Canad. J. Math. Vol. **61** (6), 2009 pp. 1239–1261

Periodicity in Rank 2 Graph Algebras

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Abstract. Kumjian and Pask introduced an aperiodicity condition for higher rank graphs. We present a detailed analysis of when this occurs in certain rank 2 graphs. When the algebra is aperiodic, we give another proof of the simplicity of $C^*(\mathbb{F}_\theta^+)$. The periodic C^*-algebras are characterized, and it is shown that $C^*(\mathbb{F}_\theta^+) \simeq C(\mathbb{T}) \otimes \mathfrak{A}$ where $\mathfrak{A}$ is a simple C^*-algebra.

1 Introduction

In this paper, we continue our study of the representation theory of rank 2 graph algebras developed in [3, 4]. Kumjian and Pask [7] introduced a family of C^*-algebras associated with higher rank graphs. They describe a property called the aperiodicity condition which implies the simplicity of the C^*-algebra. Our 2-graphs have a single vertex and are particularly amenable to analysis while exhibiting a wealth of interesting phenomena. Here we characterize when a 2-graph on one vertex is periodic, and describe the associated C^*-algebra.

The C^*-algebras of higher rank graphs have been studied in a variety of papers [5, 8, 11–13, 15]. See also [10]. The corresponding nonself-adjoint algebras were introduced by Kribs and Power [6]. The particular 2-graphs with one vertex were analyzed by Power [9], and the representation theory was developed by Power and us [3, 4]. Our work in this paper makes use of both sides of the theory; but this paper is really about the C^*-algebras. The higher rank graph algebras were inspired by the paper of Robertson and Steger [14] on higher rank Cuntz–Kreiger algebras. They also have a notion of aperiodicity that is a requirement in their case. These higher rank Cuntz–Kreiger algebras have a similar flavour to our single vertex higher rank graphs.

In the case under consideration, the 2-graph is a semigroup $\mathbb{F}_\theta^+$ given by generators and relations. We interpret the aperiodicity condition in terms of the existence of a special faithful irreducible representation of the associated C^*-algebra. The typical situation is aperiodicity. Indeed we show that periodicity only occurs under very special circumstances in which the commutation relations for words of certain lengths are given by a flip operation. Unfortunately, examples show that this periodicity may not exhibit itself except for rather long words, making a determination in specific examples difficult. We develop an algorithm for doing the computations in a more manageable way.

In the periodic case, there is also a special faithful representation. It is not irreducible, but rather decomposes as a direct integral by the methods of [4]. The special

Received by the editors June 5, 2007.
The first author partially supported by an NSERC grant.
AMS subject classification: Primary: 47L55; secondary: 47L30, 47L75, 46L05.
Keywords: higher rank graph, aperiodicity condition, simple C*-algebra, expectation.

https://doi.org/10.4153/CJM-2009-058-0 Published online by Cambridge University Press

structure in the periodic case allows us to provide a detailed analysis of this direct integral and thereby exhibit the C*-algebra as a tensor product of $C(\mathbb{T})$ with a simple C*-algebra. An important tool is a faithful approximately inner expectation onto the C*-algebra generated by the gauge invariant AF-subalgebra and the centre.

2 Background

The 2-graphs on a single vertex are semigroups which are given concretely in terms of a finite set of generators and relations of a special type. Let $\theta \in S_{m\times n}$ be a permutation of $\mathbf{m} \times \mathbf{n}$, where $\mathbf{m} = \{1, \dots, m\}$ and $\mathbf{n} = \{1, \dots, n\}$. The semigroup $\mathbb{F}_\theta^+$ is generated by $e_1, \dots, e_m$ and $f_1, \dots, f_n$. The identity is denoted as $\varnothing$. There are no relations among the e's, so they generate a copy of the free semigroup on m letters, $\mathbb{F}_m^+$; and there are no relations on the f's, so they generate a copy of $\mathbb{F}_n^+$. There are *commutation relations* between the e's and f's given by $e_i f_j = f_{j'} e_{i'}$ where $\theta(i, j) = (i', j')$.

A word $w \in \mathbb{F}_\theta^+$ has a fixed number of e's and f's regardless of the factorization, and the *degree* of w is (k, l) if there are k e's and l f's. The *length* of w is $|w| = k + l$. The commutation relations allow any word $w \in \mathbb{F}_\theta^+$ to be written with all e's first or with all f's first, say $w = e_u f_v = f_{v'} e_{u'}$. Indeed, one can factor w with any prescribed pattern of e's and f's as long as the degree is (k, l). It is straightforward to see that the factorization is uniquely determined by the pattern and that $\mathbb{F}_\theta^+$ has the unique factorization property. See also [7, 9].

A representation σ of $\mathbb{F}_\theta^+$ as operators on a Hilbert space is *row contractive* if

$$[\sigma(e_1) \cdots \sigma(e_m)] \quad \text{and} \quad [\sigma(f_1) \cdots \sigma(f_n)]$$

are contractions from $\mathcal{H}^{(m)}$ (resp. $\mathcal{H}^{(n)}$) to $\mathcal{H}$ and *row isometric* if these row operators are isometries. A row contractive representation is *defect free* if

$$\sum_{i=1}^{m} \sigma(e_i)\sigma(e_i)^* = I = \sum_{j=1}^{n} \sigma(f_j)\sigma(f_j)^*.$$

A row isometric defect free representation is called a $*$-*representation* of $\mathbb{F}_\theta^+$. The universal C*-algebra for the family of $*$-representations is denoted by $C^*(\mathbb{F}_\theta^+)$. A faithful representation of $C^*(\mathbb{F}_\theta^+)$ will be denoted as π_u.

The left regular representation λ of $\mathbb{F}_\theta^+$ is defined on $\ell^2(\mathbb{F}_\theta^+)$ with orthonormal basis $\{\xi_x : x \in \mathbb{F}_\theta^+\}$ by $\lambda(w)\xi_x = \xi_{wx}$. This is row isometric but is not defect free. The norm closed unital operator algebra generated by these operators is denoted by $\mathcal{A}_\theta$.

2.1 Gauge Automorphisms

The universal property of $C^*(\mathbb{F}_\theta^+)$ yields a family of *gauge automorphisms* $\gamma_{\alpha,\beta}$ for $\alpha, \beta \in \mathbb{T}$ determined by $\gamma_{\alpha,\beta}(\pi_u(e_i)) = \alpha\pi_u(e_i)$ and $\gamma_{\alpha,\beta}(\pi_u(f_j)) = \beta\pi_u(f_j)$. Integration around the 2-torus yields a faithful expectation

$$\Phi(X) = \int_{\mathbb{T}^2} \gamma_{\alpha,\beta}(X)\, d\alpha d\beta.$$

It is easy to check on monomials that the range is spanned by words of degree $(0,0)$ (where e_i^* and f_j^* count as degree $(-1,0)$ and $(0,-1)$, respectively).

Kumjian and Pask identify this range as an AF C*-algebra. The first observation is that any monomial in e's, f's and their adjoints can be written with all of the adjoints on the right. Clearly the isometric condition means that $\pi_u(f_i^* f_j) = \delta_{ij} = \pi_u(e_i^* e_j)$. To handle $e_i^* f_j$, observe that if $f_j e_k = e_{k'} f_{j_k}$ for $1 \le k \le m$, then

$$\pi_u(e_i^* f_j) = \pi_u\Big(e_i^* f_j \sum_k e_k e_k^* \Big) = \sum_k \pi_u(e_i^* e_{k'} f_{j_k} e_k^*) = \sum_k \delta_{ik'} \pi_u(f_{j_k} e_k^*).$$

So every word in $\mathrm{C}^*(\mathbb{F}_\theta^+)$ can be expressed as a sum of words of the form xy^* for $x, y \in \mathbb{F}_\theta^+$.

Next, they observe that for each integer $k \ge 1$, the set of words $\mathcal{S}_k$ in $\mathbb{F}_\theta^+$ of degree (k,k) determine a family of degree $(0,0)$ words $\{\pi_u(xy^*) : x, y \in \mathcal{S}_k\}$. It is clear that

$$\pi_u(x_1 y_1^*) \pi_u(x_2 y_2^*) = \delta_{y_1, x_2} \pi_u(x_1 y_2^*).$$

Thus these operators form a family of matrix units that generate a unital copy $\mathfrak{F}_k$ of the full matrix algebra $\mathfrak{M}_{(mn)^k}$. Moreover, these algebras are nested because the identity

$$\pi_u(xy^*) = \pi_u(x) \sum_i \pi_u(e_i e_i^*) \sum_j \pi_u(f_j f_j^*) \pi_u(y^*)$$

allows one to write elements of $\mathfrak{F}_k$ in terms of the basis for $\mathfrak{F}_{k+1}$. Therefore the range of the expectation Φ is the $(mn)^\infty$-UHF algebra $\mathfrak{F} = \overline{\bigcup_{k \ge 1} \mathfrak{F}_k}$. This is a simple C*-algebra.

2.2 Type 3a Representations

An important family of $*$-representations was introduced in [3]. The name refers to the classification obtained in [4].

Start with an arbitrary *tail* of $\mathbb{F}_\theta^+$, an infinite word of the form $\tau = e_{i_0} f_{j_0} e_{i_1} f_{j_1} \cdots$. Any infinite word τ with infinitely many e's and infinitely many f's may be put into this *standard form*. It may also be factored with any pattern of e's and f's, provided there are infinitely many of each. These alternate factorizations will be used later.

Let $\mathcal{G}_s = \mathcal{G} := \mathbb{F}_\theta^+$, for $s \ge 0$, viewed as a discrete set on which the generators of $\mathbb{F}_\theta^+$ act as injective maps by right multiplication, namely, $\rho(w)g = gw$ for all $g \in \mathcal{G}$. Consider $\rho_s = \rho(e_{i_s} f_{j_s})$ as a map from $\mathcal{G}_s$ into $\mathcal{G}_{s+1}$. Define $\mathcal{G}_\tau$ to be the injective limit set $\mathcal{G}_\tau = \varinjlim(\mathcal{G}_s, \rho_s)$, and let ι_s denote the injections of $\mathcal{G}_s$ into $\mathcal{G}_\tau$. Thus $\mathcal{G}_\tau$ may be viewed as the union of $\mathcal{G}_0, \mathcal{G}_1, \dots$ with respect to these inclusions.

The left regular action λ of $\mathbb{F}_\theta^+$ on itself induces corresponding maps on $\mathcal{G}_s$ by $\lambda_s(w)g = wg$. Observe that $\rho_s \lambda_s(w) = \lambda_{s+1}(w) \rho_s$. The injective limit of these actions is an action λ_τ of $\mathbb{F}_\theta^+$ on $\mathcal{G}_\tau$. Let λ_τ also denote the corresponding representation of $\mathbb{F}_\theta^+$ on $\ell^2(\mathcal{G}_\tau)$. Let $\{\xi_g : g \in \mathcal{G}_\tau\}$ denote the basis. A moment's reflection shows that this provides a defect-free, isometric representation of $\mathbb{F}_\theta^+$, *i.e.*, it is a $*$-representation.

It will be convenient to associate a directed chromatic graph with any atomic representation σ. We describe it for λ_τ. The vertices are associated with the points in

$\mathcal{G}_\tau$. For each vertex x and each $i \in \mathbf{m}$, draw a directed blue edge labelled e_i from x to y if $\lambda_\tau(e_i)\xi_x = \xi_y$. Likewise for each $j \in \mathbf{n}$, draw a directed red edge labelled f_j from x to z if $\lambda_\tau(f_j)\xi_x = \xi_z$. Observe that defect-free means that each vertex has one red and one blue edge leading into the vertex. For representations as partial isometries, row contractivity means that there is at most one edge of each colour leading into any vertex. To be isometric, there must be m blue edges and n red edges leading out of each vertex.

One of the main results of [3] is that $\mathrm{C}^*(\mathbb{F}_\theta^+)$ is the C^*-envelope of $\mathcal{A}_\theta$, and that every type 3a representation of $\mathbb{F}_\theta^+$ yields a completely isometric representation of $\mathcal{A}_\theta$ and a faithful $*$-representation of $\mathrm{C}^*(\mathbb{F}_\theta^+)$.

Therefore the gauge automorphisms are defined on $\mathrm{C}^*(\lambda_\tau(\mathbb{F}_\theta^+))$. It is shown in [3] that $\gamma_{\alpha,\beta}$ is implemented on $\ell^2(\mathcal{G}_\tau)$ by the unitary operator

$$U_{\alpha,\beta}\xi_{\iota_s(e_u f_v)} = \alpha^{|u|-s}\beta^{|v|-s}\xi_{\iota_s(e_u f_v)}.$$

2.3 Coinvariant Subspaces

The other main result of [3] is that every defect-free representation of $\mathbb{F}_\theta^+$ extends to a completely contractive representation of $\mathcal{A}_\theta$, and therefore dilates to a $*$-dilation. Moreover the minimal dilation is unique. Therefore it is possible to describe a $*$-representation completely by its compression to a coinvariant cyclic subspace, as it is then the unique $*$-dilation.

We describe such a subspace for type 3a representations. Let $\mathcal{H} = \overline{\lambda_\tau(\mathbb{F}_\theta^+)^*\xi_{\iota_0(\varnothing)}}$. This is coinvariant by construction. As it contains $\xi_{\iota_s(\varnothing)}$ for all $s \geq 1$, it is easily seen to be cyclic. Let σ_τ be the compression of λ_τ to $\mathcal{H}$.

Since λ_τ is a $*$-representation, for each $(s,t) \in (-\mathbb{N})^2$ there is a unique word $e_u f_v$ of degree $(|s|,|t|)$ such that $\xi_{\iota_0(\varnothing)}$ is in the range of $\lambda_\tau(e_u f_v)$. Set $\xi_{s,t} = \lambda_\tau(e_u f_v)^*\xi_{\iota_0(\varnothing)}$. It is not hard to see that this forms an orthonormal basis for $\mathcal{H}$.

Thus, for each $(s,t) \in (-\mathbb{N})^2$, there are unique integers $i_{s,t} \in \mathbf{m}$ and $j_{s,t} \in \mathbf{n}$ so that

$$\begin{aligned}
\sigma_\tau(e_{i_{s,t}})\xi_{s-1,t} = \xi_{s,t} &\quad \text{for } s \leq 0 \text{ and } t \leq 0,\\
\sigma_\tau(f_{j_{s,t}})\xi_{s,t-1} = \xi_{s,t} &\quad \text{for } s \leq 0 \text{ and } t \leq 0,\\
\sigma_\tau(e_i)\xi_{s,t} = 0 &\quad \text{if } i \neq i_{s+1,t} \text{ or } s = 0,\\
\sigma_\tau(f_j)\xi_{s,t} = 0 &\quad \text{if } j \neq j_{s,t+1} \text{ or } t = 0.
\end{aligned}$$

Note that we label the edges leading *into* each vertex, rather than leading out, because in the $*$-dilation the blue (or red) edge leading into a vertex is unique, while there are many leading out.

Consider how the tail $\tau = e_{i_0} f_{j_0} e_{i_1} f_{j_1} \cdots$ determines these integers. It defines the path down the diagonal, that is, $i_{s,s} := i_{|s|}$ and $j_{s-1,s} := j_{|s|}$ for $s \leq 0$. This determines the whole representation uniquely. Indeed, for any vertex $\xi_{s,t}$ with $s,t \leq 0$, take $T \geq |s|,|t|$ and select a path from $(-T,-T)$ to $(0,0)$ that passes through (s,t). The word $\tau_T = e_{i_0} f_{j_0} \cdots e_{i_{T-1}} f_{j_{T-1}}$ satisfies $\sigma_\tau(\tau_T)\xi_{-T,-T} = \xi_{0,0}$. Factor it as $\tau_T = w_1 w_2$

with $d(w_1) = (T - |s|, T - |t|)$ and $d(w_2) = (|s|, |t|)$, so that $\sigma_\tau(w_2)\xi_{-T,-T} = \xi_{s,t}$ and $\sigma_\tau(w_1)\xi_{s,t} = \xi_{0,0}$. Then $w_1 = e_{i_{s,t}} w' = f_{j_{s,t}} w''$.

It is evident that each $\sigma_\tau(e_i)$ and $\sigma_\tau(f_j)$ is a partial isometry. Moreover, each basis vector is in the range of a unique $\sigma_\tau(e_i)$ and $\sigma_\tau(f_j)$. So this is a defect-free, partially isometric representation with unique minimal $*$-dilation λ_τ.

2.4 Symmetry and Periodicity

An important part of the analysis of these atomic representations is the recognition of symmetry.

Definition 2.1 The tail τ determines the integer data $\Sigma(\tau) = \{(i_{s,t}, j_{s,t}) : s, t \le 0\}$. Two infinite words τ_1 and τ_2 with data $\Sigma(\tau_k) = \{(i^{(k)}_{s,t}, j^{(k)}_{s,t}) : s, t \le 0\}$ are said to be *tail equivalent* if the two sets of integer data eventually coincide, *i.e.,* there is an integer T so that

$$(i^{(1)}_{s,t}, j^{(1)}_{s,t}) = (i^{(2)}_{s,t}, j^{(2)}_{s,t}) \quad \text{for all } s, t \le T.$$

Say that τ_1 and τ_2 are (p, q)-*shift tail equivalent* for some $(p, q) \in \mathbb{Z}^2$ if there is an integer T so that

$$(*) \qquad (i^{(1)}_{s+p,t+q}, j^{(1)}_{s+p,t+q}) = (i^{(2)}_{s,t}, j^{(2)}_{s,t}) \quad \text{for all } s, t \le T.$$

Then τ_1 and τ_2 are *shift tail equivalent* if they are (p, q)-shift tail equivalent for some $(p, q) \in \mathbb{Z}^2$.

The *symmetry group* of τ is the subgroup of $\mathbb{Z}^2$ given by

$$H_\tau = \{(p, q) \in \mathbb{Z}^2 : \tau \text{ is } (p, q)\text{-shift tail equivalent to itself}\}.$$

A sequence τ is called *aperiodic* if $H_\tau = \{(0, 0)\}$.

The semigroup $\mathbb{F}^+_\theta$ is said to satisfy the *aperiodicity condition* if there is an aperiodic infinite word. Otherwise we say that $\mathbb{F}^+_\theta$ is *periodic*.

We also say that τ is *eventually* (p, q)-*periodic* for $(p, q) \in H_\tau$. If in fact it is fully (p, q)-periodic (that is, $(*)$ holds whenever $s, t, s + p, t + q \le 0$), then we say that τ is (p, q)-*periodic*.

In [4], the atomic $*$-representations are completely classified. One of the important steps is defining a symmetry group for the more general representations which occur. It turns out that the representation is irreducible precisely when the symmetry group is trivial. So the aperiodicity condition is equivalent to saying that there is an irreducible type 3a representation.

3 Characterization of Periodicity

Whether or not there is an irreducible type 3a representation of $\mathbb{F}^+_\theta$ depends on the semigroup. In this section, we obtain detailed information about periodic 2-graphs. In particular, a non-trivial symmetry group can only have the form $\mathbb{Z}(a, -b)$, where a, b are integers such that $m^a = n^b$ and the commutation relations are very special.

Let $\mathbf{m}^a$ denote the set of all a-tuples from the alphabet $\mathbf{m}$; and likewise $\mathbf{n}^b$ denotes b-tuples in the alphabet $\mathbf{n}$. We may suppose that $m \le n$. The case of $1 = m < n$ is of limited interest, and $m = n = 1$ is not considered.

Theorem 3.1 *If $2 \le m \le n$, then the following are equivalent for $\mathbb{F}_\theta^+$ and positive integers a and b.*

(i) *Every tail of $\mathbb{F}_\theta^+$ is eventually $(a, -b)$ periodic.*
(ii) *Every tail of $\mathbb{F}_\theta^+$ is $(a, -b)$ periodic.*
(iii) *$m^a = n^b$, and there is a bijection $\gamma \colon \mathbf{m}^a \to \mathbf{n}^b$ so that*

$$e_u f_v = f_{\gamma(u)} e_{\gamma^{-1}(v)} \quad \textit{for all } u \in \mathbf{m}^a \textit{ and } v \in \mathbf{n}^b.$$

If $1 = m < n$, then $\mathbb{F}_\theta^+$ is $(0, b)$-periodic, where b is the order of the permutation θ.

Proof Clearly, (ii) implies (i). We will show that (iii) implies (ii) and (i) implies (iii).

First consider $m \ge 2$, and suppose that condition (iii) holds. Then we also have $f_v e_u = e_{\gamma^{-1}(v)} f_{\gamma(u)}$. Now consider a type 3a representation λ_τ. Fix any standard basis vector $\xi_{s,t}$ such that $s \le -a$. Pulling back from $\xi_{s,t}$ yields a tail τ_1, which we factor as $\tau_1 = f_{v_1} e_{u_1} f_{v_2} e_{u_2} \cdots$ where $|u_k| = a$ and $|v_k| = b$. We wish to compare this with the tail τ_2 obtained from $\xi_{s+a,t-b}$.

Note that starting at the vertex $\xi_{s+a,t}$, one gets to $\xi_{s,t}$ by pulling back along a blue path a steps using a word e_u; while one obtains $\xi_{s+a,t-b}$ by pulling back b steps along the red path f_v. Hence the infinite path beginning at $\xi_{s+a,t}$ is $\tau_0 = e_u \tau_1 = f_v \tau_2$. Therefore

$$\begin{aligned}\tau_0 &= e_u f_{v_1} e_{u_1} f_{v_2} e_{u_2} f_{v_3} \cdots \\ &= f_{\gamma(u)} e_{\gamma^{-1}(v_1)} f_{\gamma(u_1)} e_{\gamma^{-1}(v_2)} f_{\gamma(u_2)} e_{\gamma^{-1}(v_3)} \cdots \\ &= f_v \tau_2.\end{aligned}$$

Hence $v = \gamma(u)$ and

$$\begin{aligned}\tau_2 &= e_{\gamma^{-1}(v_1)} f_{\gamma(u_1)} e_{\gamma^{-1}(v_2)} f_{\gamma(u_2)} e_{\gamma^{-1}(v_3)} \cdots \\ &= f_{v_1} e_{u_1} f_{v_2} e_{u_2} f_{v_3} \cdots \\ &= \tau_1.\end{aligned}$$

Therefore τ is $(a, -b)$-periodic.

Conversely, suppose that condition (iii) fails. We shall show that (i) also is false. Condition (iii) may fail for three reasons relating to the identities $e_u f_v = f_{v'} e_{u'}$:

(a) u' is not a function of v alone,
(b) v' is not a function of u alone, or
(c) there are functions $\alpha \colon \mathbf{m}^a \to \mathbf{n}^b$ and $\beta \colon \mathbf{n}^b \to \mathbf{m}^a$ so that $e_u f_v = f_{\alpha(u)} e_{\beta(v)}$ but $\beta \ne \alpha^{-1}$.

Consider (a) and select any $v \in \mathbf{n}^b$ so that there are two words u_i satisfying $e_{u_i} f_v = f_{v_i'} e_{u_i'}$ where $u_1' \ne u_2'$. Take an arbitrary word $u \in \mathbf{m}^a$ and compute $f_v e_u = e_{u'} f_{v'}$.

Pick one of the u_i's so that $u_i' \neq u'$. Without loss of generality, this is u_1. Now consider a word $e_{u_1} f_v e_u$ occurring as a segment of the tail τ, say $\tau = xe_{u_1} f_v e_u \tau'$. In the 3a representation λ_τ, there is a vertex $\xi_{s,t}$ at which the tail is $\tau_0 = e_{u_1} f_v e_u \tau' = f_{v_1'} e_{u_1'} e_u \tau'$. Moving to $\xi_{s-a,t}$ yields a vector with tail $\tau_1 = f_v e_u \tau' = e_{u'} f_{v'} \tau'$. Similarly, moving from $\xi_{s,t}$ to $\xi_{s,t-b}$ yields the tail $\tau_2 = e_{u_1'} e_u \tau'$. Since $u_1' \neq u'$, these two words do not coincide.

Hence any tail τ which contains the word $e_{u_1} f_v e_u$ infinitely often is not eventually $(a, -b)$ periodic.

Case (b) is handled in the same manner.

In case (c), note that this forces α and β to be injections. For $\alpha(u_1) = \alpha(u_2) = v_0$ implies that $e_{u_1} f_v = f_{v_0} e_{\beta(v)} = e_{u_2} f_v$; whence $e_{u_1} = e_{u_2}$ by cancellation. Similarly for β. Hence $m^a = n^b$, and α and β are bijections.

Since $\beta \neq \alpha^{-1}$, select $v \in \mathbf{n}^b$ so that $\beta(v) \neq \alpha^{-1}(v)$. Consider the tail $\tau = xe_{u_1} f_v e_{u_2} \tau'$. Again there is a vertex $\xi_{s,t}$ at which the tail is

$$\tau_0 = e_{u_1} f_v e_{u_2} \tau' = f_{\alpha(u_1)} e_{\beta(v)} e_{u_2} \tau'.$$

Moving to $\xi_{s-a,t}$ yields a vector with tail $\tau_1 = f_v e_{u_2} \tau' = e_{\alpha^{-1}(v)} f_{\beta^{-1}(u_2)} \tau'$. Similarly, moving from $\xi_{s,t}$ to $\xi_{s,t-b}$ yields the word $\tau_2 = e_{\beta(v)} e_{u_2} \tau'$. Since $\beta(v) \neq \alpha^{-1}(v)$, these two words do not coincide. The proof is finished as before.

Now consider the case $m = 1$. Then $\theta \in S_n$ and the commutation relations have the form $ef_j = f_{\theta(j)} e$ for $1 \leq j \leq n$. So $e^k f_j = f_{\theta^k(j)} e$. In particular, if b is the order of θ in S_n, then it is the smallest positive integer so that e commutes with all f_v for $v \in \mathbf{n}^b$.

In this case, a type 3a representation is determined by the infinite sequence $j_{0,t}$ for $t \leq 0$. Indeed, a simple calculation shows that $j_{s,t} = \theta^{-s}(j_{0,t})$ for all $s \leq 0$. Therefore every tail τ exhibits $(0, b)$ symmetry. Select the sequence $(j_{0,t} : t \leq 0)$ to be aperiodic (as a sequence in one variable) and to contain all n values infinitely often. It is easy to see that the data $\Sigma(\tau)$ exhibits only $(0, b)$-periodicity. ∎

Corollary 3.2 *If $\frac{\log m}{\log n}$ is irrational, then $\mathbb{F}_\theta^+$ is aperiodic for all θ in $S_{m\times n}$.*

Proof $m^a = n^b$ if and only if $\frac{\log m}{\log n} = \frac{b}{a}$. ∎

Example 3.3 Consider the following example with $m = 2$, $n = 4$, and with two 3-cycles (and two fixed points):

$$\big((1,2),(2,1),(1,3)\big) \quad \text{and} \quad \big((2,2),(2,3),(1,4)\big).$$

These relations are:

$$\begin{aligned} e_1 f_1 &= f_1 e_1, & e_1 f_2 &= f_1 e_2, & e_1 f_3 &= f_2 e_1, & e_1 f_4 &= f_2 e_2, \\ e_2 f_1 &= f_3 e_1, & e_2 f_2 &= f_3 e_2, & e_2 f_3 &= f_4 e_1, & e_2 f_4 &= f_4 e_2. \end{aligned}$$

A calculation shows that the relation between e-words of length 2 and the f's has this special symmetry. Setting

$$\gamma(11) = 1, \qquad \gamma(12) = 2, \qquad \gamma(21) = 3, \qquad \gamma(22) = 4$$

yields the relations $e_{ij} f_k = f_{\gamma(ij)} e_{\gamma^{-1}(k)}$. So this semigroup has $(2, -1)$-periodicity.

https://doi.org/10.4153/CJM-2009-058-0 Published online by Cambridge University Press

Theorem 3.1 leads to the following theorem. It is somewhat unsatisfactory because one needs to check potentially infinitely many higher commutation relations (see Example 4.8). We pose the question of whether there is a combinatorial condition on the original permutation θ which is equivalent to periodicity. A partial answer to this problem is given later in this section.

Theorem 3.4 *Suppose that $m, n \geq 2$. Then $\mathbb{F}_\theta^+$ satisfies the aperiodicity condition if and only if the technical condition* (iii) *of Theorem* 3.1 *does not hold for any* (a, b) *for which $m^a = n^b$.*

Proof List all non-zero words $(p, q) \in \mathbb{Z}^2$ in a list $\{(p_k, q_k) : k \geq 1\}$ so that each element is repeated infinitely often. For each k, we construct a word a_k in $\mathbb{F}_\theta^+$. There are two cases for the word (p, q).

If $p_k q_k \geq 0$ and $p_k \neq 0$, choose $a_k = e_1^{|p_k|} f_1^{|q_k|} e_2$. If $p_k = 0$, choose $a_k = f_1^{|q_k|} f_2$. If $p_k q_k < 0$, use the construction from the proof of Theorem 3.1. Let $\tau = a_1 a_2 a_3 \cdots$.

To see that τ is aperiodic, consider any $(p, q) \neq (0, 0)$. It occurs as (p_k, q_k) infinitely many times. If $p_k q_k \geq 0$, consider the starting point (s, t) at the beginning of the word $a_k = e_1^{|p_k|} f_1^{|q_k|} t$, where $t = e_2$ or f_2. Then moving to $(s - |p_k|, t - |q_k|)$ yields the word beginning with t, which does not coincide with the beginning of a_k. If $p_k q_k < 0$, then argue as in the previous theorem. As each (p, q) occurs infinitely often, τ is not eventually (p, q)-periodic for any period. Hence it is aperiodic. ■

The same proof works for the periodic semigroups, eliminating all symmetries except those in every representation.

Corollary 3.5 *If $\mathbb{F}_\theta^+$ is periodic with minimal period $(a, -b)$, then $\mathrm{C}^*(\mathbb{F}_\theta^+)$ has a type* 3a *representation with symmetry group $\mathbb{Z}(a, -b)$.*

Proof If $\mathbb{F}_\theta^+$ is periodic with minimal period $(a, -b)$, then a routine modification of the proof above shows that there is an infinite word whose only symmetries are $\mathbb{Z}(a, -b)$. Indeed, if $m^{a_0} = n^{b_0}$ and $\gcd(a_0, b_0) = 1$, then $a = ka_0$ and $b = kb_0$. By hypothesis, there are words a_l with no $(la_0, -lb_0)$ periodicity for $1 \leq l < k$. As in the proof above, there are always words with no (p, q) periodicity when $pq \geq 0$. So following the same process, one obtains an infinite word τ without any of these symmetries. Let λ_τ be the corresponding type 3a representation.

The symmetry group H_{λ_τ} contains $(a, -b)$, and thus $\mathbb{Z}(a, -b)$. However by construction, $H_{\lambda_\tau} \cap \mathbb{N}_0^2 = \{(0, 0)\}$ and $(la_0, -lb_0)$ are not in H_{λ_τ} for $1 \leq l < k$. So $H_{\lambda_\tau} \cap \mathbb{Z}(a_0, -b_0) = \mathbb{Z}(a, -b)$. If it contained anything else, then it would contain a non-zero element of $\mathbb{N}_0^2$. So $H_{\lambda_\tau} = \mathbb{Z}(a, -b)$. ■

4 Tests for Periodicity and Aperiodicity

We now examine a method for demonstrating aperiodicity. The permutation $\theta \in S_{mn}$ determines functions $\alpha_i : \mathbf{n} \to \mathbf{n}$ and $\beta_j : \mathbf{m} \to \mathbf{m}$ so that $\theta(i, j) = (\beta_j(i), \alpha_i(j))$. Thus if $u = i_1 \cdots i_a$ and $v = j_b \cdots j_1$,

$$e_u f_j = f_{\alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_a}(j)} e_{u'} \quad \text{and} \quad e_i f_v = f_{v'} e_{\beta_{j_1} \circ \beta_{j_2} \circ \cdots \circ \beta_{j_b}(i)}.$$

For $(a, -b)$-periodicity when $m^a = n^b$, a necessary condition is that

$$\alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_a} \quad \text{and} \quad \beta_{j_1} \circ \beta_{j_2} \circ \cdots \circ \beta_{j_b}$$

are constant maps for all u and v. So we obtain the following.

Corollary 4.1 **(Aperiodicity criterion)** *If there is a subset $B \subset \mathbf{n}$ with $|B| \geq 2$ and a word $i_1 \cdots i_k \in \mathbf{m}^k$ so that $\alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_k}(B) = B$, then $\mathbb{F}_\theta^+$ is aperiodic. Similarly, if there is a subset $A \subset \mathbf{m}$ with $|A| \geq 2$ and a word $j_1 \cdots j_k \in \mathbf{n}^k$ so that $\beta_{j_1} \circ \beta_{j_2} \circ \cdots \circ \beta_{j_k}(A) = A$, then $\mathbb{F}_\theta^+$ is aperiodic.*

Proof $(\alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_k})^a(B) = B$ is never constant. ∎

Remark 4.2 It is not hard to show that either there is a $B \subset \mathbf{n}$ with $|B| \geq 2$ and a word $i_1 \ldots i_k \in \mathbf{m}^k$ so that $\alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_k}(B) = B$ or for some sufficiently large k, all $\alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_k}$ are constant maps.

Example 4.3 For $\theta \in S_{2\times 2}$, there are nine distinct algebras up to isomorphism [9]. For example, the forward 3-cycle algebra is given by the permutation θ in $S_{2\times 2}$ given by the 3-cycle $\big((1,1),(1,2),(2,1)\big)$. This yields the relations

$$e_1 f_1 = f_2 e_1, \qquad e_1 f_2 = f_1 e_2,$$
$$e_2 f_1 = f_1 e_1, \qquad e_2 f_2 = f_2 e_2.$$

One can easily check that $e_i f_j = f_{i+j} e_j$, where addition is modulo 2. Notice that $\alpha_2 = \mathrm{id}$; so $\alpha_2(\{1,2\}) = \{1,2\}$. Hence it is aperiodic. This technique works for seven of the nine 2×2 examples.

One exception is the flip algebra, which is given by the rule $e_i f_j = f_i e_j$; and it is clearly $(1,-1)$-periodic.

The other is the square algebra given by the permutation

$$\big((1,1),(1,2),(2,2),(2,1)\big).$$

This yields maps $\alpha_1 = \beta_2 = 2$ and $\alpha_2 = \beta_1 = 1$. These maps are constant, but are not mutual inverses. So there is no $(1,-1)$ periodicity. However a calculation shows that $e_{i_1 i_2} f_{j_1 j_2} = f_{i_1' i_2} e_{j_1' j_2}$, where $i' = i+1 \pmod 2$ and $j' = j+1 \pmod 2$. So the function $\gamma(ij) = i'j$ satisfies $\gamma^{-1} = \gamma$ and $e_u f_v = f_{\gamma(u)} e_{\gamma(v)}$ for all $|u| = |v| = 2$. Thus the square algebra is $(2,-2)$-periodic.

So the periodicity can reveal itself only in the higher order commutation relations!

Example 4.4 Here is another example of this phenomenon. Consider $\theta \in S_{3\times 3}$ given by fixed points (i,i) for $1 \leq i \leq 3$, and cycles

$$\big((1,2),(2,1)\big) \quad \text{and} \quad \big((1,3),(3,2),(2,3),(3,1)\big).$$

A calculation shows that this algebra has $(2,-2)$-periodicity via the correspondence $e_{ij} f_{kl} = f_{\gamma(ij)} e_{\gamma(kl)}$ where $\gamma(23) = 13$, $\gamma(13) = 23$ and $\gamma(ij) = ij$ otherwise (so $\gamma^{-1} = \gamma$).

These calculations can be simplified somewhat by observing that there are subalgebras isomorphic to the one in Example 3.3 generated by each of the sets

$$\{e_1, e_2\,;\, f_{11}, f_{12}, f_{21}, f_{22}\}, \quad \{e_1, e_3\,;\, f_{11}, f_{23}, f_{31}, f_{33}\}, \quad \{e_2, e_3\,;\, f_{22}, f_{13}, f_{32}, f_{33}\},$$
$$\{f_1, f_2\,;\, e_{11}, e_{12}, e_{21}, e_{22}\}, \quad \{f_2, f_3\,;\, e_{22}, e_{13}, e_{32}, e_{33}\}, \quad \{f_1, f_3\,;\, e_{11}, e_{23}, e_{31}, e_{33}\}.$$

Thus there are corresponding 4×4 subsets of the 9×9 pattern of relations between words of degree 2 that must have the desired form.

In order to develop a better test, we require a refinement of condition (iii) of Theorem 3.1.

Proposition 4.5 *If $\mathbb{F}_\theta^+$ is periodic and $\gamma\colon \mathbf{m}^a \to \mathbf{n}^b$ is the bijective correspondence of Theorem* 3.1*, then for $i_0, \dots, i_a \in \mathbf{m}$, $e_{i_0} f_{\gamma(i_1\cdots i_a)} = f_{\gamma(i_0\cdots i_{a-1})} e_{i_a}$. Conversely, if there is a bijection $\gamma\colon \mathbf{m}^a \to \mathbf{n}^b$ with this property, then $\mathbb{F}_\theta^+$ is periodic.*

Similarly for $j_0, \dots, j_b \in \mathbf{n}$, $e_{\gamma^{-1}(j_0\cdots j_{b-1})} f_{j_b} = f_{j_0} e_{\gamma^{-1}(j_1\cdots j_b)}$. Again this property is equivalent to $(a, -b)$-periodicity.

Proof From the commutation relations, we know that

$$e_{i_0} f_{\gamma(i_1\cdots i_a)} = f_v e_{i'_a} = f_{\gamma(i'_0\cdots i'_{a-1})} e_{i'_a},$$

where $i'_0 \cdots i'_{a-1} = \gamma^{-1}(v)$. Let $u = ku'$ be any word of length a. Then

$$\begin{aligned} e_u e_{i_0} f_{\gamma(i_1\cdots i_a)} &= e_k e_{u' i_0} f_{\gamma(i_1\cdots i_a)} = e_k f_{\gamma(u' i_0)} e_{i_1\cdots i_a} \\ &= e_u f_{\gamma(i'_0\cdots i'_{a-1})} e_{i'_a} = f_{\gamma(u)} e_{i'_0\cdots i'_{a-1} i'_a}. \end{aligned}$$

Therefore $i'_s = i_s$ for $1 \le s \le a$. Similarly,

$$e_{i_0} f_{\gamma(i_1\cdots i_a)} e_{u'} = f_{\gamma(i'_0\cdots i'_{a-1})} e_{i'_a u'} = e_{i'_0\cdots i'_{a-1}} f_{\gamma(i'_a u')}.$$

So $i'_0 = i_0$.

Conversely, using this identity a times yields

$$e_u f_{\gamma(u')} = f_{\gamma(u)} e_{u'} \quad \text{for all } u, u' \in \mathbf{m}^a.$$

Thus $\mathbb{F}_\theta^+$ is periodic by Theorem 3.1. ∎

The next proposition shows that the periodicity of the square algebra of Example 4.3 is a consequence of the relation $e_i f_j = f_{i+1} e_j$.

Proposition 4.6 *If $m^a = n^b$ and there are maps $\alpha\colon \mathbf{m}^a \to \mathbf{n}^b$ and $\beta\colon \mathbf{n}^b \to \mathbf{m}^a$ such that $e_u f_v = f_{\alpha(u)} e_{\beta(v)}$ for all $u \in \mathbf{m}^a$ and $v \in \mathbf{n}^b$, then $\mathbb{F}_\theta^+$ is periodic.*

Proof Since there is a bijective correspondence θ' between the words $e_u f_v$ and the words $f_{v'} e_{u'}$, it is easy to verify that α and β are bijections. Thus $\beta\alpha$ is a permutation of $\mathbf{m}^a$. Let k be the order of $\beta\alpha$ in S_{m^a}; so $(\beta\alpha)^k = \mathrm{id}$. Define $\gamma\colon \mathbf{m}^{ak} \to \mathbf{n}^{bk}$ by

$$\gamma(u_1 \cdots u_k) = \alpha(u_1)\alpha\beta\alpha(u_2)\cdots\alpha(\beta\alpha)^{k-1}(u_k)$$

for $u_i \in \mathbf{m}^a$, $1 \le i \le k$. Then compute

$$\begin{aligned} e_{u_0} f_{\gamma(u_1\cdots u_k)} &= e_{u_0} f_{\alpha(u_1)} f_{\alpha\beta\alpha(u_2)} \cdots f_{\alpha(\beta\alpha)^{k-1}(u_k)} \\ &= f_{\alpha(u_0)} e_{\beta\alpha(u_1)} f_{\alpha\beta\alpha(u_2)} \cdots f_{\alpha(\beta\alpha)^{k-1}(u_k)} \\ &= f_{\alpha(u_0)} f_{\alpha\beta\alpha(u_1)} \cdots f_{\alpha(\beta\alpha)^{k-1}(u_{k-1})} e_{(\beta\alpha)^k(u_k)} \\ &= f_{\gamma(u_0 u_1 \cdots u_{k-1})} e_{u_k}. \end{aligned}$$

Therefore by Proposition 4.5, $\mathbb{F}_\theta^+$ is $(ak, -bk)$-periodic. ■

Proposition 4.5 can be used to calculate γ. In turn, this leads to a checkable algorithm for periodicity. This is captured in the following theorem.

Theorem 4.7 *Suppose that $\mathbb{F}_\theta^+$ is $(a, -b)$-periodic. Then the bijection $\gamma\colon \mathbf{m}^a \to \mathbf{n}^b$ may be calculated for $u_0 \in \mathbf{m}^a$ by starting with an arbitrary $j_0 \in \mathbf{n}$ and computing*

$$\begin{aligned} e_{u_0} f_{j_0} &= f_{j_1} e_{u_1}, \\ e_{u_1} f_{j_1} &= f_{j_2} e_{u_2}, \\ &\vdots \\ e_{u_b} f_{j_b} &= f_j e_u. \end{aligned}$$

Then $\gamma(u_0) = v_0 := j_1 j_2 \cdots j_b$, and also $j = j_0$ and $u = u_0$.

Reversing the process, start with an arbitrary $i_0 \in \mathbf{m}$ and calculate

$$\begin{aligned} e_{i_0} f_{v_0} &= f_{v_1} e_{i_1}, \\ e_{i_1} f_{v_1} &= f_{v_2} e_{i_2}, \\ &\vdots \\ e_{i_a} f_{v_a} &= f_v e_i. \end{aligned}$$

Then $\gamma^{-1}(v_0) = i_a i_{a-1} \cdots i_1 = u_0$, and also $v = v_0$ and $i = i_0$.

Conversely, if for $m^a = n^b$ and for each $u_0 \in \mathbf{m}^a$ the procedure above passes all the tests of equality for all $j_0 \in \mathbf{n}$ and all $i_0 \in \mathbf{m}$, then $\mathbb{F}_\theta^+$ is $(a, -b)$-periodic.

Proof By Proposition 4.5, if $\gamma(u_0) = j_1 j_2 \cdots j_b$, then $e_{u_0} f_{j_0} = f_{j_1} e_{u_1}$, where $u_1 = j_2 \cdots j_b j_0$. Proceeding by induction, we find that $u_i = j_{i+1} \cdots j_b j_0 \cdots j_{i-1}$ and $e_{u_i} f_{j_i} = f_{j_{i+1}} e_{u_{i+1}}$, where $u_{i+1} = j_{i+2} \cdots j_b j_0 \cdots j_i$. In the last step, we return to the beginning and obtain $e_{u_b} f_{j_b} = f_{j_0} e_{u_0}$. Hence we have calculated $\gamma(u_0) = j_1 j_2 \cdots j_b$, and $j = j_0$ and $u = u_0$.

Reversing the process works in the same manner.

Now consider the converse. Starting with each u_0 in $\mathbf{m}^a$, for each value of $j_0 \in \mathbf{n}$, we produce the same sequence $v_0 := j_1 j_2 \cdots j_b$. This defines a function $\alpha\colon \mathbf{m}^a \to \mathbf{n}^b$. Observe that with the notation from that calculation, since the sequence cycles around due to the fact that $v = v_0$ and $j = j_0$, it follows that

$$\alpha(u_i) = j_{i+1} \cdots j_b j_0 \cdots j_{i-1} \quad \text{for } 1 \le i \le b.$$

Then we reverse the process, and construct a function $\beta\colon \mathbf{n}^b \to \mathbf{m}^a$ and confirm that $\beta(v_0) = u_0$, that is, $\beta = \alpha^{-1}$. Therefore α and β are bijections. Finally, the initial calculation $e_{u_0} f_{j_0} = f_{j_1} e_{u_1}$ yields $e_{\alpha^{-1}(j_1 \cdots j_b)} f_{j_0} = f_{j_1} e_{\alpha^{-1}(j_2 \cdots j_b j_0)}$. This verifies the hypothesis of Proposition 4.5, and confirms that $\mathbb{F}_\theta^+$ is $(a, -b)$-periodic. ■

4.1 A Computer Algorithm

Theorem 4.7 provides a valid test for periodicity that is effective as a computer program. It allows a single pass through all the words in $\mathbf{m}^a$ doing several tests. Failure at any point indicates failure of $(a, -b)$-periodicity; while a completed run without failure means that $\mathbb{F}_\theta^+$ is indeed $(a, -b)$-periodic. An algorithm based on Theorem 3.1 would require checking all $m^a n^b = m^{2a}$ pairs, and this is much too computationally intensive.

Example 4.8 This example is a 4×4 example which has $(12, -12)$-periodicity. This is surprisingly high periodicity for such a small number of generators. Already it is basically impossible to calculate the multiplication table for the $4^{12} \times 4^{12}$ pairs of words. The algorithm described above reduces this example to a calculation that can be done by computer in about an hour. We first show how hand calculations allow us to deduce that there is no periodicity smaller than 12.

We call this the 8-cycle algebra. It is given by the 8-cycle:

$$\big((2,1),(1,2),(3,1),(1,3),(4,2),(2,4),(4,3),(3,4)\big),$$

two 2-cycles $\big((1,4),(4,1)\big)$ and $\big((2,3),(3,2)\big)$, and fixed points (i,i) for $1 \le i \le 4$.

We first calculate the maps α_i and β_j.

$$\alpha_1(3) = 2 \quad \text{and} \quad \alpha_1(j) = 1 \text{ otherwise},$$
$$\alpha_2(4) = 3 \quad \text{and} \quad \alpha_2(j) = 2 \text{ otherwise},$$
$$\alpha_3(4) = 1 \quad \text{and} \quad \alpha_3(j) = 3 \text{ otherwise},$$
$$\alpha_4 = 4,$$
$$\beta_1 = 1,$$
$$\beta_2(1) = 3 \quad \text{and} \quad \beta_2(i) = 2 \text{ otherwise},$$
$$\beta_3(1) = 4 \quad \text{and} \quad \beta_3(i) = 3 \text{ otherwise},$$
$$\beta_4(3) = 2 \quad \text{and} \quad \beta_4(i) = 4 \text{ otherwise}.$$

One readily calculates $\alpha_2^2 = \alpha_2\alpha_1 = \alpha_2\alpha_3 = 2$, $\alpha_3^2 = \alpha_3\alpha_1 = \alpha_3\alpha_2 = 3$, $\alpha_1^2 = 1$, and any expression involving α_4 is constant as well. However $\alpha_1\alpha_3$ and $\alpha_1\alpha_2$ are not constant. It follows though that every composition of three α_i's is constant. A similar calculation shows that the composition of any three β_j's is constant. This suggests that the 8-cycle algebra may be periodic. But one might think that it should have small order. That turns out not to be the case.

We will show by hand that if the 8-cycle algebra $\mathbb{F}_\theta^+$ is periodic with the minimal period $(a, -b)$, then $(a, -b) = (12k, -12k)$ for some $k \geq 1$. Clearly $m = n$ implies that $a = b$. A useful observation is that if $\mathbb{F}_\theta^+$ is $(k, -k)$-periodic, then, when $|u| = |v| = k$ and $e_u f_v = f_{v'} e_{u'}$, it follows that $e_{u'} f_{v'} = f_v e_u$. That is, the cycle lengths are just 1 and 2. We will show that this forces k to be a multiple of 12.

Observe that in the commutation relations between the 2-letter words $\{11, 12, 13, 24, 34\}$ remain within this set, and so we obtain a subsemigroup generated by these words that is a 2-graph with a 5×5 multiplication table.

The point $(12, 11)$ lies on the 6 cycle

$$\big((12, 11), (11, 13), (34, 11), (11, 12), (13, 11), (11, 24)\big).$$

By induction, we can obtain the following identities:

$$\begin{aligned} e_{12}^{2k+1} f_{11}^{2k+1} &= (f_{13} f_{34})^k f_{13} e_{11}^{2k+1} \quad && \text{for } k \geq 0, \\ e_{12}^{2k} f_{11}^{2k} &= (f_{13} f_{34})^k e_{11}^{2k} && \text{for } k \geq 1. \end{aligned}$$

On the other hand, we compute

$$\begin{aligned} e_{11}^{6k+1} (f_{13} f_{34})^{3k} f_{13} &= f_{11}^{6k+1} e_{34} e_{24}^{6k}, \\ e_{11}^{6k+2} (f_{13} f_{34})^{3k+1} &= f_{11}^{6k+2} e_{13} e_{12}^{6k+1}, \\ e_{11}^{6k+3} (f_{13} f_{34})^{3k+1} f_{13} &= f_{11}^{6k+3} e_{12} e_{24}^{6k+2}, \\ e_{11}^{6k+4} (f_{13} f_{34})^{3k+2} &= f_{11}^{6k+4} e_{34} e_{12}^{6k+3}, \\ e_{11}^{6k+5} (f_{13} f_{34})^{3k+2} f_{13} &= f_{11}^{6k+5} e_{13} e_{24}^{6k+4}, \\ e_{11}^{6k+6} (f_{13} f_{34})^{3k+3} &= f_{11}^{6k+6} e_{12}^{6k+6}. \end{aligned}$$

From this, we see that the required 2-cycle condition does not hold for words of length $12k + 2i$ for $1 \leq i \leq 5$. It also follows that there is no odd period $(k, -k)$, for then $(2k, -2k)$ would be a period which is not a multiple of 12. Therefore, $(k, -k)$-periodicity can only hold if 12 divides k.

The computer algorithm successfully verified that $\mathbb{F}_\theta^+$ is $(12, -12)$-periodic. Hence the symmetry group is $\mathbb{Z}(12, -12)$.

As a corollary, we see that the 5×5 algebra that we used has symmetry group $\mathbb{Z}(6, -6)$. This follows because the map γ on 4^{12} restricts to a map on the 6-letter words from the 5×5 algebra. So there is $(6, -6)$-periodicity. Our argument shows that it has no smaller period.

For a while, we had conjectured that if there is a constant k so that the composition of any k of the maps α_i is constant, as is the composition of any k of the maps β_j, then $\mathbb{F}_\theta^+$ should be periodic. However the following example shows that this is not the case. So we pose the less precise problem: *Find a computable condition on the permutation θ which is equivalent to periodicity.*

Example 4.9 Consider the 3×3 example with an 8-cycle

$$\big((1,3),(1,1),(3,1),(3,3),(2,3),(1,2),(2,1),(3,2)\big).$$

It is easy to calculate that $\alpha_i = i$ are constant for $i = 1,2,3$, as is $\beta_1 = 3$; β_2 sends 3 to 1 and $1,2$ to 2; and β_3 sends 3 to 2, and $1,2$ to 1. So $\beta_2^2 = 2 = \beta_2\beta_3$ and $\beta_3^2 = 1 = \beta_3\beta_2$. So all compositions of two maps are constant.

We claim that $e_{1^k} f_{1^k} = f_{132^{k-2}} e_{u_k}$. Indeed this is an easy calculation by induction starting with $e_{11} f_{11} = f_{13} e_{23}$, since

$$\begin{aligned} e_{1^{k+1}} f_{1^{k+1}} &= e_1 f_{132^{k-2}} e_{u_k} f_1 = f_{13} e_2 f_{2^{k-2}} f_j e_{u_k'} \\ &= f_{132^{k-2}} e_2 f_j e_{u_k'} = f_{132^{k-2}} f_2 e_{i'u_k'} = f_{132^{k-1}} e_{u_{k+1}}. \end{aligned}$$

Thus, if $\mathbb{F}_\theta^+$ were $(k,-k)$-periodic for $k \geq 4$, one would have $e_{u_k} f_{132^{k-2}} = f_{1^k} e_{1^k}$. However,

$$e_{u_k} f_{132^{k-2}} = e_{u_k'} e_i f_{132^{k-2}} = e_{u_k'} f_{j_1 j_2} e_{i'} f_{2^{k-2}} = e_{u_k'} f_{v'} e_2,$$

because $\beta_2^2 = 2$. Thus $\mathbb{F}_\theta^+$ must be aperiodic.

Example 4.10 Here is another example with a 4×4 permutation:

$$\begin{gathered} \big((1,1),(3,2),(4,4),(2,3)\big), \qquad \big((2,1),(1,2),(4,2),(2,4)\big), \\ \big((3,1),(3,4),(4,3),(1,3)\big), \qquad ((4,1),(1,4)), \quad ((2,2)), \quad ((3,3)). \end{gathered}$$

This is unusual in our experience, because $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$, $\beta_1 = \beta_3$, and $\beta_2 = \beta_4$. Compositions are constant after two compositions:

$$\alpha_1\alpha_3 = 1,\ \alpha_1^2 = 2,\ \alpha_3^2 = 3,\ \alpha_3\alpha_1 = 4$$

and

$$\beta_1\beta_2 = 1,\ \beta_2^2 = 2,\ \beta_1^2 = 3,\ \beta_2\beta_1 = 4.$$

It is $(2,-2)$-periodic.

Example 4.11 This final example gives some variants on Example 4.8. For any $m \geq 4$, consider the $m \times m$ example which consists of all flips $((i,j),(j,i))$ and fixed points $((i,i))$ except when exactly one of i, j belongs to $\{1,m\}$. These belong to the $4(m-2)$-cycle

$$\begin{aligned} \big((2,1),(1,2),(3,1),(1,3),\dots,(m-1,1),(1,m-1), \\ (m,2),(2,m),(m,3),(3,m),\dots,(m,m-1),(m-1,m)\big). \end{aligned}$$

https://doi.org/10.4153/CJM-2009-058-0 Published online by Cambridge University Press

The maps α_i and β_j become constant after three compositions.

Computer tests show that when $m = 2k + 2$ is even, the algebra has $(12k, -12k)$-periodicity for $1 \leq k \leq 9$. Simple examples show that they are not $(6k, -6k)$ or $(4k, -4k)$ periodic. Except for $k = 1$ (Example 4.8), in which an exhaustive computer check was performed, the computer tested a random set of a million words of length $12k$ and found the algebras to be $(12k, -12k)$-periodic. Experience shows that failure of periodicity exhibits itself within a small number of examples.

On the other hand, when m is odd, these examples are aperiodic. Since e_i commutes with f_i, one sees that e_i^k commutes with f_i^k. Therefore the bijection γ of m^k demonstrating periodicity must map e_i^k to f_i^k. Hence if the algebra were $(k, -k)$-periodic, one would need to have the identity $e_1^k f_2^k = f_1^k e_2^k$.

For the 5×5 12-cycle algebra, we find by induction that

$$\begin{aligned} e_1^{2k+1} f_2^{2k+1} &= f_1^{k+1} f_2 f_5^{k-1} e_5 e_2^{2k} \qquad \text{for } k \geq 1, \\ e_1^{2k} f_2^{2k} &= f_1^{k+1} f_4 f_5^{k-2} e_5 e_2^{2k-1} \quad \text{for } k \geq 2. \end{aligned}$$

Hence it is aperiodic. Similarly, for $m = 2s + 1$ one can show that

$$e_1^k f_2^k = f_1^{k-l-1} f_{2j} f_m^l e_m e_2^{k-1} \quad \text{for } k \geq m - 2,$$

where $k \equiv j - 2 \pmod{s}$ and $l = \lfloor k/s \rfloor - 2$. So again these algebras are all aperiodic.

5 Periodicity and the Structure of $\mathrm{C}^*(\mathbb{F}_\theta^+)$

We first provide a different proof of the Kumjian–Pask result that aperiodicity implies simplicity of $\mathrm{C}^*(\mathbb{F}_\theta^+)$. We have already observed that there is a faithful expectation Φ onto a $(mn)^\infty$-UHF algebra $\mathfrak{F}$. If we can show that any ideal $\mathfrak{J}$ of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ is mapped by Φ into $\mathfrak{J} \cap \mathfrak{F}$, then the simplicity of $\mathfrak{F}$ will imply that $\mathrm{C}^*(\mathbb{F}_\theta^+)$ is also simple. To do this, we copy an argument that works for the Cuntz algebra [1] (see also [2, Theorem V.4.6]). We show that in the aperiodic case, the canonical expectation onto the UHF subalgebra is approximately inner. The interested reader should look at [14], where they proved simplicity for higher rank Cuntz–Kreiger algebras.

Theorem 5.1 *Let $\mathbb{F}_\theta^+$ be aperiodic. There is a sequence of isometries $W_k \in \mathrm{C}^*(\mathbb{F}_\theta^+)$ so that*

$$\Phi(A) = \lim_{k \to \infty} W_k^* A W_k \quad \text{for all } A \in \mathrm{C}^*(\mathbb{F}_\theta^+).$$

Proof It suffices to prove the claim for elements of the form uv^* where $u, v \in \mathbb{F}_\theta^+$. Recall that $\Phi(uv^*) = uv^*$ if $d(u) = d(v)$, and $\Phi(uv^*) = 0$ otherwise. Moreover in the first case, it suffices to suppose that $d(u) = d(v) = (k_0, k_0)$ for some $k_0 \geq 1$ sufficiently large. Indeed, if we have $d(u) = (k_1, k_2) \leq (k_0, k_0)$, then

$$uv^* = \sum_{d(x) = (k_0 - k_1, k_0 - k_2)} (ux)(vx)^*$$

is the sum of words with the desired degree.

Let τ be an aperiodic tail constructed by Theorem 3.4. Then there is a finite segment, say τ_k, which has no (a, b)-periodicity for $|a| \le m^k$ and $|b| \le n^k$. Let $\mathcal{S}_k = \{x \in \mathbb{F}_\theta^+ : d(x) = (k, k)\}$. Then set $W_k = \sum_{x\in\mathcal{S}_k} x\tau_k x^*$.

Suppose that $d(u) = d(v) = (k, k)$. Then

$$W_k^* uv^* W_k = \sum_{x\in\mathcal{S}_k}\sum_{y\in\mathcal{S}_k} x\tau_k^* x^* uv^* y\tau_k y^* = u\tau_k^*\tau_k v^* = uv^*.$$

On the other hand, suppose that

$$d(u) \vee d(v) \le (k, k) \quad \text{and} \quad d(v) - d(u) = (a, b) \ne (0, 0).$$

Then x^*uv^*y will either be 0 or will have the form $x_0^*y_0$ in reduced form of total degree (a, b). Therefore $\tau_k^* x_0^* y_0 \tau_k = 0$ because τ_k does not have (a, b) periodicity. So an examination of the calculation above yields $W_k^* uv^* W_k = 0$. ■

Corollary 5.2 *If $\mathbb{F}_\theta^+$ is aperiodic, then $\mathrm{C}^*(\mathbb{F}_\theta^+)$ is simple.*

Proof If $\mathfrak{J}$ is a non-zero ideal in $\mathrm{C}^*(\mathbb{F}_\theta^+)$, let A be a non-zero positive element. Then $\Phi(A) = \lim_{k\to\infty} W_k^* A W_k$ belongs to $\mathfrak{J} \cap \mathfrak{F}$. Since Φ is faithful, $\Phi(A) \ne 0$. It follows that $\mathfrak{J}$ contains the ideal of $\mathfrak{F}$ generated by $\Phi(A)$, which contains the identity because $\mathfrak{F}$ is simple. Therefore $\mathfrak{J} = \mathrm{C}^*(\mathbb{F}_\theta^+)$. ■

5.1 Periodic Algebras

We now turn to the structure of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ when $\mathbb{F}_\theta^+$ is periodic. Our goal is to show that $\mathrm{C}^*(\mathbb{F}_\theta^+) \simeq \mathrm{C}(\mathbb{T}) \otimes \mathfrak{A}$ where $\mathfrak{A}$ is simple.

Assume that the minimum period is $(a, -b)$ for $a, b > 0$. By Theorem 3.1, there is a bijection $\gamma\colon \mathbf{m}^a \to \mathbf{n}^b$ so that $e_u f_v = f_{\gamma(u)} e_{\gamma^{-1}(v)}$ for all $u \in \mathbf{m}^a$ and $v \in \mathbf{n}^b$.

Lemma 5.3 *Let $\mathbb{F}_\theta^+$ be periodic with minimal period $(a, -b)$ and define*

$$W := \sum_{u\in\mathbf{m}^a} f_{\gamma(u)} e_u^*.$$

Then W is a unitary in the centre of $\mathrm{C}^*(\mathbb{F}_\theta^+)$.

Proof It is clear that W is unitary since $\{e_u : u \in \mathbf{m}^a\}$ is a set of Cuntz isometries, as are $\{f_v : v \in \mathbf{n}^b\}$. By Proposition 4.5,

$$\begin{aligned} e_i W &= \sum_{u\in\mathbf{m}^a} e_i f_{\gamma(u)} e_u^* = \sum_{u'\in\mathbf{m}^{a-1}, j\in\mathbf{m}} e_i f_{\gamma(u'j)} e_{u'j}^* \\ &= \sum_{u'\in\mathbf{m}^{a-1}} f_{\gamma(iu')} \sum_{j=1}^{m} e_j e_j^*\, e_{u'}^* = \sum_{u'\in\mathbf{m}^{a-1}} f_{\gamma(iu')} e_{u'}^*. \end{aligned}$$

https://doi.org/10.4153/CJM-2009-058-0 Published online by Cambridge University Press

Thus we compute

$$\begin{aligned} We_i &= \sum_{u \in \mathbf{m}^a} f_{\gamma(u)} e_u^* e_i = \sum_{u' \in \mathbf{m}^{a-1}, j \in \mathbf{m}} f_{\gamma(ju')} e_{ju'}^* e_i \\ &= \sum_{u' \in \mathbf{m}^{a-1}, j \in \mathbf{m}} f_{\gamma(ju')} e_{u'}^* (e_j^* e_i) = \sum_{u' \in \mathbf{m}^{a-1}} f_{\gamma(iu')} e_{u'}^* = e_i W. \end{aligned}$$

Similarly, W commutes with each f_j, and hence it lies in the centre of $\mathrm{C}^*(\mathbb{F}_\theta^+)$. ■

Corollary 5.4 *Let $\mathbb{F}_\theta^+$ be periodic with minimal period $(a, -b)$. Then $f_{\gamma(u)} = e_u W$ for all $u \in \mathbf{m}^a$. Also if $u \in \mathbf{m}^a$ and $v \in \mathbf{n}^b$, then $e_u^* f_v = \delta_{u,\gamma^{-1}(v)} W$.*

Proof $e_u W = W e_u = \sum_{u' \in \mathbf{m}^a} f_{\gamma(u')} (e_{u'}^* e_u) = f_{\gamma(u)}$. Therefore

$$e_u^* f_{\gamma(u')} = e_u^* e_{u'} W = \delta_{u,u'} W.$$ ■

5.2 The Direct Integral Decomposition

Let λ_τ be the representation constructed in Corollary 3.5. This representation is a faithful representation of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ by [3]. By construction, the symmetry group $H_\tau = \mathbb{Z}(a, -b)$. By [4], λ_τ decomposes as a direct integral of irreducible representations of type 3bii. It will be helpful to see how this is done in this case, using the extra structure in our possession.

Following the explicit construction of Example 2.2, we have described λ_τ as an inductive limit of copies of the left regular representation. Indeed, $\mathbb{F}_\theta^+$ acts on the set $\mathcal{G}_\tau = \varinjlim(\mathcal{G}_s, \rho_s)$, where the ρ_s are injections of $\mathcal{G}_s$ into $\mathcal{G}_{s+1}$ determined by the word τ, and ι_s are injections of $\mathcal{G}_s$ into $\mathcal{G}_\tau$. We formed $\mathcal{H}_\tau = \ell^2(\mathcal{G}_\tau)$ and obtained a faithful representation of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ via the action

$$\lambda_\tau(w)\xi_{\iota_s(x)} = \xi_{\iota_s(wx)} \quad \text{for all } w, x \in \mathbb{F}_\theta^+ \text{ and } s \geq 0.$$

To understand the way that W of Lemma 5.3 acts on a basis vector ξ, observe that there is a unique word $u \in \mathbf{m}^a$ with $\xi \in \operatorname{Ran} \lambda_\tau(e_u)$. The unitary W acts on ξ by pulling back a steps along the blue edges to $\lambda_\tau(e_u)^*\xi$, and then pushing forward via $\lambda_\tau(f_{\gamma(u)})$. This can be computed at a basis vector ξ by representing it as $\xi_{\iota_s(x)}$ with $d(x) \geq (a, 0)$ by choosing s sufficiently large. Then x factors as $x = e_u x'$ with $|u| = a$, and $\lambda_\tau(W)\xi_{\iota_s(x)} = \xi_{\iota_s(f_{\gamma(u)} x')}$.

Put an equivalence relation $\sim$ on $\mathcal{G}_\tau$ by taking the equivalence classes to be the orbits of each basis vector under powers of W. That is, $x \sim y$ if and only if there is an integer $k \in \mathbb{Z}$ so that $W^k \xi_x = \xi_y$. Let the equivalence classes be denoted by $[x]$, and set $\mathcal{W}_{[x]} = \operatorname{span}\{\xi_y : y \in [x]\}$. Identify each $\mathcal{W}_{[x]}$ with $\ell^2(\mathbb{Z})$ by fixing a representative $x' \in [x]$ and sending the standard basis $\{\eta_k : k \in \mathbb{Z}\}$ for $\ell^2(\mathbb{Z})$ to $\mathcal{W}_{[x]}$ by setting $J_{[x]}\eta_k = W^k \xi_{x'}$.

We wish to choose the representative for each equivalence class in a consistent way. So begin with the element $x_0 = \iota_0(\varnothing)$. Let

$$\mathcal{S} = \{x \in \mathcal{G}_\tau : \xi_x = \lambda_\tau(e_u e_v^* f_w^*)\xi_{x_0} \text{ for } u, v \in \mathbb{F}_m^+,\ w \in \mathbb{F}_n^+,\ |w| < b\}.$$

We claim that $\mathcal{S}$ intersects each equivalence class of $\mathcal{G}_\tau/{\sim}$ in a single point.

Let $x \in \mathcal{G}_\tau$, and choose s so that $\iota_s(\mathcal{G}_s)$ contains x. Then with $\xi = \xi_{\iota_s(\varnothing)}$, we have words $e_u f_v$ and $f_w e_{u'}$ so that $\lambda_\tau(e_u f_v)\xi = \xi_x$ and $\lambda_\tau(f_w e_{u'})\xi = \xi_{x_0}$. If $|v| \equiv -k \pmod b$ for $1 \le k < b$, we replace ξ by $\xi_{\iota_{s+k}(\varnothing)}$ so that $|v|$ is a multiple of b. Factor $f_v = f_{v_1} \cdots f_{v_s}$ where $|v_i| = b$ and factor $f_w = f_{w_0} f_{w_1} \cdots f_{w_t}$ where $0 \le |w_0| < b$ and $|w_j| = b$ for $1 \le j \le t$. Then set $u_i = \gamma^{-1}(v_i)$ and $x_i = \gamma^{-1}(w_i)$ and use Corollary 5.4:

$$\begin{aligned}\xi_x &= \lambda_\tau(e_u f_v e_{u'}^* f_w^*)\xi_{x_0}\\ &= \lambda_\tau(e_u)\lambda_\tau(f_{v_1}\cdots f_{v_s})\lambda_\tau(e_{u'}^*)\lambda_\tau(f_{w_t}^*\cdots f_{w_1}^*)\lambda_\tau(f_{w_0})^*\xi_{x_0}\\ &= \lambda_\tau(e_u)\lambda_\tau(W^s e_{u_1}\cdots e_{u_s})\lambda_\tau(e_{u'}^*)\lambda_\tau(W^{t*}e_{x_t}^*\cdots e_{x_1}^*)\lambda_\tau(f_{w_0}^*)\xi_{x_0}\\ &= \lambda_\tau(W^{s-t})\lambda_\tau(e_{u''}e_{x''}^* f_{w_0}^*)\xi_{x_0} =: \lambda_\tau(W^{s-t})\xi_y,\end{aligned}$$

where $u'' = uu_1\cdots u_s$ and $x'' = x_1\cdots x_t u'$ and $\xi_y = \lambda_\tau(e_{u''}e_{x''}^* f_{w_0}^*)\xi_{x_0}$. Therefore $[x] \cap \mathcal{S}$ contains y.

Uniqueness follows because there is an essentially unique way to write

$$\xi_x = \lambda_\tau(e_u f_v e_{u'}^* f_w^*)\xi_{x_0},$$

except for reducing the word because of redundancies. As

$$\xi_x = \lambda_\tau(W^k)\xi_y = \lambda_\tau(W^k e_{u''}e_{v''}^* f_{w_0}^*)\xi_{x_0},$$

one sees $W^k e_{u''}e_{v''}^* f_{w_0}^*$ has degree $(|u''| - |v''| - ka, kb - |w_0|)$. Since $kb - |w_0|$ is not in $[1-b, 0]$, this point does not lie in $\mathcal{S}$.

The fact that W lies in the centre of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ means that if $w \in \mathbb{F}_\theta^+$ and $\lambda_\tau(w)\xi_x = \xi_y$ then $\lambda_\tau(wW^k)\xi_x = \lambda_\tau(W^k)\xi_y$. Thus $\lambda_\tau(w)$ maps each subspace $\mathcal{W}_{[x]}$ onto another subspace $\mathcal{W}_{[y]}$.

Let U be the bilateral shift on $\ell^2(\mathbb{Z})$. Then one sees that $\lambda_\tau(W)$ acts as a shift $J_{[x]}UJ_{[x]}^*$ on every $\mathcal{W}_{[x]}$. Hence $\lambda_\tau(W) \simeq U \otimes I$ is a bilateral shift of infinite multiplicity. In particular, the spectrum of W is the whole circle $\mathbb{T}$, and the spectral measure of $\lambda_\tau(W)$ is Lebesgue measure.

Let us consider how $\lambda_\tau(e_i)$ acts on $\mathcal{W}_{[x]}$, say $[x] \cap \mathcal{S} = \{y\}$, where

$$\xi_y = \lambda_\tau(e_u e_v^* f_w^*)\xi_{x_0}.$$

Then $[e_i x] \cap \mathcal{S}$ is z where $\xi_z = \lambda_\tau(e_{iu}e_v^* f_w^*)\xi_{x_0}$. This is in reduced form unless $u = \varnothing$ and $v = v'i$, in which case, after cancellation, $\xi_z = \lambda_\tau(e_{v'}^* f_w^*)\xi_{x_0}$. Either way, we see that $z \in \mathcal{S}$. Hence $\lambda_\tau(e_i)|_{\mathcal{W}_{[x]}} = J_{[z]}J_{[y]}^* = J_{[e_i x]}J_{[x]}^*$.

Similarly we analyze $\lambda_\tau(f_j)$. This comes down to understanding the representative $[f_j x] \cap \mathcal{S}$. Again we use the representative $y \in [x]$ with $\xi_y = \lambda_\tau(e_u e_v^* f_w^*)\xi_{x_0}$. If $|w| \ge 1$, write $w = w'i$. Then there is a unique word $\tilde v \in \mathbb{F}_n^+$ with $|\tilde v| = b-1$ so that $\lambda_\tau(e_v^* f_w^*)\xi_{x_0} = \lambda_\tau(f_{\tilde v} f_{\tilde v}^* e_v^* f_w^*)\xi_{x_0}$ is in the range of $\lambda_\tau(f_{\tilde v})$. Therefore using the commutation relations $f_j e_u = e_{u'} f_{j'}$ and $e_v^* f_i^* = f_{i'}^* e_{v'}^*$,

$$\begin{aligned}\lambda_\tau(f_j e_u e_v^* f_w^*)\xi_0 &= \lambda_\tau(f_j e_u f_{\tilde v} f_{\tilde v}^* e_v^* f_w^*)\xi_0 \\ &= \lambda_\tau(e_{u'} f_{j'\tilde v} f_{i'\tilde v}^* e_{v'}^* f_{w'})\xi_0 \\ &= \lambda_\tau(e_{u'} e_{j'\tilde v} W W^* e_{i'\tilde v}^* e_{v'}^* f_{w'})\xi_0 \\ &= \lambda_\tau(e_{u'j'\tilde v} e_{v'i'\tilde v}^* f_{w'})\xi_0.\end{aligned}$$

So again, we see that the canonical representative y in $[x]$ is carried by f_j to $f_j y$, the canonical representative in $[f_j x]$, whence $\lambda_\tau(f_j)|_{\mathcal{W}_{[x]}} = J_{[f_j x]} J_{[x]}^*$.

However, when $w = \varnothing$, one obtains

$$\begin{aligned}\lambda_\tau(f_j e_u e_v^*)\xi_0 &= \lambda_\tau(e_{u'} f_{j'\tilde v} f_{\tilde v}^* e_{v'}^*)\xi_0 \\ &= \lambda_\tau(e_{u'} e_{j'\tilde v} W f_{\tilde v}^* e_{v'}^*)\xi_0 \\ &= \lambda_\tau(W e_{u'j'\tilde v} e_{v''}^* f_{w'})\xi_0,\end{aligned}$$

where $|w'| = |\tilde v| = b - 1$. So when $\lambda_\tau(f_j)$ maps $\mathcal{W}_{[x]}$ to $\mathcal{W}_{[f_j x]}$, it is acting like the bilateral shift, namely $\lambda_\tau(f_j)|_{\mathcal{W}_{[x]}} = J_{[f_j x]} U J_{[x]}^*$.

Form a Hilbert space $\mathcal{K} = \ell^2(\mathcal{G}_\tau/\sim)$ with basis $\{\zeta_{[x]}; [x] \in \mathcal{G}_\tau/\sim\}$. One can see that λ_τ is unitarily equivalent to a representation π_τ on $\ell^2(\mathcal{G}_\tau/\sim) \otimes L^2(\mathbb{T})$ given by

$$\begin{aligned}&\pi_\tau(e_i)\zeta_{[x]} \otimes h = \zeta_{[e_i x]} \otimes h, \\ &\pi_\tau(f_j)\zeta_{[x]} \otimes h = \begin{cases} \zeta_{[f_j x]} \otimes h & \text{if } [x] = [e_u e_v^* f_w^* x_0],\ 1 \le |w| < b, \\ \zeta_{[f_j x]} \otimes zh & \text{if } [x] = [e_u e_v^* x_0], \end{cases}\end{aligned}$$

for all $[x] \in \mathcal{G}_\tau/\sim$ and $h \in L^2(\mathbb{T})$.

Define a representation σ_1 by $\sigma_1(w)\zeta_{[x]} = \zeta_{[y]}$ if $\lambda_\tau(w)\mathcal{W}_{[x]} = \mathcal{W}_{[y]}$, *i.e.*, $\sigma_1(w)\zeta_{[x]} = \zeta_{[wx]}$, Then for each $z \in \mathbb{T}$, define $\sigma_z(e_i) = \sigma_1(e_i)$ and $\sigma_z(f_j) = z\sigma_1(f_j)$ and extend to $\mathbb{F}_\theta^+$. It is not difficult to see that σ_z is unitarily equivalent to another atomic representation ρ_{z^b}, given by

$$\begin{aligned}&\rho_{z^b}(e_i)\zeta_{[x]} = \zeta_{[e_i x]}, \\ &\rho_{z^b}(f_j)\zeta_{[x]} = \begin{cases} \zeta_{[f_j x]} & \text{if } [x] = [e_u e_v^* f_w^* x_0],\ 1 \le |w| < b, \\ z^b \zeta_{[f_j x]} & \text{if } [x] = [e_u e_v^* x_0]. \end{cases}\end{aligned}$$

In particular, $\sigma_z \simeq \sigma_w$ if and only if $z^b = w^b$.

The representations σ_z are irreducible by [4, Lemma 8.14] because the symmetry group $H_{\sigma_z} = \{0\}$ is the trivial subgroup of $\mathbb{Z}^2/\mathbb{Z}(a, -b)$. Note that $\sigma_z(W) = z^b I$.

From this picture, one can see how to decompose the representation λ_τ as a direct integral. Indeed,

$$\lambda_\tau \simeq \int_0^{2\pi/b\,\oplus} \sigma_{e^{it}}\, dt \simeq \int_0^{2\pi\,\oplus} \rho_{e^{it}}\, dt.$$

In particular, notice that the C*-algebra $\mathfrak{A} = \sigma_z(\mathrm{C}^*(\mathbb{F}_\theta^+))$ is independent of z.

5.3 An Expectation

We need to build a somewhat different expectation in the periodic case.

Theorem 5.5 *If $\mathbb{F}_\theta^+$ is a periodic semigroup with minimal period $(a,-b)$, define*

$$\Psi(X) = \int_{\mathbb{T}} \gamma_{z^b,z^a}(X)\,dz.$$

Then Ψ is a faithful, approximately inner expectation onto

$$\mathrm{C}^*(\mathfrak{F}, W) \simeq \mathrm{C}(\mathbb{T}) \otimes \mathfrak{F} \simeq \mathrm{C}(\mathbb{T}, \mathfrak{F}).$$

Proof As an integral of automorphisms, Ψ is evidently a faithful completely positive map. Suppose that $w = e_u f_v e_{u'}^* f_{v'}^*$ is a word of degree (k,l) where $k = |u| - |u'|$ and $l = |v| - |v'|$. Then

$$\Psi(e_u f_v e_{u'}^* f_{v'}^*) = \int_{\mathbb{T}} z^{kb+la} e_u f_v e_{u'}^* f_{v'}^*\,dz = \begin{cases} e_u f_v e_{u'}^* f_{v'}^* & \text{if } kb+la=0,\\ 0 & \text{otherwise.}\end{cases}$$

That is, $\Psi(w) = w$ if $d(w) \in \mathbb{Z}(a,-b)$ and is 0 otherwise. Therefore this is an idempotent map, and so it is an expectation. Since the degree map is a homomorphism, the range of Ψ is a C^*-subalgebra of $\mathrm{C}^*(\mathbb{F}_\theta^+)$.

The range contains $\mathfrak{F}$ as this is spanned by words of degree $(0,0)$ and W, which has degree $(-a,b)$. The typical word of degree $(-pa,pb)$ is $w = e_u f_v e_{u'}^* f_{v'}^*$ where $|u| - |u'| = -pa$ and $|v| - |v'| = pb$. For convenience, consider $p \geq 0$. If $|u| \equiv -k \pmod a$ for $0 \leq k < a$ and $|v| \equiv -l \pmod b$ for $0 \leq l < b$, then

$$w = \sum_{x \in \mathbb{F}_\theta^+, d(x)=(k,l)} e_u f_v x x^* e_{u'}^* f_{v'}^*.$$

Hence w is in the span of words of the form $e_u f_v e_{u'}^* f_{v'}^*$ for which

$$|u| \equiv |u'| \equiv 0 \pmod a \quad \text{and} \quad |v| \equiv |v'| \equiv 0 \pmod b.$$

For such words, we may split each word u, u' into words of length a and each v, v' into words of length b as

$$\begin{aligned} w' &= e_{u_1\cdots u_s} f_{v_1\cdots v_{t+p}} e^*_{u'_1\cdots u'_{s+p}} f^*_{v'_1\cdots v'_t} \\ &= W^p e_{u_1\cdots u_s \gamma^{-1}(v_1)\cdots\gamma^{-1}(v_{t+p})} e^*_{\gamma^{-1}(v'_1)\cdots\gamma^{-1}(v'_t) u'_1\cdots u'_{s+p}}. \end{aligned}$$

Therefore all of these words lie in $\mathrm{C}^*(\mathfrak{F}, W)$. So this identifies the range of Ψ as $\mathrm{C}^*(\mathfrak{F}, W)$.

Now W lies in the centre of $\mathrm{C}^*(\mathbb{F}_\theta^+)$; and it is easy to check that $\mathrm{C}^*(W) \cap \mathfrak{F} = \mathbb{C}I$. So a dense subalgebra of $\mathrm{C}^*(\mathfrak{F}, W)$ is given by the polynomials $\sum_k F_k W^k$ where the sum is finite and $F_k \in \mathfrak{F}$. Since the spectrum of W is $\mathbb{T}$,

$$\Big\| \sum_k F_k W^k \Big\| = \sup_{z\in\mathbb{T}} \Big\| \sum_k F_k z^k \Big\|.$$

Now a routine modification of the proof of Féjer's Theorem shows that $\mathrm{C}^*(\mathfrak{F}, W) \simeq \mathrm{C}(\mathbb{T}, \mathfrak{F}) \simeq \mathrm{C}(\mathbb{T}) \otimes \mathfrak{F}$.

The last part of the proof is to establish that Ψ is approximately inner. The argument is a modification of the proof of Theorem 5.1.

Let τ be the infinite tail used above whose only symmetries are $\mathbb{Z}(a, -b)$. Then there is a finite segment, say τ_k, such that $\tau_k^* u^* v \tau_k = 0$ whenever $u, v \in \mathbb{F}_\theta^+$ with

$$d(u) - d(v) \notin \mathbb{Z}(a, -b) \quad \text{and} \quad \max\{d(u), d(v)\} \leq (ka, kb).$$

Let $\mathcal{S}_k = \{x \in \mathbb{F}_\theta^+ : d(x) = (ak, bk)\}$. Then set $W_k = \sum_{x \in \mathcal{S}_k} x \tau_k x^*$.

Suppose that $w = uv^*$ for $u, v \in \mathbb{F}_\theta^+$ such that $d(w) \in \mathbb{Z}(a, -b)$ and

$$\max\{d(u), d(v)\} \leq (ka, kb).$$

Then as before, we may write w as a sum of words with the same property and the additional stipulation that $d(u) \vee d(v) = (ka, kb)$. Say, as a typical case, that $d(u) = (ka, (k-p)b)$ and $d(v) = ((k-p)a, kb)$. Then $uv^* = W^{-p}(uW^p v^*)$, and the term $uW^p v^*$ splits as a sum of words xy^* with $d(x) = d(y) = (pa, pb)$. Thus if $d(x_0) = d(y_0) = (pa, pb)$,

$$\begin{aligned} W_k^* W^{*p} x_0 y_0^* W_k &= W^{*p} \sum_{x \in \mathcal{S}_k} \sum_{y \in \mathcal{S}_k} x \tau_k^* x^* x_0 y_0^* y \tau_k y^* \\ &= W^{*p} x_0 \tau_k^* \tau_k y_0^* = W^{*p} x_0 y_0^*. \end{aligned}$$

On the other hand, suppose that $w = uv^*$ for $u, v \in \mathbb{F}_\theta^+$ with $d(w) \notin \mathbb{Z}(a, -b)$ and $\max\{d(u), d(v)\} \leq (ka, kb)$. Then $x^* uv^* y$ will either be 0 or will have the form $x_0^* y_0$ in reduced form with

$$\begin{aligned} d(y_0) - d(x_0) &= (d(v) - d(y)) - (d(u) - d(x)) \\ &= d(v) - d(u) = -d(w) \notin \mathbb{Z}(a, -b). \end{aligned}$$

Therefore $\tau_k^* x_0^* y_0 \tau_k = 0$, whence $W_k^* uv^* W_k = 0$. ∎

Now we "evaluate at 1" and obtain a faithful, approximately inner expectation of $\mathfrak{A} = \sigma_1(\mathrm{C}^*(\mathbb{F}_\theta^+))$ onto $\mathfrak{F}$.

Theorem 5.6 *There is a faithful, approximately inner expectation Ψ_1 of $\mathfrak{A}$ onto $\mathfrak{F}$ so that the following diagram commutes, where ε_1 is evaluation at 1:*

$$\begin{tikzcd} \mathrm{C}^*(\mathbb{F}_\theta^+) \arrow[r, "\sigma_1"] \arrow[d, "\Psi"'] & \mathfrak{A} \arrow[d, "\Psi_1"] \\ \mathrm{C}(\mathbb{T}, \mathfrak{F}) \arrow[r, "\varepsilon_1"] & \mathfrak{F} \end{tikzcd}$$

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Proof The first step is to observe that [3, Lemma 3.4] shows that there is a unitary $U_z := U_{z^b,z^a}$ given by

$$U_z \xi_{\iota_s(e_u f_v)} = z^{b(|u|-s)+a(|v|-s)} \xi_{\iota_s(e_u f_v)},$$

which implements γ_{z^b,z^a} in that

$$\gamma_{z^b,z^a}(X) = U_z \lambda_\tau(X) U_z^* \quad \text{for } X \in \mathrm{C}^*(\mathbb{F}_\theta^+).$$

Observe that U_z is constant on the subspaces $\mathcal{W}_{[x]}$, and thus it determines a unitary V_z on $\ell^2(\mathcal{G}_\tau/\sim)$ by $V_z \zeta_{[x]} = t\zeta_{[x]}$ if $U_z|_{\mathcal{W}_{[x]}} = tP_{\mathcal{W}_{[x]}}$.

Clearly $V_z \sigma_1(X) V_z^* = \sigma_1(U_z \lambda_\tau(X) U_z^*) = \sigma_1(\gamma_{z^b,z^a}(X))$. This determines an automorphism ψ_z of $\mathfrak{A}$. Define a map $\Psi_1(A) = \int_{\mathbb{T}} \psi_z(A)\, dz$ for $A \in \mathfrak{A}$. It follows that $\Psi_1(\sigma_1(X)) = \sigma_1(\Psi(X))$ for $X \in \mathrm{C}^*(\mathbb{F}_\theta^+)$. In particular, Ψ_1 is a faithful expectation of $\mathfrak{A}$ onto $\sigma_1(\mathrm{C}^*(\mathfrak{F}, W))$.

Now $\sigma_1(W) = I$. So $\sigma_1(\mathrm{C}^*(\mathfrak{F}, W)) = \sigma_1(\mathfrak{F}) \simeq \mathfrak{F}$, because $\mathfrak{F}$ is simple. The ideal of $\mathrm{C}^*(\mathfrak{F}, W)$ generated by $W - I$ is contained in $\ker \sigma_1$. But this ideal is evidently $\mathrm{C}_0(\mathbb{T} \setminus \{1\}) \otimes \mathfrak{F}$. Thus $\sigma_1|_{\mathrm{C}^*(\mathfrak{F},W)} = \varepsilon_1$ is evaluation at 1.

Finally, since Ψ is approximately inner, we obtain for any $A = \sigma_1(X)$ in $\mathfrak{A}$ that

$$\Psi_1(A) = \sigma_1(\Psi(X)) = \lim_{k\to\infty} \sigma_1(W_k^* X W_k) = \lim_{k\to\infty} \sigma_1(W_k^*) A \sigma_1(W_k).$$

So Ψ_1 is approximately inner. ■

Corollary 5.7 *$\mathfrak{A}$ is simple.*

Proof If $\mathfrak{J}$ is a non-zero ideal, then it contains a positive element A. Since Ψ_1 is faithful, $\Psi_1(A) \neq 0$. As Ψ_1 is approximately inner, $\Psi_1(A)$ belongs to $\mathfrak{F} \cap \mathfrak{J}$. But $\mathfrak{F}$ is simple and unital, so $\mathfrak{J}$ contains $\mathfrak{F}$ and thus is all of $\mathfrak{A}$. ■

Corollary 5.8 *Let $\mathbb{F}_\theta^+$ have minimal period $(a, -b)$. Then $\mathrm{C}^*(\mathbb{F}_\theta^+) \simeq \mathrm{C}(\mathbb{T}) \otimes \mathfrak{A}$.*

Proof Here it is more convenient to use the representations ρ_z. The analysis above applied to each representation ρ_z shows that $\ker \rho_z = \langle W - zI \rangle$. Define a map φ from $\mathrm{C}^*(\mathbb{F}_\theta^+)$ to $\mathrm{C}(\mathbb{T}, \mathfrak{A})$ by $\varphi(X)(z) = \rho_z(X)$. Checking continuity is straightforward, as is surjectivity. That the map is injective follows from the direct integral decomposition of the faithful representation λ_τ. ■

As a corollary, we obtain a special case of the result of Robertson and Sims [13] on the simplicity of higher rank graph algebras.

Corollary 5.9 *Let $\mathbb{F}_\theta^+$ be a rank 2 graph with a single vertex. Then $\mathrm{C}^*(\mathbb{F}_\theta^+)$ is simple if and only if $\mathbb{F}_\theta^+$ is aperiodic.*

Corollary 5.10 *Let $\mathbb{F}_\theta^+$ be a periodic rank 2 graph with a single vertex. Then the centre of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ is $\mathrm{C}^*(W) \simeq \mathrm{C}(\mathbb{T})$.*

Acknowledgements We would like to thank Stephen Power for helpful conversations. We also thank Colin Davidson and Shaoquan Jiang for assistance in writing computer code for some of our calculations.

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