-1}) \subseteq \mathbf{R}^{2^{r+1}+1}$. When $n = 6$, $\mathbf{R}F(1, 1, 4) \not\subseteq \mathbf{R}^{13}$ and Lam's result in [4] shows that $\mathbf{R}F(1, 1, 4) \subseteq \mathbf{R}^{15}$. In fact, Stong showed in [9] that $\mathbf{R}F(1, 1, 4) \subseteq \mathbf{R}^{14}$ and $\mathbf{R}F(1, 1, 4) \not\subseteq \mathbf{R}^{13}$, so that this is the best possible result.

2. If $n = 4$, part (b) of the theorem becomes

$$\mathbf{R}F(1, 1, 2) \not\subset \mathbf{R}^6, \qquad \mathbf{R}F(1, 1, 2) \not\subseteq \mathbf{R}^5.$$

Also if $n = 5$, part (c) of the theorem becomes

$$\mathbf{R}F(1, 1, 4) \not\subset \mathbf{R}^{10}, \qquad \mathbf{R}F(1, 1, 4) \not\subseteq \mathbf{R}^9.$$

Thus Lam's immersion results given in [4] that

$$\mathbf{R}F(1, 1, 2) \subseteq \mathbf{R}^6 \quad \text{and} \quad \mathbf{R}F(1, 1, 4) \subseteq \mathbf{R}^{10}$$

are the best possible.

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