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PROOF OF THE RADICAL CONJECTURE FOR HOMOGENEOUS KAHLER MANIFOLDS

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Introduction

In 1967 Gindikin and Vinberg stated the Fundamental Conjecture for homogeneous Kähler manifolds. It (roughly) states that every homogeneous Kähler manifold is a fiber space over a bounded homogeneous domain for which the fibers are a product of a flat with a simply connected compact homogeneous Kähler manifold. This conjecture has been proven in a number of cases (see [6] for a recent survey). In particular, it holds if the homogeneous Kähler manifold admits a reductive or an arbitrary solvable transitive group of automorphisms [5]. It is thus tempting to think about the general case. It is natural to expect that lack of knowledge about the radical of a transitive group G of automorphisms of a homogeneous Kähler manifold M is the main obstruction to a proof of the Fundamental Conjecture for M. Thus it is of importance to consider the Kähler algebra generated by the radical of the Lie algebra of G. Computations in this context suggest that one rather considers Kähler algebras generated by an arbitrary solvable ideal. In this context the Radical Conjecture for Kähler algebras was formulated [6]: Assume that the Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ is generated by a solvable ideal $\mathfrak{r}$ of $\mathfrak{g}$, i.e. $\mathfrak{g}=\mathfrak{r}+j\mathfrak{r}+\mathfrak{k}$, then $\mathfrak{g}=\mathfrak{s}+\mathfrak{k}$, where $\mathfrak{s}\cap\mathfrak{k}=0$, $j\mathfrak{s}\subset\mathfrak{s}$ (after an inessential change of j), and $\mathfrak{s}$ is a solvable Kähler algebra.

If $\mathfrak{r}$ is abelian, a direct proof of the Radical Conjecture can be given (following closely a proof of Gindikin, Piatetskii-Shapiro and Vinberg [8]. A proof of the Radical Conjecture proceeds by induction on $\dim\mathfrak{r}$. It was started in [6]. Since the case that $\mathfrak{r}$ is abelian was already settled one considers a maximal ideal $\mathfrak{n}$ of $\mathfrak{g}$ properly contained in $\mathfrak{r}$ and sets $\mathfrak{g}'=\mathfrak{n}+j\mathfrak{n}+\mathfrak{k}$. For this Kähler algebra the Radical Conjecture already holds. Generalizing constructions of Gindikin, Piatetskii-Shapiro and Vinberg it

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was shown in [6] that only three cases have to be considered (one is that $\mathfrak{n}+j\mathfrak{n}$ is essentially an abelian Kähler algebra, the other two are characterized by the distributions of eigenvalues of a maximal idempotent in $\mathfrak{s}'$, where $\mathfrak{s}'$ is associated with $\mathfrak{g}'$ via the Radical Conjecture, $\mathfrak{g}'=\mathfrak{s}'+\mathfrak{k}$).

In the present paper, we continue and finish the proof of the Radical Conjecture.

The details are rather technical and involved. We thus only want to point out that in Case 1 (where $\mathfrak{n}+j\mathfrak{n}$ is essentially abelian) we prove a statement which is stronger than the Radical Conjecture. In Case 3 we combine the description of the representations of $sl(2, Z)$ with results on the Kantor-Koecher-Tits construction of Lie algebras with more standard techniques of Kähler algebras to prove the Radical Conjecture.

Finally, we would like to note that the Radical Conjecture and a substantial part of its proof have been used in the recent proof of the Fundamental Conjecture (jointly with K. Nakajima).

I would like to thank K. Johnson for making me aware of [16] and E. Neher for helpful discussions and information about Jordan triples (the structure of which is used in case 3).

I would like to express great appreciation to K. Nakajima for his careful reading of a preliminary version of this paper, in particular for many critical and helpful remarks and for simplifying Lemma 1.5, Lemma 2.17 and section 3.7. He also pointed out to me a subcase of case 3 which was not discucssed originally.

§ 1. Case 1: The Lie algebra $\mathfrak{n}+j\mathfrak{n}$ is the modification of an abelian Kähler algebra

1.1. As in [6; 4.33] we consider a Kähler algbra $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ and a solvable ideal $\mathfrak{r}$ of $\mathfrak{g}$ satisfying $\mathfrak{g}=\mathfrak{r}+j\mathfrak{r}+\mathfrak{k}$. We assume that the Radical Conjecture holds if $\dim \mathfrak{r}\leq N-1$. We assume $\dim \mathfrak{r}=N$ and we may assume $\mathfrak{r}\subset \mathrm{nil}(\mathfrak{g})$. Moreover we can assume that the dimension of $\mathfrak{r}$ is minimal among those solvable ideals $\mathfrak{u}$ of $\mathfrak{g}$ for which $\mathfrak{g}=\mathfrak{u}+j\mathfrak{u}+\mathfrak{k}$ holds. We choose an ideal $\mathfrak{n}$ of $\mathfrak{g}$ which satisfies $\mathfrak{r}\supsetneqq\mathfrak{n}\supset[\mathfrak{r}, \mathfrak{r}]$ and is maximal with this property. Because the case of an abelian $\mathfrak{r}$ has been settled in [6] we can also assume $\mathfrak{n}\neq 0$.

We set $\mathfrak{g}'=\mathfrak{n}+j\mathfrak{n}+\mathfrak{k}$ and apply the Radical Conjecture to $\mathfrak{g}'$. Hence $\mathfrak{g}'=\mathfrak{a}+\mathfrak{t}+\mathfrak{k}$ where $\mathfrak{a}+\mathfrak{t}$ is a solvable Kähler algebra. The case under consideration in this paper is defined by $\mathfrak{t}=0$, i.e. $\mathfrak{g}'=\mathfrak{a}+\mathfrak{k}$ where $\mathfrak{a}$ is the modification of an abelian Kähler algebra. Such algebras have been investigated in [5].

1.2. From [5; 3.3] we know $\mathfrak{a} = \mathfrak{n} + j\mathfrak{n} = \hat{\mathfrak{a}}_0 + \hat{\mathfrak{a}}_1$ where $\hat{\mathfrak{a}}_0$ is an abelian ideal of $\mathfrak{a}$ and $\hat{\mathfrak{a}}_1$ is an abelian subalgebra. Moreover $\hat{\mathfrak{a}}_0 = [\mathfrak{a}, \mathfrak{a}]$. Because $\mathfrak{n}$ acts nilpotently on $\mathfrak{g}$ we have $[\mathfrak{n}, \mathfrak{n}] = 0$. Set $\mathfrak{n}_0 = \hat{\mathfrak{a}}_0 \cap \mathfrak{n}$ and $\mathfrak{n}_1 = \hat{\mathfrak{a}}_1 \cap \mathfrak{n}$.

LEMMA. $\mathfrak{n} = \mathfrak{n}_0 + \mathfrak{n}_1$.

Proof. Let $n \in \mathfrak{n}$ and $n = a_0 + a_1$, $a_j \in \hat{\mathfrak{a}}_j$. From the fact that $\mathfrak{a}$ is the modification of an abelian Kähler algebra we get $[b, n] = D(b)n - D(n)b$ for all $b \in \mathfrak{a}$. As $ad n$ is nilpotent we have $D(n) = 0$. Further, from [5: 3.3] we know $D(b)\hat{\mathfrak{a}}_1 = 0$. Therefore $[b, n] = D(b)a_0 \in \mathfrak{n}$ for all $b \in \mathfrak{a}$. Using $\hat{\mathfrak{a}}_0 = [\mathfrak{a}, \mathfrak{a}]$ we derive from this $a_0 \in \mathfrak{n}$. The lemma follows.

1.3. We note that $\mathfrak{n}_0 + j\mathfrak{n}_0$ is contained in the commutator $\hat{\mathfrak{a}}_0 = [\mathfrak{a}, \mathfrak{a}]$ of the solvable Lie algebra $\mathfrak{a}$. Hence $\mathrm{ad}(\mathfrak{n}_0 + j\mathfrak{n}_0)$ is a commuting family of nilpotent derivations of $\mathfrak{g}$. In particular we have $[\mathfrak{n}_0 + j\mathfrak{n}_0, \mathfrak{n}] = 0$.

In contrast to this the family $\mathrm{ad}(\mathfrak{n}_1 + j\mathfrak{n}_1)$ is abelian but does not consist in general of nilpotent derivations.

1.4. Consider $\mathfrak{g}' = \mathfrak{a} + \mathfrak{k}$. From the Radical Conjecture we know $\mathfrak{a} \cap \mathfrak{k} = 0$ and $\mathfrak{a}$ is a solvable Kähler subalgebra of $\mathfrak{g}'$. Let $\mathfrak{k}_0 \subset \mathfrak{k}$ be an ideal of $\mathfrak{g}'$. We know that $\mathfrak{k}$ is the Lie algebra of a compact group and $\mathfrak{k}_0$ is an ideal of $\mathfrak{k}$. Hence $\mathfrak{k} = (\mathfrak{z}_1 + \mathfrak{z}_0) \oplus \mathfrak{k}'_1 \oplus \mathfrak{k}'_0$ where $\mathfrak{k}_0 = \mathfrak{k}'_0 \oplus \mathfrak{z}_0$, $\mathfrak{k}'_1$, $\mathfrak{k}'_0$ semisimple, $\mathfrak{z}_0$ $\mathfrak{z}_1$ abelian. Let $\mathfrak{h}$ be a maximal semisimple subalgebra of $\mathfrak{g}'$ containing $\mathfrak{k}'_1 \oplus \mathfrak{k}'_0$. Then $\mathfrak{k}'_0$ is an ideal of $\mathfrak{h}$ and we get $\mathfrak{h} = \mathfrak{h}'_1 \oplus \mathfrak{k}'_0$. Moreover, $[\mathrm{rad}\, \mathfrak{g}' + \mathfrak{h}'_1, \mathfrak{k}'_0] = 0$. Since $\mathfrak{n} \subset \mathrm{rad}\, \mathfrak{g}'$ we can assume w.r.g. that $j\mathfrak{n}$ projects trivially onto $\mathfrak{k}'_0$. Hence $[\mathfrak{a}, \mathfrak{k}'_0] = 0$ and $\mathfrak{g}' = (\mathfrak{a} + \mathfrak{z}_1 + \mathfrak{z}_0 + \mathfrak{k}'_1) \oplus \mathfrak{k}'_0$ is a direct sum of Lie agebras. Moreover, $\mathfrak{z}_0$ is an ideal of $\mathfrak{g}'$ and $\mathfrak{z}_0 \subset \mathrm{rad}\, \mathfrak{g}'$. But $\mathfrak{z}_0 \cap \mathrm{nil}(\mathfrak{g}') = 0$ since $\mathrm{nil}(\mathfrak{g}')$ operates nilpotently on $\mathfrak{g}$ whereas $\mathfrak{k}_0$ acts semisimply. From this we conclude $[\mathfrak{g}', \mathfrak{z}_0] = 0$.

Therefore, when considering $\mathfrak{g}'$ only, we may assume that $\mathfrak{k}$ does not contain any ideal of $\mathfrak{g}'$. But then we have a faithful representation ψ of $\mathfrak{g}'$ as affine transformations of the complex vector space $\mathfrak{a}$. The elements of $\mathfrak{k}$ act linearly on $\mathfrak{a}$ and the elements of $\mathfrak{a}$ by $z \mapsto a + D(a)z$, $z \in \mathfrak{a}$, $a \in \mathfrak{a}$ (see [5; 3.5]). Considering the linear parts of these transformations we see that $D(\mathfrak{a}) + \psi(\mathfrak{k}) = \mathfrak{S}$ is a Lie algebra of skewadjoint endomorphisms of $\mathfrak{a}$; moreover, $D(\mathfrak{a})$ is an abelian subalgebra, $\psi(\mathfrak{k})$ is a subalgebra and (possibly changing j inessentially) we also can assume $D(\mathfrak{a}) \cap \psi(\mathfrak{k}) = 0$. We

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split $\mathfrak{G} = \oplus \mathfrak{G}_i + \mathfrak{Z}$ into simple summands $\mathfrak{G}_i$ and its center $\mathfrak{Z}$. Applying [16: Theorem 1.1] to the projections $D_i(\mathfrak{a})$ and $\psi_i(\mathfrak{k})$ of $D(\mathfrak{a})$ and $\psi(\mathfrak{k})$ onto $\mathfrak{G}_i$ shows that $D_i(\mathfrak{a}) \subset \psi_i(\mathfrak{k})$ holds. Therefore $D(a) \subset \psi(\mathfrak{k})$ mod center $(D(\mathfrak{a}) + \psi(\mathfrak{k}))$. This implies:

LEMMA. *After an inessential change of j we can assume* $[D(\mathfrak{a}), \psi(\mathfrak{k})] = 0$.

For the original Lie Algebra $\mathfrak{g}'$ (ideal in $\mathfrak{k}$ permitted) this implies (after some inessential change of j):

COROLLARY 1. a) $[\mathfrak{k}, \mathfrak{a}] \subset \mathfrak{a}$,
b) $D([\mathfrak{k}, \mathfrak{a}]) = 0$,
c) $[D(\mathfrak{a}), \operatorname{ad} \mathfrak{k} | \mathfrak{a}] = 0$,
d) $[\operatorname{ad} \mathfrak{k} | \mathfrak{a}, j] = 0$.

Proof. Write $\mathfrak{k} = \mathfrak{z} + \mathfrak{k}_1' + \mathfrak{k}_0'$ and represent $\mathfrak{a}$ by affine transformations on some complex vector space $(\cong \mathfrak{a})$. Then $[k, a] = [(0, \psi(k)), (a, D(a))] = (\psi(k)a, [\psi(k), D(a)]) = (\psi(k)a, 0) = (\psi(k)a, D(\psi(k)a)) - (0, D(\psi(k)a))$ where the last summand has to be in $\mathfrak{k}$. By our assumption, we may assume $a = jn$ for some $n \in \mathfrak{n}$, whence $\psi(k)a = i(\psi(k)n)$. Changing j on $\psi(k)\mathfrak{n}$ inessentially we obtain $D(i(\psi(k)\mathfrak{n})) = 0$, proving a) and b). Part c) has been known before and d) follows from a).

We retain the notation of 1.2 and 1.3 and write $\hat{\mathfrak{a}}_1 = \hat{\mathfrak{a}}_{10} + \hat{\mathfrak{a}}_{11}$ as in [5: 3.3].

COROLLARY 2. a) $[\mathfrak{k}, \hat{\mathfrak{a}}_{10}] \subset \hat{\mathfrak{a}}_{10}$,
b) $[\mathfrak{k}, \hat{\mathfrak{a}}_{11}] = 0$,
c) $[\mathfrak{k}, \hat{\mathfrak{a}}_0] \subset \hat{\mathfrak{a}}_0$.

Proof. Since $[\mathfrak{k}, \mathfrak{a}] \subset \mathfrak{a}$ and $\hat{\mathfrak{a}}_0 = [\mathfrak{a}, \mathfrak{a}]$ part c) is clear. We know that $\mathfrak{k}$ acts skewadjoint on $\mathfrak{a}$, whence $[\mathfrak{k}, a_1] \subset a_1$. By Corollary 1 we have $[\mathfrak{k}, \mathfrak{a}_1] \subset \hat{\mathfrak{a}}_{10}$ and a) and b) follow.

1.5. Let $n \in \mathfrak{n}_1$ and write $\operatorname{ad} jn = D + N$ where D is the semisimple part of $\operatorname{ad} jn$ and N its nilpotent part.

The proof of the following lemma is a simplified version of the author's original proof. We follow a suggestion of K. Nakajima.

LEMMA. *For $n \in \mathfrak{n}_1$, the semisimple part D of* ad jn *has only imaginary eigenvalues.*

Proof. Let R be the real part of D and $\mathfrak{g} = \oplus \mathfrak{g}_\lambda$ be the decomposition of $\mathfrak{g}$ into eigenspaces for R. Because $D|\mathfrak{g}'$ has only imaginary eigenvalues we get

(1) $\mathfrak{g}' \subset \mathfrak{g}_0$.

The integrability condition of $\mathfrak{g}$ implies $[jn, jx_\lambda] = j[jn, x_\lambda] \bmod \mathfrak{g}'$ for all $x_\lambda \in \mathfrak{g}_\lambda$. Therefore

(2) $j\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda + \mathfrak{g}'$.

Moreover, by an inessential change of j we may even assume

(3) $j\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda + \mathfrak{n} + j\mathfrak{n}$.

Let λ be the eigenvalue of R with maximal absolute value. We may assume $\lambda \geq 0$. Suppose $\lambda > 0$. Then $[\mathfrak{g}_\lambda, \mathfrak{g}_\lambda] = 0$. Therefore

(4) $\mathfrak{g}_\lambda + \mathfrak{n} + j\mathfrak{n}$ is a solvable Kähler algebra.

By [5] we know that $\operatorname{ad} jn$ has only purely imaginary eigenvalues in this algebra. This is a contradiction; hence we obtain $\lambda = 0$, finishing the proof of the lemma.

1.6. In the last section we have seen that the semisimple parts of elements of $\mathfrak{n}_1$ have only imaginary eigenvalues. In the following sections we show that, by a change of ρ and a modification, we can remove these semisimple parts entirely.

We consider the eigenspace decomposition of $\mathfrak{g}$ relative to $-D^2$, D the semisimple part of $\operatorname{ad} jn$, $n \in \mathfrak{n}_1$. Then there exist subspaces $\mathfrak{g}_\alpha$, $\alpha \geq 0$, and an endomorphism I of $\mathfrak{g}$ such that

1) $I^2 = -\operatorname{id}$,
2) $D|\mathfrak{g}_\alpha = \alpha I$,
3) $\mathfrak{g} = \bigoplus_{\alpha \geq 0} \mathfrak{g}_\alpha$.

From Corollary 1, a) of 1.4 we know that $\mathfrak{a}$ is an ideal of $\mathfrak{g}' = \mathfrak{k} + \mathfrak{a}$. Since $\mathfrak{a}$ is solvable we have $\mathfrak{a} \subset \operatorname{rad} \mathfrak{g}'$. Therefore $[\mathfrak{k}, \mathfrak{a}] \subset [\mathfrak{g}', \mathfrak{g}'] \cap \operatorname{rad} \mathfrak{g}' = \operatorname{nil}(\mathfrak{g}')$. In particular $\operatorname{ad}[\mathfrak{k}, \mathfrak{a}]$ is nilpotent on $\mathfrak{g}$. Moreover, from Corollary 1. d) we derive $j[\mathfrak{k}, \mathfrak{n}_1] = [\mathfrak{k}, j\mathfrak{n}_1] \subset [\mathfrak{k}, \mathfrak{a}]$. Hence $\operatorname{ad} j[\mathfrak{k}, \mathfrak{n}_1]$ is nilpotent. *Hence we can (and will) assume that n is taken from some $\mathfrak{k}$-invariant complement of $[\mathfrak{k}, \mathfrak{n}_1]$ in $\mathfrak{n}_1$. But then $[\mathfrak{k}, n] = 0$.*

1.7. In this section we want to prove that (after an inessential change of j) we have

LEMMA. a) $j\mathfrak{g}_\alpha \subset \mathfrak{g}_\alpha$ *for all* α.
b) $Djx_\alpha = jDx_\alpha$ *for all* α *and all* $x_\alpha \in \mathfrak{g}_\alpha$.

Proof. From the integrability condition we get

$$(1) \qquad [jn, jx_\alpha] = j[jn, x_\alpha] \bmod \mathfrak{g}' \text{ for all } x_\alpha \in \mathfrak{g}_\alpha .$$

Because ad jn leaves $\mathfrak{g}'$ invariant, we also have

$$(2) \qquad Djx_\alpha = jDx_\alpha \bmod \mathfrak{g}' .$$

This implies

$$(3) \qquad j\mathfrak{g}_\alpha \subset \mathfrak{g}_\alpha + \mathfrak{g}' .$$

Hence, for $x_\alpha \in \mathfrak{g}_\alpha$ we have $jx_\alpha = y_\alpha + g_0' + \sum_{\beta\neq 0} g_\beta'$ where $g_\lambda' \in \mathfrak{g}_\lambda'$ and — w.r.g. — $g_0' \in \mathfrak{a}$. We want to use this expansion of jx_α in the integrability condition $[jn, jx_\alpha] = j[jn, x_\alpha] + j[n, jx_\alpha] + [n, x_\alpha] + k$. Therefore, we have to study $[n, jx_\alpha]$. It is easy to see that $\mathfrak{g}_\beta' \subset \hat{\mathfrak{a}}_0$ for all $\beta \neq 0$. Hence $[n, \mathfrak{g}_\beta'] = 0$ for all $\beta \neq 0$. From the description of $\mathfrak{a}$ we derive immediately $\mathfrak{g}_0' = \hat{\mathfrak{a}}_1 + (\mathfrak{g}_0' \cap \hat{\mathfrak{a}}_0) + \mathfrak{k}$.

But $n \in \hat{\mathfrak{a}}_1$, whence $[n, \hat{\mathfrak{a}}_1] = 0$. Moreover, $n \in \operatorname{nil}(\mathfrak{g})$ implies $[n, \hat{\mathfrak{a}}_0] = 0$. We have thus shown $[n_1, \mathfrak{a}] = 0$, and in particular

$$(4) \qquad [n, jx_\alpha] \in \mathfrak{g}_\alpha' \qquad \text{for all } x_\alpha \in \mathfrak{g}_\alpha .$$

But $\mathfrak{g}_\lambda'$ is j-invariant for all λ, whence

$$(5) \qquad [jn, jx_\alpha] - j[jn, x_\alpha] \in \mathfrak{g}_\alpha' + \mathfrak{k} .$$

From Corollar 1, d) of 1.4 and our choice of n we know $[jn, \mathfrak{k}] = j[n, \mathfrak{k}] = 0$. In particular, $D\mathfrak{k} = 0$. Because $[jn, \mathfrak{g}_\alpha'] \subset \mathfrak{g}_\alpha'$ we derive from (5)

$$(6) \qquad Djx_\alpha = jDx_\alpha + g_\alpha'' + k'' \qquad \text{for some } g_\alpha'' \in g_\alpha',\ k'' \in \mathfrak{k} .$$

Applying D to (6) we get $D^2jx_\alpha = -\alpha^2 jx_\alpha + g_\alpha''' + k'''$. We compare the μ-components and see $-\mu^2(jx_\alpha)_\mu = -\alpha^2(jx_\alpha)_\mu$ for $\mu \neq \alpha, 0$. This implies $j\mathfrak{g}_\alpha \subset \mathfrak{g}_\alpha + g_0$ and in particular, $j\mathfrak{g}_0 \subset \mathfrak{g}_0$. Let $\alpha \neq 0$. Comparing for $\mu = \alpha$ and $\mu = 0$ gives $\mathfrak{g}_\alpha''' = 0$ and $(jx_\alpha)_0 \in \mathfrak{k}$. This implies $j\mathfrak{g}_\alpha \subset \mathfrak{g}_\alpha + \mathfrak{k}$. After an inessential change of j we can assume (proving a)).

$$(7) \qquad j\mathfrak{g}_\alpha \subset \mathfrak{g}_\alpha \qquad \text{for all } \alpha .$$

It is easy to verify that for every α, $\mathfrak{g}(\alpha) = \bigoplus_{n\in Z} \mathfrak{g}_{\nu\alpha}$ is a j-invariant

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subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{g}_\lambda = \mathfrak{r}_\lambda + j\mathfrak{r}_\lambda$ for $\lambda \neq 0$ and $\mathfrak{g}_0 = \mathfrak{r}_0 + j\mathfrak{r}_0 + \mathfrak{k}$. Therefore we can apply the induction hypothesis to $g(\alpha)$ when $\mathfrak{g}(\alpha) \neq \mathfrak{g}$. In case $\mathfrak{g}(\alpha) \neq \mathfrak{g}$ for all α, the lemma follows. Assume now $g(\alpha) = \mathfrak{g}$. We can assume $\alpha = 1$. Applying the induction hypothesis to $\mathfrak{g}(m)$, $m = 2, 3, \cdots$ we see that the lemma holds for all $m \neq 1$. We write $\operatorname{ad} jn = D + N$, where N is the nilpotent part of $\operatorname{ad} jn$, and use the induction hypothesis for $\mathfrak{g}(2)$ to see for $x_1, y_1 \in \mathfrak{g}_1$: $(d/dt)\rho(e^{t\operatorname{ad} jn}x_1, e^{t\operatorname{ad} jn}y_1) = \rho(jn, e^{t\operatorname{ad} jn} \times [x_1, y_1]) = \rho(jn, e^{tD}e^{tN}[x_1, y_1]) = \rho(jn, e^{tN}[x_1, y_1])$. Here the last term is a polynomial in t. An integration yields

$$\rho(e^{t\operatorname{ad} jn}x_1, e^{t\operatorname{ad} jn}y_1) = \rho(x_1, y_1) + c_1t + c_2t^2 + \cdots . \tag{8}$$

We have $e^{t\operatorname{ad} jn} = e^{tN}e^{tD}$ where $e^{tD}x$ is bounded for all $x \in \mathfrak{g}$ and $e^{tN}y$ is a polynomial in t. Comparing like powers of t on both sides gives

$$\rho(e^{tD}x_1, e^{tD}y_1) = \rho(x_1, y_1) \qquad \text{for all } x_1, y_1 \in \mathfrak{g}_1 . \tag{9}$$

In particular, we obtain $\rho(Dx_1, y_1) + \rho(x_1, Dy_1) = 0$ for all $x_1, y_1 \in \mathfrak{g}_1$. Applying this to $y_1 \in \mathfrak{g}_1'$ and $x_1 \in \mathfrak{g}_1^\perp = \{x_1 \in \mathfrak{g}_1;\ \rho(x_1, \mathfrak{g}_1') = 0\}$ we get $D\mathfrak{g}_1^\perp \subset \mathfrak{g}_1^\perp$ (since $D\mathfrak{g}_1' \subset \mathfrak{g}_1'$ anyway). We also have $j\mathfrak{g}_1^\perp \subset \mathfrak{g}_1^\perp$. Finally, from (6) and (7) we know $Djx_1 = jDx_1 + g_1'$, where $g_1' \in \mathfrak{g}_1'$. For $x_1 \in \mathfrak{g}_1^\perp$ we obtain $g_1' \in \mathfrak{k}$. If $x_1 \in \mathfrak{g}_\lambda$, $\lambda \neq 0$, then $g_1' = 0$. If $x_1 \in \mathfrak{g}_0$, then $g_1' = 0$, since D commutes with j in $\mathfrak{g}_0$, This finishes the proof of the lemma.

1.8. Let $\mathscr{D}$ be the closure of $\{\exp D;\ D$ is the semisimple part of some $\operatorname{ad} jn,\ n \in \hat{\mathfrak{a}}_1\}$ in $\mathrm{Gl}(\mathfrak{g})$. Then $\mathscr{D}$ is an abelian compact group of automorphisms of $\mathfrak{g}$. Moreover, $\mathscr{D}$ acts trivially on $\mathfrak{k}$ and commutes with $\operatorname{ad}\mathfrak{k}$ on $\mathfrak{a}$ by Corollary 1, c) of 1.4 and with j by 1.7. Also, if $x \notin \mathfrak{k}$ then $Wx \notin \mathfrak{k}$ for all $W \in \mathscr{D}$.

We consider the skew from $\tilde{\rho}$ on $\mathfrak{g}$ given by

$$\tilde{\rho}(u, v) = \int_{\mathscr{D}} \rho(Wu, Wv)\,dW , \tag{1}$$

where dW denotes the normalized bi-invariant Haar measure of $\mathscr{D}$.

It is easy to verify (3.4) through (3.7) of [6; § 3]. Hence we have

LEMMA. a) $(\mathfrak{g}, \mathfrak{k}, j, \tilde{\rho})$ *is a Kähler algebra.*

b) *The semisimple parts of* $\operatorname{ad} jn$, $n \in \hat{\mathfrak{a}}_1$, *are skew-symmetric relative to* $\tilde{\rho}$ *and commute with* j.

1.9. We want to consider modifications of $\mathfrak{g}$, for which $\operatorname{ad} jn$ is

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nilpotent for all $n \in \mathfrak{n}$. The modification map has to vanish on $[\mathfrak{g}, \mathfrak{g}]$. We therefore prove

LEMMA. *After an inessential change of* j *we may assume that* $\operatorname{ad}(j\mathfrak{n} \cap [\mathfrak{g}, \mathfrak{g}])$ *consists of nilpotent elements.*

Proof. Suppose $jn \in [\mathfrak{g}, \mathfrak{g}]$ and denote by D the semisimple part of $\operatorname{ad} jn$, then $\operatorname{ad} jn \in [\operatorname{ad}\mathfrak{g}, \operatorname{ad}\mathfrak{g}] = \mathfrak{m}$. But $\mathfrak{m}$ is an algebraic Lie algebra, whence $D \in \mathfrak{m}$. Hence, by the appendix we can write $D = D_h + D_r$, $[D_h, D_r] = 0$ and we can find a maximal semisimple subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and $h_0 \in \mathfrak{h}$ such that $\operatorname{ad} h_0\ D_h$. Moreover, $D_r \in \operatorname{rad}[\operatorname{ad}\mathfrak{g}, \operatorname{ad}\mathfrak{g}]$. But this radical consists only of nilpotent endomorphisms, whence $D_r = 0$. Clearly $h_0 \in \mathfrak{g}_0$ where $\mathfrak{g}_0$ is defined in 1.6. If $D \neq 0$ we have $\mathfrak{g}_0 \neq \mathfrak{g}$. Moreover, from 1.7 we can easily derive that $\mathfrak{g}_0$ is a Kähler algebra and $\mathfrak{g}_0 = (\mathfrak{r} \cap \mathfrak{g}_0) + j(\mathfrak{r} \cap \mathfrak{g}_0) + \mathfrak{k}$. We can apply the induction hypothesis to $\mathfrak{g}_0$. Since $\operatorname{ad} h_0$ is skew adjoint relative to $\tilde{\rho}$ we have $\tilde{\rho}(h_0, [\mathfrak{g}, \mathfrak{g}]) = 0$ by the closedness condition. This implies in particular $\tilde{\rho}(h_0, \mathfrak{h} + \operatorname{nil}(\mathfrak{g})) = 0$. Let $\mathfrak{q}$ be a complement of $\operatorname{nil}(\mathfrak{g})$ in $\operatorname{rad}\mathfrak{g}$ which is $\mathfrak{h}$-invariant. Then $[\mathfrak{h}, \mathfrak{g}] = 0$. But then $\tilde{\rho}(\mathfrak{q}, \mathfrak{h}) = \tilde{\rho}(\mathfrak{q}, [\mathfrak{h}, \mathfrak{h}]) = 0$. In particular $\tilde{\rho}(\mathfrak{h}_0, \mathfrak{g}) = 0$. Therefore $\tilde{\rho}(h_0, \mathfrak{g}) = 0$. In particular $0 = \tilde{\rho}(jh_0, h_0)$, whence $h_0 \in \mathfrak{k}$. From $[h_0, \mathfrak{k}] = D\mathfrak{k} = 0$ we get $h_0 \in \operatorname{center}(\mathfrak{k})$. Now we introduce an inessential change of j by redefining j on a $\mathfrak{k}$-invariant complement $\mathfrak{b}_1$ of $[\mathfrak{k}, \mathfrak{n}_1]$ in $\mathfrak{n}_1$. For $n \in \mathfrak{b}_1$ we set $j'n = jn - h_0$. Then $\operatorname{ad} j'n$ is nilpotent for all $n \in \mathfrak{n}$ such that $\operatorname{ad} jn \in \operatorname{ad}[\mathfrak{g}, \mathfrak{g}]$. We will also assume that the center of $\mathfrak{k}$ is j'-invariant.

We claim that $\mathfrak{n} + j'\mathfrak{n}$ is a solvable algebra having all properties of Corollary 1 of 1.4. We note $\hat{\mathfrak{a}}_0 + [\mathfrak{k}, \hat{\mathfrak{a}}_1] \subset \mathfrak{n} + j'\mathfrak{n}$ and $[\mathfrak{n} + j'\mathfrak{n}, \mathfrak{n} + j'\mathfrak{n}] \subset [\mathfrak{a} + \mathfrak{k}_0, \mathfrak{a} + \mathfrak{k}_0]$ where $\mathfrak{k}_0$ denotes the center of $\mathfrak{k}$. Hence the last commutator is contained in $\hat{\mathfrak{a}}_0 + [\mathfrak{k}_0, \mathfrak{a}]$. But $[\mathfrak{k}_0, \mathfrak{a}] \subset \hat{\mathfrak{a}}_0 + [\mathfrak{k}, \hat{\mathfrak{a}}_1]$ and $\mathfrak{n} + j'\mathfrak{n}$ is a solvable subalgebra of $\mathfrak{g}$. Moreover, $[\mathfrak{k}, \mathfrak{n} + j'\mathfrak{n}] \subset [\mathfrak{k}, \mathfrak{a} + \mathfrak{k}_0] = [\mathfrak{k}, \mathfrak{a}] \subset \hat{\mathfrak{a}}_0 + [\mathfrak{k}, \hat{\mathfrak{a}}_1] \subset \mathfrak{n} + j'\mathfrak{n}$ and $[k, j'x] = [k, jx + k_0] = [k, jx] = j[k, x] = j[k, x]$ for all $x \in \mathfrak{n} + j'\mathfrak{n}$ since $\operatorname{ad}[\mathfrak{k}, \mathfrak{a}]$ consists of nilpotent endomorphisms. Therefore, using j' instead of j we may assume that $j\mathfrak{n} \cap [\mathfrak{g}, \mathfrak{g}]$ operates nilpotently on $\mathfrak{g}$. This proves the lemma.

1.10. From 1.9 we see that the elements of $\operatorname{ad}(j\mathfrak{n} \cap [\mathfrak{g}, \mathfrak{g}])$ have no semisimple parts. Denote by $\mathfrak{v}$ an algebraic complement of $j\mathfrak{n} \cap [\mathfrak{g}, \mathfrak{g}]$ in $j\mathfrak{n}$. We can assume $\mathfrak{v} \subset \hat{\mathfrak{a}}_1$. We split $\operatorname{ad} jn = D(jn) + N(jn)$ for every $jn \in \mathfrak{v}$ into semisimple part $D(jn)$ and nilpotent part $N(jn)$ and define a (modification) map $D : \mathfrak{g} \to \operatorname{Der}\mathfrak{g}$ by $D([\mathfrak{g}, \mathfrak{g}]) = 0$ and $D(\mathfrak{v})$ as above and

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trivial otherwise. It is easy to verify the properties (3.1) to (3.4) of [5] for the map D and the Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \tilde{\rho})$. Moreover, as $D(jn)$ is the semisimple part of $\operatorname{ad} jn$ we know $D(jn)\mathfrak{g} \subset [\mathfrak{g}, \mathfrak{g}]$. Hence [5: 3.5] is satisfied.

We define a new algebra structure on $\mathfrak{g}$ by setting $[x, y]^1 = [x, y] - D(x)y + D(y)x$, $x, y \in \mathfrak{g}$.

A straightforward computation shows

LEMMA. $(\mathfrak{g}, [\cdot, \cdot]^1, \mathfrak{k}, j, \tilde{\rho})$ *is a Kähler algebra.*

It is clear that $\mathfrak{r}$ and $\mathfrak{n}$ are also ideals of $(\mathfrak{g}, [\cdot, \cdot]^1)$. The subalgebra $\mathfrak{n} + j\mathfrak{n}$ is now abelian and acts nilpotently on $\mathfrak{g}$. Moreover, it is easy to verify that the Radical Conjecture holds for $(\mathfrak{g}, [\cdot, \cdot], \mathfrak{k}, j, \tilde{\rho})$ iff it holds for $(\mathfrak{g}, [\cdot, \cdot]^1, \mathfrak{k}, j, \tilde{\rho})$.

Therefore, from now on *we assume* (w.r.g.) *that* $\mathfrak{a} = \mathfrak{n} + j\mathfrak{n}$ *is abelian and* $\operatorname{ad} a$ *is nilpotent for all* $a \in \mathfrak{a}$.

1.11. We want to follow the proof of [5; § 6]. Therefore, we consider the set $\mathfrak{w} = \{x \in \mathfrak{g};\ \operatorname{ad} x|\mathfrak{n}$ and $\operatorname{ad} jx|\mathfrak{n}$ are skew adjoint relative to $\langle \cdot, \cdot \rangle = \rho(j\cdot, \cdot)$ and commute with $j\}$. Put $\tilde{\mathfrak{w}} = \{x \in \mathfrak{w};\ [x, \mathfrak{n}] = 0\}$. Clearly, $\mathfrak{w}$ is j-invariant (since $\mathfrak{k}$ is skewadjoint on $\mathfrak{a}$ and commutes with j.)

LEMMA. $[\mathfrak{a}, \mathfrak{g}] \subset \tilde{\mathfrak{w}}$.

Proof. Note $[[jn, x], m] = [jn, [x, m]] = 0$ for all $n, m \in \mathfrak{n}$, $x \in \mathfrak{g}$. Also $[j[jn, x], m] = [[jn, jx] - j[n, jx] - [n, x] - k, m] = -[k, m]$ for all $n, m \in \mathfrak{n}$. Hence $[j\mathfrak{n}, \mathfrak{g}] \subset \tilde{\mathfrak{w}}$. Clearly, $[\mathfrak{n}, \mathfrak{g}] \subset \mathfrak{n} \subset \tilde{\mathfrak{w}}$. Hence the lemma.

1.12.

LEMMA. $\rho([\mathfrak{a}, \tilde{\mathfrak{w}}], \mathfrak{a}) = 0$.

Proof. Let $w \in \tilde{\mathfrak{w}}$ and $n, n^1, m, m^1 \in \mathfrak{n}$. Then $\rho([n^1 + jn, w], m^1 + jm) = \rho([jn, w], m^1 + jm) = -\rho([w, m^1 + jm], jn) = -\rho([w, jm], jn)$ where we have used the fact that $\mathfrak{a}$ is abelian. Using the integrability condition for $\mathfrak{g}$ we get $\rho([w, jm]), jn) = -\rho(j[w, jm], n) = -\rho([jw, jm] - j[jw, m] - [w, m] - k, n) = -\rho([jw, jm], n) - \rho([jw, m], jn) = \rho([n, jw], jm) - \rho([jw, m], jn) = A$. Set $S(jw) = \operatorname{ad} jw|\mathfrak{n}$. Then we get $A = -\rho(S(jw)n, jm) - \rho(S(jw)m, jn) = -\rho(S(jw)n, jm) + \rho(jS(jw)m, n) = 0$ because $S(jw)$ is skewadjoint relative to $\langle n, m \rangle = \rho(jn, m)$.

COROLLARY. $\rho((\operatorname{ad} \mathfrak{a})^2\mathfrak{g}, \mathfrak{a}) = 0$.

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1.13. Using the above corollary we see

LEMMA. *$\rho(e^{t\,\mathrm{ad}\,h}x, e^{t\,\mathrm{ad}\,h}y) = at^2 + bt + c$ for $h \in \mathfrak{a}$, $x, y \in \mathfrak{g}$ and some $a, b, c \in \boldsymbol{R}$.*

Proof. We know $\dfrac{d^3}{dt^3}\rho(e^{t\,\mathrm{ad}\,h}x, e^{t\,\mathrm{ad}\,h}y) = \dfrac{d^2}{dt^2}\rho(h, e^{t\,\mathrm{ad}\,h}[x, y]) = \rho(h, (\mathrm{ad}\,h)^2 *)$ $= 0$. Hence the lemma.

1.14. Next we want to prove the following easy

LEMMA. *$(\mathrm{ad}\,a)^r jx = j(\mathrm{ad}\,a)^r x \bmod \mathfrak{g}'$ for all $a \in \mathfrak{a}$, $x \in \mathfrak{g}$.*

Proof. The claim is trivial for $r = 0$. For $r = 1$ we split $a = n' + jn$, $n, n' \in \mathfrak{n}$ and see that it suffices to consider $a = jn$. But here the claim follows immediately from the integrability condition. Let now $r \geq 2$. Then $(\mathrm{ad}\,a)^r jx = (\mathrm{ad}\,a)(\mathrm{ad}\,a)^{r-1} jx = (\mathrm{ad}\,a)\{j(\mathrm{ad}\,a)^{r-1}x + g'\} = j(\mathrm{ad}\,a)^r x + g''$ with some $g', g'' \in \mathfrak{g}'$.

1.15. The trivial results above can be combined to give.

LEMMA. *$(\mathrm{ad}\,\mathfrak{a})^2\mathfrak{g} \subset \mathfrak{k}$.*

Proof. $bt^2 + ct + d = \rho(e^{t\,\mathrm{ad}\,a}[a, [a, x]], e^{t\,\mathrm{ad}\,a}j[a, y]) = \rho(e^{t\,\mathrm{ad}\,a}[a, [a, x]], je^{t\,\mathrm{ad}\,a}[a, y] + g^1) = -\langle e^{t\,\mathrm{ad}\,a}[a, [a, x]], e^{t\,\mathrm{ad}\,a}[a, y]\rangle$ where $x, y \in \mathfrak{g}$, $a \in \mathfrak{a}$. With $y = [a, x]$ we see that $B = |e^{t\,\mathrm{ad}\,a}(\mathrm{ad}\,a)^2 x|^2$ is a quadratic polynomial in t. We know that $\mathrm{ad}\,a$ is nilpotent, hence B is a polynomial in t. Let $\mathrm{ad}\,a$ be nilpotent of degree s, i.e. $(\mathrm{ad}\,a)^s = 0$, $(\mathrm{ad}\,a)^{s-1} \neq 0$. Then B is a polynomial of degree $\leq 2(s-3)$ and the coefficient of $t^{2(s-3)}$ is $|(\mathrm{ad}\,a)^{s-1}x|^2$. If $2(s-3) \leq 2$, then $s \leq 4$. If $2(s-3) > 2$, then $(\mathrm{ad}\,a)^{s-1}\mathfrak{g} \subset \mathfrak{k}$ and the highest term in B is at most of degree $2(s-4)$. Suppose also $2(s-4) > 2$, then by the same argument as above, $(\mathrm{ad}\,a)^{s-2}\mathfrak{g} \subset \mathfrak{k}$. But then $(\mathrm{ad}\,a)^{s-1}\mathfrak{g} \subset [\mathfrak{a}, \mathfrak{k}] \cap \mathfrak{k} = 0$. Therefore $2(s-4) \leq 2$, i.e. $s \leq 5$. As seen above, this implies $(\mathrm{ad}\,a)^4\mathfrak{g} \subset \mathfrak{k}$. Hence in both cases, $2(s-3) \leq 2$ and $2(s-3) > 2$, we have $(\mathrm{ad}\,a)^4\mathfrak{g} \subset \mathfrak{k}$. Choosing $y = x$ above and again comparing highest terms in t we get even $(\mathrm{ad}\,a)^3\mathfrak{g} \subset \mathfrak{k}$. Finally, we consider $C = -\langle e^{t\,\mathrm{ad}\,a}[a, x], e^{t\,\mathrm{ad}\,a}x\rangle$ $= \rho([a, x] + t[a, [a, x]], jx + tj[a, x] + (1/2)t^2 j[a, [a, x]]$. We know $j[a, x] = [a, jx] + g'$ and $j(\mathrm{ad}\,a)^2 x = (\mathrm{ad}\,a)^2 jx + g''$ where $g', g'' \in \mathfrak{g}'$. We note that the coefficient of t^3 equals $-(1/2)|(\mathrm{ad}\,a)^2 x|^2 = (1/2)\rho((\mathrm{ad}\,a)^2 x, (\mathrm{ad}\,a)^2 jx)$ by Corollary 1.12. Hence, altogether, we get $C = \rho(e^{t\,\mathrm{ad}\,a}[a, x], e^{t\,\mathrm{ad}\,a}jx) + p(t)$ where $p(t)$ is a quadratic polynomial. Applying Lemma 1.13 we see that actually C is a quadratic polynomial. Therefore $(\mathrm{ad}\,a)^2\mathfrak{g} \subset \mathfrak{k}$.

1.16. Later we will frequently use the following.

LEMMA. a) $\mathfrak{a} \subset \mathrm{rad}(\mathfrak{g})$
b) $[\mathfrak{a}, \mathfrak{g}] \subset \mathrm{nil}(\mathfrak{g})$.
c) $(\mathrm{ad}\, \mathfrak{a})^2 = 0$.

Proof. It suffices to prove a) for an element of type jn, $n \in \mathfrak{n}$. We work in $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathrm{rad}\, \mathfrak{g}$. Suppose $\mathrm{ad}\, jn \neq 0$ on $\tilde{\mathfrak{g}}$. Then $0 \neq \tilde{x} = jn \bmod \mathrm{rad}(g)$ is nilpotent in the semisimple algebra $\tilde{\mathfrak{g}}$. Therefore, by the Jacobson-Morozov-Theorem there exist $\tilde{y}$, $\tilde{h} \in \tilde{\mathfrak{g}}$ such that $\boldsymbol{R}\tilde{x} + \boldsymbol{R}\tilde{h} + \boldsymbol{R}\tilde{y} \cong sl(2, \boldsymbol{R})$. In particular $(\mathrm{ad}\, \tilde{x})^2 \tilde{y} = 2\tilde{x}$. On the other hand we know $(\mathrm{ad}\, jn)^2 \mathfrak{g} \subset \mathfrak{k}$ mod rad($\mathfrak{g}$). Hence $\tilde{x} \in \mathfrak{k} \bmod \mathrm{rad}(\mathfrak{g})$. Since $\mathfrak{k}$ acts semisimple on $\mathfrak{g}$ (and on $\tilde{\mathfrak{g}}$) and $\tilde{x}$ acts nilpotent, $\tilde{x} = 0$. This is a contradiction and $\tilde{x} = 0$ follows, whence a). Part b) is a trivial consequence of a). To prove c) we combine $\mathfrak{a} \subset \mathrm{rad}(\mathfrak{g})$ with Lemma 1.15 and get $(\mathrm{ad}\, \mathfrak{a})^2 \mathfrak{g} \subset \mathrm{nil}(\mathfrak{g}) \cap \mathfrak{k} = 0$.

1.17. Eventually we want to generalize the proof of [5; § 6] to our setting. We therefore consider $\check{\mathfrak{g}}^{(1)} = [j\mathfrak{n}, \mathfrak{g}] + \mathfrak{g}'$. Note $[j\mathfrak{n}, \mathfrak{g}] \subset \mathrm{nil}(\mathfrak{g})$ by 1.16.

LEMMA. a) $\check{\mathfrak{g}}^{(1)}$ *is a Kähler algebra.*
b) $\check{\mathfrak{g}}^{(1)} = (\mathfrak{r} \cap \check{\mathfrak{g}}^{(1)}) + j(\mathfrak{r} \cap \check{\mathfrak{g}}^{(1)}) + \mathfrak{k}$.

Proof. a) Clearly $[\mathfrak{n}, \check{\mathfrak{g}}^{(1)}] \subset \check{\mathfrak{g}}^{(1)}$. Because $(\mathrm{ad}\, jn)^2 = 0$, we also have $[j\mathfrak{n}, \check{\mathfrak{g}}^{(1)}] \subset [j\mathfrak{n}, \mathfrak{g}'] \subset \mathfrak{g}' \subset \check{\mathfrak{g}}^{(1)}$. For $k \in \mathfrak{k}$ we get $[k, [j\mathfrak{n}, \mathfrak{g}]] \subset [j[k, \mathfrak{n}], \mathfrak{g}] + [j\mathfrak{n}, \mathfrak{g}] \subset \check{\mathfrak{g}}^{(1)}$. Finally $[[jn, x], [jm, y]] = [jn, [x, [jm, y]]] \in \check{\mathfrak{g}}^{(1)}$ because $(\mathrm{ad}\, j\mathfrak{n})^2 = 0$. This shows that $\check{\mathfrak{g}}^{(1)}$ is a subalgebra of $\mathfrak{g}$. The integrability condition implies that $\check{\mathfrak{g}}^{(1)}$ is j-invariant.

b) We have $[jn, jr] + j[jn, r] \bmod \mathfrak{g}'$. Therefore $[j\mathfrak{n}, \mathfrak{r}] + \mathfrak{n} + j([j\mathfrak{n}, \mathfrak{r}]\ \mathfrak{n}) + \mathfrak{k} = \check{\mathfrak{g}}^{(1)}$. From this the assertion follows.

1.18. From the definition of $\check{\mathfrak{g}}^{(1)}$ it follows $\check{\mathfrak{g}}^{(1)} \subset \mathrm{nil}_0(\mathfrak{g}) + \mathfrak{k}$, where $\mathrm{nil}_0(\mathfrak{g})$ denotes the greatest nilpotent ideal of $\mathfrak{g}$ [2; § 4.4 and § 5.3]. We may assume that $\mathfrak{a}^{(1)} = \check{\mathfrak{g}}^{(1)} \cap \mathrm{nil}_0(\mathfrak{g})$ is invariant under j. In particular, as $\mathrm{ad}\, \mathfrak{a}^{(1)}$ consists only of nilpotent derivations of $\mathfrak{g}$, we see that $\mathfrak{a}^{(1)}$ is abelian.

We would like to point out that $\mathfrak{a}^{(1)} = [j\mathfrak{n}, \mathfrak{g}] + \mathfrak{n} + j\mathfrak{n} \subset \mathrm{nil}_0(\mathfrak{g})$ holds.

1.19. Next we want to consider $\mathfrak{g}^{(0)} = \{x \in \mathfrak{g}; [x, \mathfrak{g}^{(2)}] \subset \mathfrak{g}^{(2)}\}$ where $\mathfrak{g}^{(2)} = \mathfrak{n} + j\mathfrak{n}$. Before we can show that $\mathfrak{g}^{(0)}$ satisfies the induction hypothesis we have to consider the vector space $\mathfrak{r} \cap j\mathfrak{r}$.

Let $\mathfrak{v} = \{x \in \mathfrak{r}; jx \in \mathfrak{r} + \mathfrak{k}\}$.

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After an inessential change of j we can assume $jx \in \mathfrak{v}$ for all $x \in \mathfrak{v}$. We note that this can be done without changing j on $\mathfrak{r} \cap \breve{\mathfrak{g}}^{(1)}$.

LEMMA. a) $\mathfrak{v} = \mathfrak{r} \cap j\mathfrak{r}$
b) $j\mathfrak{v} \subset \mathfrak{v}$

Proof. Let $x \in \mathfrak{v}$, then $x, jx \in \mathfrak{r}$ and $j(jx) = -x + k$. Therefore $jx \in \mathfrak{v}$. But then $j(jx) \in \mathfrak{r}$, whence $k \in \mathfrak{r}$ and $k = 0$. Hence $x = j(-jx) \in \mathfrak{r} \cap j\mathfrak{r}$ and $\mathfrak{v} \subset \mathfrak{r} \cap j\mathfrak{r}$. Let $x \in \mathfrak{r} \cap j\mathfrak{r}$; then $x \in \mathfrak{r}$ and $x = jy$ for some $y \in \mathfrak{r}$. Hence $jx = -y + k$ and $x \in \mathfrak{v}$ follows. This proves the lemma.

1.20. LMMA. $\mathfrak{r} \cap j\mathfrak{r} + \mathfrak{n}$ *is an ideal of* $\mathfrak{g}$.

Proof. We have to verify that $\mathfrak{v}$ is mapped into $\mathfrak{v} + \mathfrak{n}$ by $\mathfrak{k}$, $\mathfrak{r}$ and $j\mathfrak{r}$. Let $x \in \mathfrak{v} = \mathfrak{r} \cap j\mathfrak{r}$ and $y \in \mathfrak{r}$ such that $x = jy$ holds. Clearly $[\mathfrak{k}, x] \subset \mathfrak{r}$; moreover, $[k, jy] = j[k, y] + k'$ by the definition of a Kähler algebra. Hence $j[k, y] + k' \in \mathfrak{r}$, yielding $[k, y] \in \mathfrak{v}$. Therefore $k' = 0$ and $[k, x] = j[k, y] \in \mathfrak{r} \cap j\mathfrak{r}$. Next note $[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{n}$. Finally, we consider $[jr, \mathfrak{v}]$. Then, with x and y as above, we have $[jr, jy] = j([jr, y] + [r, jy]) + [r, y] + k \in \mathfrak{r}$. Because $[r, y] \in \mathfrak{n}$, we obtain $jz + k \in \mathfrak{r}$, where $z = [jr, y] + [r, jy] \in \mathfrak{r}$. This yields $z \in \mathfrak{v}$ and $k = 0$. Therefore $[jr, jy] \in j\mathfrak{v} + n \subset \mathfrak{v} + \mathfrak{n}$. This finishes the proof.

1.21. From the last lemma we see that $\mathfrak{n} \subset \mathfrak{v} + \mathfrak{n} \subset \mathfrak{r}$ is a chain of ideals. We had chosen $\mathfrak{n}$ maximal. Therefore either $\mathfrak{r} \cap j\mathfrak{r} \subset \mathfrak{n}$ or $\mathfrak{r} \cap j\mathfrak{r} + \mathfrak{n} = \mathfrak{r}$. In the latter case we get $\mathfrak{g} = \mathfrak{r} \cap j\mathfrak{r} + \mathfrak{n} + j\mathfrak{n} + \mathfrak{k}$. Hence $\mathfrak{g} = \operatorname{rad} \mathfrak{g} + \mathfrak{k}$. After an inessential change of j we may assume that $\operatorname{rad} \mathfrak{g}$ is j-invariant and the Radical Conjecture holds.

Hence, *from now on we will assume* $\mathfrak{r} \cap j\mathfrak{r} \subset \mathfrak{n}$.

1.22. Now we can return to the consideration of the subalgebra $\mathfrak{g}^{(0)} = \{x \in \mathfrak{g}; [x, \mathfrak{g}^{(2)}] \subset \mathfrak{g}^{(2)}\}$ where $\mathfrak{g}^{(2)} = \mathfrak{n} + j\mathfrak{n}$. Clearly, $\mathfrak{g}' \subset \mathfrak{g}^{(0)}$.

LEMMA. a) $\mathfrak{g}^{(0)}$ *is a Kähler subalgebra of* $\mathfrak{g}$.
b) $\mathfrak{g}^{(0)} = (\mathfrak{g}^{(0)} \cap \mathfrak{r}) + j(\mathfrak{g}^{(0)} \cap \mathfrak{r}) + \mathfrak{k}$.

Proof. a) Let $x \in \mathfrak{g}^{(0)}$. Then we have to prove that jx leaves $\mathfrak{g}^{(2)}$ invariant. This follows from $[jx, \mathfrak{n}] \subset \mathfrak{n} \subset \mathfrak{g}^{(2)}$ and $[jx, j\mathfrak{n}] = j[jx, n] + j[x, jn] + [x, n] + k$; note that in the last expression all summands but k are in the greatest nilpotent ideal of $\mathfrak{g}$. Therefore $\operatorname{ad} k = 0$. Hence $k = 0$, because we have assumed that $\mathfrak{g}$ acts effectively on some manifold. But this implies $[jx, jn] \in \mathfrak{g}^{(2)}$.

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b) Because $\mathfrak{k} \subset \mathfrak{g}^{(0)}$ we have only to consider elements of type $x = \tilde{r} + jr \in \mathfrak{g}^{(0)}$. Then $[jn, x] \in \mathfrak{g}^{(2)}$, i.e. $[jn, \tilde{r}] + [jn, jr] = [jn, \tilde{r}] + j[jn, r] + j[n, jr] + [n, r] + k \in \mathfrak{g}^{(2)}$. Therefore $[jn, \tilde{r}] + j[jn, r] \in \mathfrak{g}'$. Hence there exist $a, b \in \mathfrak{n}$, $k' \in \mathfrak{k}$ such that $[jn, \tilde{r}] + j[jn, r] = a + jb + k'$. This implies $j([jn, r] - b) = a - [jn, \tilde{r}] + k'$, whence $[jn, r] - b \in \mathfrak{v}$ and $k' = 0$. In particular $[jn, r] - b \in \mathfrak{n}$. Hence we obtain $[jn, r]$, $[jn, \tilde{r}] \in \mathfrak{n}$. This implies $[r, \mathfrak{g}^{(2)}] \subset \mathfrak{g}^{(2)}$ and $[\tilde{r}, \mathfrak{g}^{(2)}] \subset \mathfrak{g}^{(2)}$, whence $r, \tilde{r} \in \mathfrak{g}^{(0)}$. We have shown $\mathfrak{g}^{(0)} \subset (\mathfrak{g}^{(0)} \cap \mathfrak{r}) + j(\mathfrak{g}^{(0)} \cap \mathfrak{r}) + \mathfrak{k}$. The opposite inclusion is trivial and the lemma is proven.

1.23. To be able to use the induction hypothesis for $\mathfrak{g}^{(0)}$ we have to exclude the case $\mathfrak{r} \cap \mathfrak{g}^{(0)} = \mathfrak{r}$. But in this case $\mathfrak{g} = \mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(2)}$ is an abelian ideal of $\mathfrak{g}$. Consider $\mathfrak{u} = \{x \in \mathfrak{g}, \rho(x; \mathfrak{g}^{(2)}) = 0\}$. Then $\mathfrak{u}$ is a Kähler subalgebra of $\mathfrak{g}$ and $\mathfrak{u} \cap \mathfrak{g}^{(2)} = 0$. Clearly $\mathfrak{u} \cong \mathfrak{g}/\mathfrak{g}^{(2)}$. Therefore we can apply the induction hypothesis to $\mathfrak{u}$. So after an inessential change of j on $\mathfrak{u}$ we get a solvable Kähler subalgebra $\tilde{\mathfrak{s}}$ of $\mathfrak{u}$ satisfying $\mathfrak{u} = \tilde{\mathfrak{s}} + \mathfrak{k}$, $\tilde{\mathfrak{s}} \cap \mathfrak{k} = 0$. But the the Radical Conjecture follows with $\mathfrak{s} = \mathfrak{g}^{(2)} + \tilde{\mathfrak{s}}$.

Hence, *from now on we will assume* $\mathfrak{g}^{(0)} \neq \mathfrak{g}$. From the last lemma it follows that we can apply the induction hypothesis. So after an inessential change of j we have $\mathfrak{g}^{(0)} = \mathfrak{t}^{(0)} + \mathfrak{a}^{(0)} + \mathfrak{k}$, where $\mathfrak{t}^{(0)} + \mathfrak{a}^{(0)}$ is a solvable Kähler algebra, $t^{(0)}$ is a modification of a normal j-algebra and $\mathfrak{a}^{(0)}$ is a modification of an abelian Kähler algebra.

From 1.18 we know $\mathfrak{a}^{(1)} \subset \mathrm{nil}_0(\mathfrak{g}) \cap \mathfrak{g}^{(0)}$. Therefore also $\mathfrak{a}^{(1)} + [\mathfrak{a}^{(1)}, \mathfrak{g}] \subset \mathrm{nil}_0(\mathfrak{g}) \cap \mathfrak{g}^{(0)} \subset \mathrm{nil}_0(\mathfrak{g}^{(0)})$. It is easy to see that one can choose $\mathfrak{t}^{(0)} + \mathfrak{a}^{(0)}$ so that $\mathrm{nil}_0(\mathfrak{g}^{(0)}) \subset \mathfrak{t}^{(0)} + \mathfrak{a}^{(0)}$ holds.

LEMMA. a) $\mathfrak{a}^{(1)} \subset \mathfrak{a}^{(0)}$.

b) $\check{\mathfrak{g}}^{(1)}$ *is a subalgebra of* $\mathfrak{g}^{(0)}$.

Proof. It suffices to prove $[j\mathfrak{n}, \mathfrak{g}] \subset \mathfrak{g}^{(0)}$. It is easy to see that we only have to note $[j\mathfrak{n}, [j\mathfrak{n}, \mathfrak{g}]] = 0 \subset \mathfrak{g}^{(2)}$.

1.24. We want to generalize the proof of [5; § 6] to our setting. We define $\mathfrak{g}^{(-1)} = \mathfrak{g}$, $\mathfrak{g}^{(0)} = \mathfrak{t}^{(0)} + \mathfrak{a}^{(0)} + \mathfrak{k}$, $\mathfrak{g}^{(1)} = \mathfrak{a}^{(1)}$ and $\mathfrak{g}^{(2)} = \mathfrak{n} + j\mathfrak{n}$.

We have seen above $\mathfrak{g}^{(-1)} \supset \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)}$. Also, $j\mathfrak{g}^{(k)} \subset \mathfrak{g}^{(k)}$. By the definition of $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(0)}$ we have $[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(2)}] \subset \mathfrak{g}^{(1)}$ and $[\mathfrak{g}^{(0)}, \mathfrak{g}^{(2)}] \subset \mathfrak{g}^{(2)}$. Further we know $[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = 0$ and $[\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \subset \mathfrak{g}^{(0)}$ hence, in particular $[\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}] = 0$.

LEMMA. *The subspaces* $\mathfrak{g}^{(i)}$ *form a j-invariant filtration of the Lie algebra* $\mathfrak{g}$.

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Proof. We have only to consider two types of commutators.

$[\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(1)}$: let $x_0 \in \mathfrak{g}^{(0)}$ and $x_1 \in \mathfrak{g}^{(1)}$. Because $[\mathfrak{g}^{(0)}, \mathfrak{g}^{(2)}] \subset \mathfrak{g}^{(2)} \subset \mathfrak{g}^{(1)}$ we can assume $x_1 = [jn, x]$ for some $n \in \mathfrak{n}$, $x \in \mathfrak{g}$. Then $[x_0, x_1] = [x_0, [jn, x]] = [jn, [x_0, x]] - [[jn, x_0], x] \in \mathfrak{g}^{(1)}$.

$[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(0)}$: let $x \in \mathfrak{g}$ and $x_1 \in \mathfrak{g}^{(1)}$. Then for all $n \in \mathfrak{n}$ we have $[jn, [x, x_1]] = [[jn, x], x_1] + [x, [jn, x_1]] = 0$ since $\mathfrak{g}^{(1)}$ is abelian. This proves the claim.

1.25. From this point on we can use large parts of the proof of [5; § 6]. We have the obvious identifications: $\mathfrak{g}^{(i)} \leftrightarrow \mathfrak{S}^{(i)}$, $\mathfrak{a}^{(0)} \leftrightarrow \mathfrak{a}$, $\mathfrak{k}^{(0)} \leftrightarrow \mathfrak{S}_D = \mathfrak{h}$ and $\mathfrak{n} \leftrightarrow \mathfrak{g}$.

(1) Let $x \in \mathfrak{a}^{(0)}$ and $\operatorname{ad} x = S + N$ be the decomposition of $\operatorname{ad} x$ into semisimple part S and nilpotent part N. Then $N\mathfrak{g} \subset \mathfrak{g}^{(0)}$.

Proof. See [5].

(2) We denote by $\mathfrak{s}$ the principal idempotent of $\mathfrak{k}^{(0)}$ and define H_0, $\mathfrak{g}_\lambda$, $\bar{\mathfrak{g}}$, $\bar{\mathfrak{g}}^{(\mu)} \cdots$ as in [5]. For an arbitrary $n \in \mathfrak{n}$ we define (as in [5]) $\{abc\} = [[[\overline{jn}, a], b], c]$, $a, b, c \in \bar{\mathfrak{g}}^{(-1)}$.

(3) $\{abc\}$ is invariant under permutations of a, b, c.

Proof. As in [7] one notes $\{abc\} - \{bca\} = [[[\overline{jn}, a], b], c] - [[[\overline{jn}, b], a], c] = [[\overline{jn}, [a, b]], c] = 0$ since $[\bar{\mathfrak{g}}^{(-1)}, \bar{\mathfrak{g}}^{(-1)}] = 0$. Hence $\{abc\} = \{bac\}$. Moreover, $[[[\overline{jn}, a], b], c] - [[[\overline{jn}, a], c], b] = [[\overline{jn}, a], [b, c]] = 0$.

We also have

(4) $[\overline{jm}, \{abc\}] = [[[\overline{jn}, a], b], [\overline{jm}, c]]$ for all $m \in \mathfrak{n}$.

(5) $\overline{H}_0 \overline{jn}_\alpha = -\alpha \overline{jn}_\alpha$ where $\alpha \in \{0, \pm 1/2\}$, $n_\alpha \in \mathfrak{n}_\alpha$.

The standard argument yields:

(6) $[jn_\alpha, \mathfrak{g}_\lambda^{(-1)}] \subset \mathfrak{g}_{\lambda-\alpha}^{(1)}$.

This implies

(7) $\lambda \in \{\alpha, \alpha \pm 1/2\}$ where $\alpha \in \{0, \pm 1/2\}$ if $[jn_\alpha, \mathfrak{g}_\lambda^{(-1)}] \neq 0$. Eventually we want to prove $\{abc\} = 0$ for all $n \in \mathfrak{n}$. It clearly suffices to prove this for $\mathfrak{n} = n_\alpha \in \mathfrak{n}_\alpha$, $\alpha \in \{0, \pm 1/2.\}$. If $jn \in \mathfrak{n}$, then $\{abc\} = 0$. This implies that we can assume $\alpha \in \{0, 1/2\}$ since for $\alpha = -1/2$ we have $jn = [s, n] \in \mathfrak{n}$.

Suppose now that we can choose $a \in \bar{\mathfrak{g}}_\lambda^{(-1)}$, $b \in \bar{\mathfrak{g}}_\mu^{(-1)}$, $c \in \bar{\mathfrak{g}}_\nu^{(-1)}$, so that $\{abc\} \neq 0$. From the definition and the symmetry of $\{abc\}$ we derive that the commutators $[jn_\alpha, a]$, $[jn_\alpha, b]$ and $[jn_\alpha, c]$ do not vanish. Hence, by (7),

(8) $\lambda, \mu, \nu \in \{\alpha, \alpha \pm 1/2\}$ where $\alpha \in \{0, 1/2\}$.

The argument of [5] carries over without change to prove

(9) $\lambda + \mu$, $\mu + \nu$, $\nu + \lambda \in \{\alpha, 1 + \alpha, 1/2 + \alpha\}$, where $\alpha \in \{0, 1/2\}$.

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Next we prove a result similar to [5; § 6.15].

(10) $$[\mathfrak{g}, \mathfrak{g}^{(1)}] \subset \{a \in \mathfrak{a}^{(0)};\ \mathrm{ad}\, a \,|\, \mathfrak{a}^{(0)} + \mathfrak{k}^{(0)} \text{ is nilpotent}\} + [\mathfrak{k}^{(0)}, \mathfrak{k}^{(0)}]\,.$$

Proof. Let $x \in [\mathfrak{g}, \mathfrak{g}^{(1)}]$. Then $x \in \mathrm{nil}_0(\mathfrak{g}^{(0)}) \subset \mathfrak{s}^{(0)} = \mathfrak{k}^{(0)} + \mathfrak{a}^{(0)}$ by the convention 1.23. Hence $[\mathfrak{g}^{(0)}, x] \subset \mathfrak{s}^{(0)}$. We write $x = t + a$; since $ad\, x \,|\, \mathfrak{s}^{(0)}$ is nilpotent, $t \in [\mathfrak{k}^{(0)}, \mathfrak{k}^{(0)}]$ and $\mathrm{ad}\, a \,|\, \mathfrak{s}^{(0)}$ is nilpotent. This proves the claim.

The next result is [5; 6.16] and is proven as there. Let

(11) Let $g_\nu \in \mathfrak{g}_\nu$, $u_\eta \in \bar{\mathfrak{g}}^{(1)}_\eta$, $v_\xi \in \bar{\mathfrak{g}}^{(1)}_\xi$ and assume

$$\eta + \xi > 0 \text{ or } \eta = \xi = 0, \text{ then } [[g_\nu, u_\eta], v_\xi] = 0\,.$$

This implies

(12) $$[[\bar{\mathfrak{g}}^{(-1)}, \bar{\mathfrak{g}}^{(1)}_\eta], \bar{\mathfrak{g}}^{(1)}_\xi] = 0 \text{ if } \eta + \xi > 0 \text{ or } \eta = \xi = 0\,.$$

We want to apply (12) to $[\overline{jm}, \{abc\}]$. First we note that there exists some $m = m_\beta \in \mathfrak{n}_\beta$ so that $Q = [\overline{jm}, \{abc\}] \neq 0$; otherwise all representatives of $\{abc\}$ are in $\mathfrak{g}^{(0)}$, whence $\{abc\} = 0$. We note that Q is symmetric in a, b, c and in n, m. Therefofre $\beta \neq -1/2$. Moreover, (8) and (9) hold for B as well and comparing (4) and (12) we see that

(13) $$(\lambda - \alpha) + (\nu - \beta) \leq 0 \text{ and } \lambda - \alpha \neq 0 \text{ or } \nu - \beta \neq 0\,.$$

The same relations hold for every permutation of λ, μ, ν and transposition of α, β.

If $\alpha \neq \beta$, then we can assume $\alpha = 0$, $\beta = 1/2$. In this case relations (8), (9) and (13) become

(8)′ $\lambda, \mu, \nu \in \{0, \pm 1/2\} \cap \{0, 1/2, 1\} = \{0, 1/2\}$

(9)′ $\lambda + \mu, \mu + \nu, \lambda + \nu \in \{0, 1/2, 1\} \cap \{1/2, 1, 3/2\} = \{1/2, 1\}$

(13)′ $\lambda + u, \mu + \nu, \lambda + \nu \leq 1/2$.

Hence $\lambda + \mu = \mu + \nu = \lambda + \nu = 1/2$. Using (8)′ we get a contradiction. Therefore, we can assume $\alpha = \beta \in \{0, 1/2\}$. For $\alpha = 0$ the relations (8), (9), (13) give

(8)″ $\lambda, \mu, \nu \in \{0, \pm 1/2\}$

(9)″ $\lambda + \mu, \mu + \nu, \lambda + \nu \in \{0, 1/2, 1\}$

(13)″ $\lambda + \mu, \mu + \nu, \lambda + \nu \leq 0$.

Hence $\lambda + \mu, \mu + \nu, \lambda + \nu = 0$. Therefore $\mu = -\lambda$ and $\nu = -\mu = \lambda$. Since (13) also implies $\lambda \neq 0$ or $\nu \neq 0$, $\lambda, \mu, \nu \neq 0$. Therefore $\lambda + \nu = 2\lambda \neq 0$, a contrradiction.

Finally, for 1/2 we get

(8)‴ $\lambda, \mu, \nu \in \{0, 1/2, 1\}$
(9)‴ $\lambda+\mu, \mu+\nu, \lambda+\nu \in \{1/2, 1, 3/2\}$
(13)‴ $\lambda+\mu, \mu+\nu, \lambda+\nu \leq 1$ and at most one of the numbers λ, μ, ν is equal to 1/2.

This implies in particular that 1 is not attained in (13)‴. Therefore $\lambda+\mu = \mu+\nu = \lambda+\nu = 1/2$. Hence $\lambda, \mu, \nu \in \{0, 1/2\}$. By (13)‴ two of these numbers have to be 0, but then their sum is not 1/2, a contradiction. Therefore we have proven altogether

(14) $$\{abc\} = 0\,.$$

This implies

(15) $$[[\mathfrak{g}^{(1)}, \mathfrak{g}], \mathfrak{g}] \subset \mathfrak{g}^{(0)}\,.$$

From this one derives as in [5]

(16) $$[[\mathfrak{g}^{(1)}, \mathfrak{g}], \mathfrak{g}^{(1)}] = 0\,.$$

1.26. We consider $\mathfrak{z} = Z(\mathfrak{g}^{(1)}) = \{x \in \mathfrak{g}\,;\, [x, \mathfrak{g}^{(1)}] = 0\}$. Because $\mathfrak{g}^{(2)} \subset \mathfrak{g}^{(1)}$ we have in particular $[x, \mathfrak{g}^{(2)}] = 0 \subset \mathfrak{g}^{(2)}$ for all $x \in \mathfrak{z}$. Hence $\mathfrak{z} \subset \mathfrak{g}^{(0)}$. Again we can take over the first part of the proof of [7; part III, Lemma 18] without change and get

(1) $\mathfrak{z} \subset \mathfrak{g}^{(0)}$ is an ideal of $\mathfrak{g}$.

Obviously

(2) $\mathfrak{g}^{(1)} \subset \mathfrak{z}$.
(3) The Radical Conjecture holds for $\mathfrak{g}$.

Proof. We consider the solvable ideal $\mathfrak{z} \cap \mathfrak{r}$ of $\mathfrak{g}$. From (2) we know $\mathfrak{n} \subset \mathfrak{z} \cap \mathfrak{r} \subset \mathfrak{r}$. Hence we obtain that $\mathfrak{z} \cap \mathfrak{r}$ coincides either with $\mathfrak{n}$ or with $\mathfrak{r}$. In the latter case $\mathfrak{r} \subset \mathfrak{z} \subset \mathfrak{g}^{(0)}$ and $\mathfrak{g} = \mathfrak{g}^{(0)}$ follows. But we have settled this case already in 1.23 and are considering here only the case $\mathfrak{g} \neq \mathfrak{g}^{(0)}$. Therefore $\mathfrak{z} \cap \mathfrak{r} = \mathfrak{n}$. Because $\mathfrak{g}^{(1)} \subset \mathfrak{z}$ we obtain $\mathfrak{g}^{(1)} \cap \mathfrak{r} \subset \mathfrak{n}$. In particular $[j\mathfrak{n}, \mathfrak{r}] \subset \mathfrak{n}$. But then $[jn, jr] = j[jn, r] + j[n, jr] + [n, r] + k$ with some $k \in \mathfrak{k}$. It is easy to check that here all summands but k are contained in $\mathrm{nil}_0(\mathfrak{g})$. Hence $\mathrm{ad}\, k = 0$, whence $k = 0$. Therefore $[j\mathfrak{n}, j\mathfrak{r}] \subset \mathfrak{g}^{(2)}$. Finally, $[j\mathfrak{n}, \mathfrak{k}] = j[\mathfrak{n}, \mathfrak{k}] \subset j\mathfrak{n} \subset \mathfrak{g}^{(2)}$ (where we have used once more Corollary 1, d) of 1.4). Altogether we have shown $[j\mathfrak{n}, \mathfrak{g}] \subset \mathfrak{g}^{(2)}$. From this we obtain $[\mathfrak{g}^{(2)}, \mathfrak{g}] \subset \mathfrak{g}^{(2)}$. Hence $\mathfrak{g} \subset \mathfrak{g}^{(0)}$, a contradiction. This finishes the proof of (3) and also shows that the Radical Conjecture holds in "Case 1".

§ 2. Case 2: $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$

2.1. In this section we use $\mathfrak{g}$, $\mathfrak{r}$, $\mathfrak{n}$, $\mathfrak{g}'$, $\mathfrak{t}$ and $\mathfrak{a}$ as introduced in 1.1. Here $\mathfrak{t}$ is a modification of a normal j-algebra and $\mathfrak{a}$ the modification of an abelian Kähler algebra. Let e be the principal idempotent of $\mathfrak{t}$ and $\mathfrak{g} = \oplus \mathfrak{g}_\lambda$ the eigenspace decomposition of $\mathfrak{g}$ relative to $\operatorname{Re}(\operatorname{ad} je)$. In this section we consider the case where $\operatorname{Re}(\operatorname{ad} je)$ has only the eigenvalues 0 and 1. We know that $\operatorname{ad} je$ leaves invariant $\mathfrak{t}$, $\mathfrak{a}$, $\mathfrak{r}$, $\mathfrak{n}$ and $\mathfrak{g}'$. Therefore these spaces have an eigenspace decomposition as well.

2.2. In [6; 4.10] we have seen $\mathfrak{t}_1 = \mathfrak{n}_1$. Here we want to prove

LEMMA. $\mathfrak{n}_1 = \mathfrak{r}_1 = \mathfrak{g}_1$.

Proof. Clearly, $\mathfrak{g}_1$ is an abelian ideal of $\mathfrak{g}$. Therefore (independent of the induction) $\mathfrak{g}_1 + j\mathfrak{g}_1 + \mathfrak{k}$ is a Kähler algebra for which the Radical Conjecture holds. Hence (after an inessential change of j) we can assume that $\mathfrak{g}_1 + j\mathfrak{g}_1$ is a solvable Kähler algebra. Let $\tilde{e}$ be the maximal idempotent of $\mathfrak{g}_1 + j\mathfrak{g}_1$. Then $\tilde{e} = x_1 + jy_1$ and $[j\tilde{e}, je] = 0$. But this implies $y_1 = 0$ and $\tilde{e} \in g_1$. Therefore $[je, \tilde{e}] = \tilde{e}$. Hence we obtain $e = \tilde{e}$ and $\mathfrak{g}_1 = \mathfrak{n}_1$ follows.

COROLLARY. *$\mathfrak{g}_1 + j\mathfrak{g}_1$ is a modification of a normal j-algebra with principal idempotent e.*

2.3. Let c be a minimal idempotent in $\mathfrak{g}_1$. Then $[jc, c] = c$ and (in the underlying normal j-algebra) $\{x \in \mathfrak{g}_1; (jc, x) = x\} = \boldsymbol{R}c$.

We consider the eigenspace decomposition of $\mathfrak{g}$ relative to $\operatorname{Re}(\operatorname{ad} jc)$, $\mathfrak{g} = \bigoplus_{a \in \boldsymbol{R}} \mathfrak{g}^{(a)}$. Where necessary we write $\mathfrak{g}^{(a)} = \mathfrak{g}^{(a)}(c)$. We recall that subscripts refer to weights relative to $\operatorname{ad} je$, $\mathfrak{g}_\lambda^{(a)}$, etc.

We know that jc leaves $\mathfrak{g}_1$, $\mathfrak{g}_0$, $\mathfrak{r}$, $\mathfrak{n}$, $\mathfrak{t}$, $\mathfrak{a}$ and $\mathfrak{g}'$ invariant. Hence we also have a decomposition of each of these spaces relative to $\operatorname{ad} jc$.

Note that the weights of $\operatorname{ad} jc$ in $\mathfrak{g}_1 + j\mathfrak{g}_1$ and in $j\mathfrak{g}_1 + \mathfrak{k}$ are 0, $\pm 1/2$, 1. Hence, if $a \neq 0, \pm 1/2, 1$, then $\mathfrak{g}^{(a)} \subset \mathfrak{g}_0$. Moreover, by the usual argument $j\mathfrak{g}_\lambda^{(a)} \subset \mathfrak{g}_\lambda^{(a)} + \mathfrak{g}'$.

2.4. We can use the proof of [9; Lemma 4.2] and obtain

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LEMMA ([9]). *Let* $g \in \mathfrak{g}^{(a)}$ *and* $jg = \tilde{g} + x + jy + k$ *where* $\tilde{g} \in \mathfrak{g}^{(a)}$, $x, y \in \mathfrak{g}_1$ *and* $k \in \mathfrak{k}$. *Then* $x, y \in (\mathfrak{g}^{(a)} + \mathfrak{g}^{(a+1)}) \cap \mathfrak{g}_1$.

2.5. In this section we prove

LEMMA. $\mathfrak{g}^{(a)} = 0$ *if* $a \notin \frac{1}{2}\boldsymbol{Z}$.

Proof. Let $M = \{a \notin (1/2)\boldsymbol{Z}; \mathfrak{g}^{(a)} \neq 0\}$. We choose $a \in M$ so that $|a|$ is maximal. Then $[\mathfrak{g}^{(a)}, \mathfrak{g}_1] = 0$ and in particular $[\mathfrak{g}^{(a)}, c] = 0$. We will show that $\mathrm{ad}(\boldsymbol{R} + \boldsymbol{R}jc)$ is a "symplectic representation" of $\boldsymbol{R}c + \boldsymbol{R}jc$ on $\mathfrak{g}^{(a)}$. Then, by [8, sect. 2,3] we know that $\mathrm{ad}\, jc$ has only the weights 0, $\pm 1/2$ on $\mathfrak{g}^{(a)}$ yielding a contradiction and proving the lemma.

$$j\mathfrak{g}^{(a)} \subset \mathfrak{g}^{(a)} + \mathfrak{k}. \tag{1}$$

This follows from Lemma 2.4. From (1) we obtain that (after an inessential change of j) *we can assume* $j\mathfrak{g}^{(a)} \subset g^{(a)}$ for the chosen $a \in M$. Since $|a|$ is maximal in M we also have

$$[\mathfrak{g}^{(a)}, \mathfrak{g}^{(a)}] = 0 \qquad \text{if } 2a \notin \frac{1}{2}\boldsymbol{Z}. \tag{2}$$

$$\rho([\mathfrak{g}^{(a)}, \mathfrak{g}^{(a)}], jc) = 0 \qquad \text{if } 2a \in \frac{1}{2}\boldsymbol{Z}. \tag{3}$$

To prove (3) we note first $\mathfrak{g}^{(a)} \subset \mathfrak{g}_0$, since $a \in M$. Hence $[\mathfrak{g}^{(a)}, \mathfrak{g}^{(a)}] \subset \mathfrak{g}^{(2a)} \cap \mathfrak{g}_0 \subset \tilde{\mathfrak{g}} = \bigoplus_{b \in (1/2)\boldsymbol{Z}} \mathfrak{g}^{(b)}$. Since $\mathfrak{g}' \subset \tilde{\mathfrak{g}}$, Lemma 2.4 shows $j\tilde{\mathfrak{g}} \subset \tilde{\mathfrak{g}}$. Hence $\tilde{\mathfrak{g}}$ is a Kähler subalgebra of $\mathfrak{g}$. By assumption, $\mathfrak{g}^{(a)} \neq 0$, $a \notin (1/2)\boldsymbol{Z}$; therefore $\mathfrak{g} \neq \tilde{\mathfrak{g}}$. Finally, $\mathfrak{g}^{(b)} = \mathfrak{r}^{(b)} + j\mathfrak{r}^{(b)} \bmod \mathfrak{g}'$ since $\mathfrak{g} = \mathfrak{r} + j\mathfrak{r} + \mathfrak{k}$. Hence we can apply the induction hypothesis to $\tilde{\mathfrak{g}}$. Therefore $\rho(\tilde{\mathfrak{g}} \cap \mathfrak{g}_0, jc) = 0$ and in particular $\rho([\mathfrak{g}^{(a)}, \mathfrak{g}^{(a)}], jc) = 0$. From (3) it follows immediately that $\mathrm{ad}\, jc$ is symplectic on $\mathfrak{g}^{(a)}$:

$$\rho([jc, x], y) + \rho(x, [jc, y]) = 0 \qquad \text{for all } x, y \in \mathfrak{g}^{(a)} \tag{4}$$

Since $[\mathfrak{g}^{(a)}, c] = 0$, $\mathrm{ad}\, c$ is symplectic on $\mathfrak{g}^{(a)}$ as well. Because $[jc, c] = c$, we have only left to verify (and do it by a straightforward computation)

$$[j, \mathrm{ad}\, jc - 1/2[j, \mathrm{ad}\, c]]\mathfrak{g}^{(a)} = 0. \tag{5}$$

This finishes the proof of the lemma.

2.6. We sharpen the last result and get

LEMMA. $\mathfrak{g}^{(a)} = 0$ *if* $a \notin \{0, \pm 1/2, \pm 1\}$.

Proof. Let $a \in (1/2)\boldsymbol{Z}$, $a \notin \{0, \pm 1/2, \pm 1\}$ and suppose $\mathfrak{g}^{(a)} \neq 0$. We assume that $|a|$ is maximal. Then $[\mathfrak{g}^{(a)}, \mathfrak{g}_1] = 0$, since $\operatorname{ad} jc$ has only the weights 0, 1/2, 1 in $\mathfrak{g}_1$. Moreover, $j\mathfrak{g}^{(a)} \subset \mathfrak{g}^{(a)} + \mathfrak{k}$ follows from Lemma 2.4 since $\mathfrak{g}^{(a)} \cap \mathfrak{g}_1 = 0$ and $\mathfrak{g}^{(a+1)} \cap \mathfrak{g}_1 = 0$. We can again assume that even $j\mathfrak{g}^{(a)} \subset \mathfrak{g}^{(a)}$ holds. From $[\mathfrak{g}^{(a)}, c] = 0$ and $[\mathfrak{g}^{(a)}, \mathfrak{g}^{(a)}] = 0$ we conclude that $\operatorname{ad} c$ and $\operatorname{ad} jc$ are symplectic maps of $\mathfrak{g}^{(a)}$. As at the end of the proof of the last lemma one finishes the verification that ad is a symplectic representation of $\boldsymbol{R}c + \boldsymbol{R}jc$ on $\mathfrak{g}^{(a)}$. Hence $a \in \{0, \pm 1/2\}$, a contradiction.

2.7. From Lemma 2.4 it is easy to derive that after an inessential change of j we can assume $j\hat{\mathfrak{g}} \subset \hat{\mathfrak{g}}$ where $\hat{\mathfrak{g}} = \mathfrak{g}^{(-1)} + \mathfrak{g}^{(0)} + \mathfrak{g}^{(1)}$. Therefore $\hat{\mathfrak{g}}$ is a Kähler subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{g}^{(n)} = \mathfrak{r}^{(n)} + j\mathfrak{r}^{(n)} \bmod (\mathfrak{g}' \cap \hat{\mathfrak{g}})$. Hence we can apply the induction hypothesis to $\hat{\mathfrak{g}}$ in case $\hat{\mathfrak{g}} \neq \mathfrak{g}$. In this case we have $\mathfrak{g}^{(-1)} = 0$, whence

$$\mathfrak{g} = \mathfrak{g}^{(-1/2)} + \mathfrak{g}^{(0)} + \mathfrak{g}^{(1/2)} + \mathfrak{g}^{(1)} \qquad \text{if } \hat{\mathfrak{g}} \neq \mathfrak{g}\,.$$

Before we continue to consider this case more closely we want to finish the possibility $\mathfrak{g} = \hat{\mathfrak{g}}$.

LEMMA. *If $\mathfrak{g} = \hat{\mathfrak{g}}$, then the Radical Conjecture holds.*

Proof. By our assumption, jc has only the weights 0 and 1 in $\mathfrak{g}_1$. Hence $\mathfrak{g}_1 = \mathfrak{g}_1^{(0)} + \mathfrak{g}_1^{(0)}$ and since c is a minimal idempotent $\mathfrak{g}_1^{(1)} = \boldsymbol{R}c$. Because there is no weight 1/2 we know that the underlying normal j-algebra $\mathfrak{g}_1 + j\mathfrak{g}_1$ is the product of the subalgebras $\boldsymbol{R}c + \boldsymbol{R}jc$ and $\mathfrak{g}_1^{(0)} + j\mathfrak{g}_1^{(0)}$. Since the modification derivations $D(x)$ annihilate c and jc we conclude that $\mathfrak{g}_1^{(0)} + j\mathfrak{g}_1^{(0)}$ is a subalgebra of the given Kähler algebra $\mathfrak{g}_1 + j\mathfrak{g}_1$. We also know that $\mathfrak{k}$ leaves both algebras invariant. Set $\mathfrak{g}^* = \mathfrak{g}^{(-1)} + j\mathfrak{g}_1^{(0)} + \mathfrak{g}_1^{(0)} + \mathfrak{k}$. Then Lemma 2.4 shows that $\mathfrak{g}^*$ is j-invariant. It is easy to verify that $\mathfrak{g}^*$ is a subalgebra of $\mathfrak{g}$. It is straightforward to check that $\mathfrak{g}^{(-1)} + \mathfrak{g}_1^{(0)}$ is an ideal of $\mathfrak{g}^*$. Clearly, $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}_1^{(0)}$ are both abelian and since $[\mathfrak{g}^{(-1)}, \mathfrak{g}_1^{(0)}]$ $= [\mathfrak{g}_0^{(-1)}, \mathfrak{g}_1^{(0)}] \subset \mathfrak{g}_1^{(-1)} = 0$ we see that $\mathfrak{g}^{(-1)} + \mathfrak{g}_1^{(0)}$ is an abelian ideal of the Kähler algebra $\mathfrak{g}^*$. Therefore the Radical Conjecture holds for $\mathfrak{g}^*$. Next we want to show $\mathfrak{g}^{(-1)} = 0$. We consider the idempotent $e - c$ of $\mathfrak{g}_1^{(0)} + j\mathfrak{g}_1^{(0)}$. It is easy to see that the real part of $\operatorname{ad} j(e - x)$ acts are identity map on $\mathfrak{g}^{(-1)}$. From the solvable theory, applied to $\mathfrak{g}^*$, we know that $j\mathfrak{g}^{(-1)}$ is annihilated by the real part of $\operatorname{ad} j(e - c)$. But from Lemma 2.4 we obtain $i\mathfrak{g}^{(-1)} \subset \mathfrak{g}^{(-1)} + \mathfrak{g}_1^{(0)} + j\mathfrak{g}_1^{(0)} + \mathfrak{k}$, whence $j\mathfrak{g}^{(-1)} \subset j\mathfrak{g}_1^{(0)} + \mathfrak{k}$ and $\mathfrak{g}^{(-1)} \subset \mathfrak{g}_1^{(0)} + \mathfrak{k}$. This implies $\mathfrak{g}^{(-1)} = 0$. Hence, by our assumption $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 = \mathfrak{g}^{(0)} + \mathfrak{g}^{(1)}$.

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To finish our argument we consider as in [8; sect. 2.5] the space $\mathfrak{u} = \{x \in \mathfrak{g}; [x, c] = 0, [jx, c] = 0\}$. Clearly, $\mathfrak{k} \subset \mathfrak{u}$ and $j\mathfrak{u} \subset \mathfrak{u}$. Since $\boldsymbol{R}c$ is a one dimensional ideal of $\mathfrak{g}$ we have for $x \in \mathfrak{g}$: $[x, c] = ac$ and $[jx, c] = bc$ for some $a, b \in \boldsymbol{R}$. A straightforward computation shows $x - ajc - bc \in \mathfrak{u}$. Hence $\mathfrak{g} = \mathfrak{u} + \boldsymbol{R}c + \boldsymbol{R}jc$. From the definition of $\mathfrak{u}$ we derive $\mathfrak{g}_1^{(0)} \subset \mathfrak{u}$, whence $\mathfrak{g}_1^{(0)} + j\mathfrak{g}_1^{(0)} + \mathfrak{k} \subset \mathfrak{u}$. Next we show $[jc, \mathfrak{u}] \subset \mathfrak{u}$. Let $u \in \mathfrak{u}$, then $[[jc, u], c] = 0$ and $[j[jc, u], c] = [[jc, ju] - j[c, ju] - [c, u] - k, c] = 0$. Therefore $\mathfrak{u} = \mathfrak{u}^{(0)} + \mathfrak{u}^{(1)}$. Since $\mathfrak{g}^{(1)} = \boldsymbol{R}c$, $\mathfrak{u} \subset \mathfrak{g}^{(0)}$. Let $u = u_1 + u_0$, $u_i \in \mathfrak{g}_i$. Then $u \in \mathfrak{u}$ iff $[u_0, c] = 0$ and $[ju_1 + ju_0, c] = 0$. In particular $u_0 \in \mathfrak{v} = \{x \in \mathfrak{g}_0; [x, c] = 0\}$. On the other hand, let $v \in v$, then $[v, c] = 0$ and $[jv, c] = bc$. Therefore $v - bc \in \mathfrak{u}$. But $\mathfrak{u} \subset \mathfrak{g}^{(0)}$ and $v \in \mathfrak{g}_0 \subset \mathfrak{g}^{(0)}$, whence $b = 0$. Therefore $\mathfrak{u} = \mathfrak{g}_1^{(0)} + \mathfrak{v}$. To see that $\mathfrak{u}$ is a Kähler algebra we have to show $[\mathfrak{v}, \mathfrak{g}_1^{(0)}] \subset \mathfrak{g}_1^{(0)}$. But this follows since $\mathfrak{u} \subset \mathfrak{g}^{(0)}$, $\mathfrak{g}^{(0)}$ is a subalgebra and $\mathfrak{g}^{(0)} \cap \mathfrak{g}_1 = \mathfrak{g}_1^{(0)}$. Finally, $\mathfrak{r}_0 = \mathfrak{r} \cap \mathfrak{g}_0$ acts nilpotently on $\mathfrak{g}$, therefore $[\mathfrak{r}_0, c] = 0$ and $\mathfrak{r}_0 \subset \mathfrak{v} \subset \mathfrak{u}$ follows. Now it is easy to see that $\mathfrak{u} = (\mathfrak{r}_0 + \mathfrak{g}_1^{(0)}) + j(\mathfrak{r}_0 + \mathfrak{g}_1^{(0)}) + \mathfrak{k}$ holds, where $\mathfrak{r}_0 + \mathfrak{g}_1^{(0)}$ is a nilpotent ideal of $\mathfrak{u}$. This implies that the Radical Conjecture holds for $\mathfrak{u}$. From this we will derive that the Radical Conjecture holds for $\mathfrak{g}$. Let $\mathfrak{h}$ be a maximal semisimple subalgebra of $\mathfrak{g}_0$. Then $\mathfrak{h}$ is maximal semisimple in $\mathfrak{g}$. Moreover, $[\mathfrak{h}, \boldsymbol{R}c] \subset \boldsymbol{R}c$ implies $[\mathfrak{h}, c] = 0$, i.e. $\mathfrak{h} \subset \mathfrak{v}$. Let $\mathfrak{t}_h$ be a maximal split solvable subalgebra of $\mathfrak{h}$. Then $\mathfrak{t}_u = \mathfrak{t}_h + \mathrm{rad}(\mathfrak{v}) + \mathfrak{g}_1^{(0)}$ is a solvable subalgebra of $\mathfrak{u}$ and $\mathfrak{u} = \mathfrak{t}_u + \mathfrak{k}$ by the Radical Conjecture applied to $\mathfrak{u}$. Moreover, $\mathfrak{h} \subset \mathfrak{t}_h + \mathfrak{k}$. Therefore, $\mathfrak{t}_0 = \mathfrak{t}_h + \mathrm{rad}(\mathfrak{g}_0)$ is a solvable subalgebra of $\mathfrak{g}_0$ and $\mathfrak{g}_0 = \mathfrak{t}_0 + \mathfrak{k}$ holds. Hence $\mathfrak{t} = \mathfrak{t}_0 + \mathfrak{g}_1$ is a solvable subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{t} + \mathfrak{k}$. From this the Radical Conjecture follows.

2.8. In the last subsection we have seen that the Radical Conjecture holds if $\hat{\mathfrak{g}}(c) = \mathfrak{g}$ for some minimal idempotent c of $\mathfrak{g}_1$. Therefore, from now on we can assume $\mathfrak{g} \neq \hat{\mathfrak{g}}(c)$ for all minimal idempotents c of $\mathfrak{g}_1$. *Hence* $\mathfrak{g} = \mathfrak{g}^{(-1/2)}(c) + \mathfrak{g}^{(0)}(c) + \mathfrak{g}^{(1/2)}(c) + \mathfrak{g}^{(1)}(c)$ *for all minimal idempotents* $c \in \mathfrak{g}_1$. Applying the induction hypothesis to $\hat{\mathfrak{g}}(c) = \mathfrak{g}^{(0)}(c) + \mathfrak{g}^{(1)}(c)$ once more shows $\mathfrak{g}^{(1)}(c) \subset \mathfrak{g}_1$. Therefore $\mathfrak{g}_0^{(a)}(c) \neq 0$ implies $-1/2 \leq a \leq 1/2$. This shows that [9; Lemma 4.3] holds. This together with Lemma 2.4, (which is [9; Lemma 4.2]) enables us to carry out the rest of the arguments of [9; § 2] with few changes.

We choose minimal tripotents $c_1, \cdots, c_k$ in $\mathfrak{g}_1$ satisfying

$$c_1 + \cdots + c_k = e \tag{1}$$

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(2) $$[jc_i, c_k] = \delta_{ik} c_k.$$

These conditions determine the c_k's uniquely (up to permutation). For the purposes of this subsection we order $c_1, \cdots, c_k$ as in [9]. From the solvable theory (applied to $\mathfrak{g}_1 + j\mathfrak{g}_1$) we know

(3) $$[jc_i, jc_k] = 0 \quad \text{for all } i, k.$$

Therefore we get a simultaneous eigenspace decomposition of $\mathfrak{g}$ relative to $R_1, \cdots, R_l, R_k = \mathrm{Re}(\mathrm{ad}\, jc_k)$. We thus get $\mathfrak{g} = \oplus \mathfrak{g}^{(\Lambda)}$ where $R_i x^{(\Lambda)} = \Lambda(i) x^{(\Lambda)}$ for all $x^{(\Lambda)} \in \mathfrak{g}^{(\Lambda)}$. Clearly, $\mathfrak{g}^{(\Lambda)} = \mathfrak{g}_1^{(\Lambda)} + \mathfrak{g}_0^{(\Lambda)}$.

We know $\Lambda = \Delta_i$ or $\Lambda = (1/2)(\Delta_i + \Delta_j)$ on $\mathfrak{g}_1$ where $\Delta_i(k) = \delta_{ik}$. Moreover, we have seen above that $\Lambda(k) = 1$ for some $k \in \{1, \cdots, l\}$ implies $\mathfrak{g}^{(\Lambda)} \subset \mathfrak{g}_1$ and that $\Lambda(k) \in \{0, \pm 1/2, 1\}$ for all k.

As in [9] we introduce the subalgebra $\mathfrak{s}$ of $\mathfrak{g}_0$, $\mathfrak{s} = \{x \in \mathfrak{g}_0;\ [x, e] = 0\}$. Since $x - j[x, e] \in \mathfrak{s}$ for each $x \in \mathfrak{g}_0$ we have $\mathfrak{g}_0 = \mathfrak{s} + j\mathfrak{g}_1$.

Using these fact and definitions, the proof of [9; Lemma 4.4] carries over and yields

(4) $$\mathfrak{g}_0^{(\Lambda)} \subset s \text{ if } \Lambda \neq 1/2(\Delta_i - \Delta_k) \quad \text{for all } i, k.$$

Next we consider $s \in \mathfrak{s}$ and decompose it relative to $\mathfrak{g} = \oplus \mathfrak{g}^{(\Lambda)}$. From (4) we conclude that $s^{(0)} + \sum_{a \neq b} s^{((1/2)\Delta_a - 1/(2)\Delta_b)} \in \mathfrak{s}$. It is easy to see that $[s^{(0)}, c_i] \in \boldsymbol{R} c_i$ holds. Moreover, assume $0 \neq x = [s^{((1/2)\Delta_a - (1/2)\Delta_b)}, c_i]$, $a \neq b$, is a multiple of some c_k. Then an application of R_k yields $1 = (1/2)(\delta_{ak} - \delta_{bk}) + \delta_{ik}$. Hence $i = k$, $i \neq a, b$. But then $R_a x = (1/2)x$, whence $x = 0$, a contradiction. Therefore x is perpendicular to $\bigoplus_{k=1}^{l} \boldsymbol{R} c_k$. Altogether $[s^{(0)}, e] = 0$ follows. We thus have proven

(5) ([9]) Let $s \in \mathfrak{s}$, then $s^{(0)} \in \mathfrak{s}$.

Next we show

(6) ([8], [9]) Let $\Lambda = (1/2)(\Delta_a - \Delta_b)$, $a \leq b$, then we have

$$\mathfrak{g}_0^{(\Lambda)} = (\mathfrak{s} \cap \mathfrak{g}_0^{(\Lambda)}) + (j\mathfrak{g}_1)^{(\Lambda)}.$$

For $\Lambda = 0$ this follows from (5) and the case $\Lambda \neq 0$ can be shown as in [8; Sect. 4.4].

We set $\sigma(x) = \mathrm{trace}(\mathrm{ad}\, x | \mathfrak{g}_1)$, $x \in \mathfrak{g}$. The proof of [9; Lemma 4.6] carries over without change and yields ($y_{\mathfrak{s}}$ denoting the $\mathfrak{s}$-component of y)

(7) ([9], [8]) let $\Lambda = 1/2(\Delta_a - \Delta_b)$, $a \leq b$, and let $s \in \mathfrak{s} \cap g_0^{(\Lambda)}$. Then $\sigma([s, jx]_{\mathfrak{s}}) = 0$ for all $x \in \mathfrak{g}_1$.

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LEMMA ([8]). trace$(\operatorname{ad} s|\mathfrak{g}_1) = 0$ *for all* $s \in \mathfrak{s}$.

Proof. By (5) we can assume $s \in \mathfrak{s}^{(0)}$. Using (6) and (7) we can carry out the proof of [8; Lemma 8, sect. 4] without further changes and obtain the claim.

2.9. We collect the properties of $\mathfrak{s}$ which will be used in the following sections.

LEMMA. a) $\mathfrak{s} = \{x \in \mathfrak{g}_0; [x, e] = 0\}$ *is a Kähler subalgebra of* $\mathfrak{g}_0$ *and* $\mathfrak{k} \subset \mathfrak{s}$.

b) $\mathfrak{g}_0 = \mathfrak{s} + j\mathfrak{g}$ *is a direct sum of vectcr spaces.*

c) trace$(\operatorname{ad} \mathfrak{s}|\mathfrak{g}_1) = 0$

d) $\operatorname{ad} \mathfrak{s}|\mathfrak{g}_1$ *is a contained in the isotropy algebra of the homogeneous cone* K *in* $\mathfrak{g}_1$ *which is associated with the Kähler algebra* $\mathfrak{g}_1 + j\mathfrak{g}_1$ *and the point* $e \in K$.

Proof. a) Following [8; Sec. 2, 5] we let $\mathfrak{m} = \{x \in \mathfrak{g}; [x, e] = 0, [jx, e] = 0\}$. Clearly, $j\mathfrak{m} \subset \mathfrak{m}$. As in loc. cit. one proves $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}_1 + j\mathfrak{g}_1$ and $[je, \mathfrak{m}] \subset \mathfrak{m}$. Hence $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$. But from the definition of $\mathfrak{m}$ it follows $0 = [jm_1, e] = m_1$, whence $\mathfrak{m} \subset \mathfrak{g}_0$. Obviously, $\mathfrak{m} \subset \mathfrak{s}$. But $\mathfrak{m}$ and $\mathfrak{s}$ have the same dimension since both are algebraic complements of $j\mathfrak{g}_1$ in $\mathfrak{g}$. Therefore $\mathfrak{m} = \mathfrak{s}$ and a) follows. b) and c) have been shown in the last subsection. d) This follows from [17; Proposition 4] and c).

2.10. Put $\check{\mathfrak{s}} = \{x \in \mathfrak{s}; [x, \mathfrak{g}_1] = 0\}$. Then $\check{\mathfrak{s}} \subset \mathfrak{s}$ is an ideal of $\mathfrak{g}$.

LEMMA. $\mathfrak{s} = \check{\mathfrak{s}} + j\check{\mathfrak{s}} + \mathfrak{k}$ *and* $\mathfrak{r}_0 \subset \check{\mathfrak{s}} + j\mathfrak{g}_1$.

Proof. Let $r \in \mathfrak{r}_0 = \mathfrak{r} \cap \mathfrak{g}_0$; then $r = s + jg_1$, $s \in \mathfrak{s}$, $g_1 \in \mathfrak{g}_1$. From Lemma 2.9, we know $\operatorname{ad} \mathfrak{g}_0|\mathfrak{g}_1 \subset \operatorname{Lie Aut} K$, the infinitesimal linear automorphisms of the cone K. Hence $\operatorname{ad} \mathfrak{r}_0|\mathfrak{g}_1$ is an ideal in $\operatorname{ad} \mathfrak{g}_0|\mathfrak{g}_1$ which consists of nilpotent endomorphisms. This implies $\operatorname{ad} \mathfrak{r}_0|\mathfrak{g}_1 \subset \operatorname{ad} j\mathfrak{g}_1|\mathfrak{g}_1$, whence. $\mathfrak{r}_0 \subset \check{\mathfrak{s}} + j\mathfrak{g}_1$. But this implies $\mathfrak{s} \subset \mathfrak{r}_0 + j\mathfrak{r}_0 + \mathfrak{g}_1 + j\mathfrak{g}_1 + \mathfrak{k} \subset \check{\mathfrak{s}} + j\check{\mathfrak{s}} + \mathfrak{k} + \mathfrak{g}_1 + j\mathfrak{g}_1 \subset \mathfrak{s} + \mathfrak{g}_1 + j\mathfrak{g}_1$ and the assertion follows.

2.11. Set $\mathfrak{x} = \{jg_1;\ s + jg_1 \in \mathfrak{r}_0 \text{ for some } s \in \check{\mathfrak{s}}\}$. Denote by $\mathfrak{p}$ the Kähler subalgebra of $\mathfrak{g}_1 + j\mathfrak{g}_1$ generated by $\mathfrak{x}$. Then $\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{g}_0) + (\mathfrak{p} \cap \mathfrak{g}_1)$.

LEMMA. a) $\mathfrak{x}$ *is an ideal of* $j\mathfrak{g}_1$.

b) $[\mathfrak{s}, \mathfrak{p}] \subset \mathfrak{s} + \mathfrak{p}$.

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c) $\mathfrak{p} + \mathfrak{s}$ *is a Kähler subalgebra of* $\mathfrak{g}$.

d) $\mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{g}_1 \neq \mathfrak{g}_1$.

Proof. a) Let $jb \in \check{\mathfrak{x}}$ and $s \in \check{\mathfrak{s}}$ so that $s + jb \in r_0$. Then $[ja, s] + [ja, jb] = [ja, s + jb] \in \mathfrak{r}_0$. Since $[ja, s] \in \check{\mathfrak{s}}$ and $[ja, jb] \in j\mathfrak{g}_1$ we conclude $[ja, jb] \in \check{\mathfrak{x}}$.

b) First we prove

(1) Let $\mathfrak{u}$ be a subalgebra of $\mathfrak{g}$ satisfying $[\mathfrak{s}, \mathfrak{u}] \subset \mathfrak{u} + \mathfrak{s}$. Then the Lie algebra $\bar{\mathfrak{u}}$ generated by $\mathfrak{u} + j\mathfrak{u}$ satisfies $[\mathfrak{s}, \bar{\mathfrak{u}}] \subset \bar{\mathfrak{u}} + \mathfrak{s}$.

Proof. Since $\check{\mathfrak{s}}$ is an ideal of $\mathfrak{g}$ the condition $[\mathfrak{s}, \mathfrak{u}] \subset \mathfrak{u} + \mathfrak{s}$ is equivalent to $[j\check{\mathfrak{s}}, \mathfrak{u}] \subset \mathfrak{u} + \mathfrak{s}$ and $[\mathfrak{k}, \mathfrak{u}] \subset \mathfrak{u} + \mathfrak{s}$. But then $[j\check{s}, ju] = j[j\check{s}, u] + j[\check{s}, ju] + [\check{s}, u] + k$ shows $[j\check{\mathfrak{s}}, j\mathfrak{u}] \subset j\mathfrak{u} + \mathfrak{s}$ and $[k, ju] = j[k, u] + k'$ implies $[\mathfrak{k}, j\mathfrak{u}] \subset j\mathfrak{u} + \mathfrak{s}$. Hence altogether $[\mathfrak{s}, \mathfrak{u} + j\mathfrak{u}] \subset \mathfrak{u} + j\mathfrak{u} + s$. A simple induction finishes now the proof of (1).

From (1) follows immediately

(2) Let $\mathfrak{u}$ be a subalgebra of $\mathfrak{g}$ satisfying $[\mathfrak{s}, \mathfrak{u}] \subset \mathfrak{u} + \mathfrak{s}$. Then the j-algebra $\tilde{\mathfrak{u}}$ generated by $\mathfrak{u}$ satisfies $[\mathfrak{s}, \tilde{\mathfrak{u}}] \subset \tilde{\mathfrak{u}} + \mathfrak{s}$.

To prove b) it suffices now to show

(3) $[\mathfrak{s}, \check{\mathfrak{x}}] \subset \check{\mathfrak{x}} + \mathfrak{s}$.

Proof. We note that $[k, s + jg_1] = [k, s] + k' + j[k, g_1] \in r_0$ if $s + jg_1 \in r_0$; clearly $[k, s] \in \check{\mathfrak{s}}$ if $s \in \check{\mathfrak{s}}$, whence $k' \in \check{\mathfrak{s}}$ by Lemma 2.10. Hence $[\mathfrak{k}, \check{\mathfrak{x}}] \subset \check{\mathfrak{x}}$. Also, $[js', s + jg_1] = [js', s] + [js', jg_1] = [js', s] + j[js', g_1] + j[s', jg_1] + [s', g_1] + k = ([js', s] + j[s', jg_1] + k) + j[js', g_1] \in r_0 \cap (\mathfrak{s} + j\mathfrak{g}_1)$ for all $s' \in \check{\mathfrak{s}}$ and s, g_1 as in the definition of $\check{\mathfrak{x}}$. By Lemma 2.10, we conclude $[js', s] + j[s', jg_1] + k \in \check{\mathfrak{s}}$. Hence $j[js', g_1] \in \check{\mathfrak{x}}$. This finishes the proof of (3) and thus of b).

c) Since $\mathfrak{p}$ and $\mathfrak{s}$ are Kähler algebras the assertion follows from b).

d) We know $\mathfrak{r} \subset \mathrm{nil}(\mathfrak{g})$. Therefore $\mathrm{ad}\, r$ is nilpotent for all $r \in \mathfrak{r}$. Since $\check{\mathfrak{s}}$ is an ideal of $\mathfrak{g}$ we derive from $\mathrm{ad}(s + jg_1)^n b = (\mathrm{ad}\, jg_1 | \mathfrak{g}_1)^n b$, $b \in \mathfrak{g}_1$, that $\mathrm{ad}\, jg_1 | \mathfrak{g}_1$ is nilpotent for all $jg_1 \in \check{\mathfrak{x}}$. This implies that $\check{\mathfrak{x}}$ is perpendicular to all jc_i, c_i a minimal idempotent in $\mathfrak{g}_1$. We order the minimal idempotents as in [4; p. 5] and see that the last minimal idempotent c is perpendicular to the clan generated by $j\check{\mathfrak{x}}$. From this it follows that $\check{\mathfrak{x}} + j\check{\mathfrak{x}}$ is contained in the j-algebra of elements of $\mathfrak{g}_1 + j\mathfrak{g}_1$ which are perpendicular to $\boldsymbol{R}c + \boldsymbol{R}jc$. Therefore c is perendicular to $\mathfrak{p}_1$ and d) follows.

2.12. We have $\mathfrak{r}_0 \subset \check{\mathfrak{s}} + \check{\mathfrak{x}} \subset \mathfrak{s} + \mathfrak{p}$ and $\mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{g}_1 \subset \mathfrak{r}_1$. It is easy to verify that $\mathfrak{r}_0 + \mathfrak{p}_1$ is a (solvable) ideal of $\mathfrak{p} + \mathfrak{s}$. Moreover, $\dim(\mathfrak{r}_0 + \mathfrak{p}_1) < \dim \mathfrak{r}$. Consider the Kähler subalgebra $\mathfrak{w} = (\mathfrak{r}_0 + \mathfrak{p}_1) + j(\mathfrak{r}_0 + \mathfrak{p}_1) + \mathfrak{k}$ of $\mathfrak{s} + \mathfrak{p}$.

LEMMA. $\mathfrak{s} \subset \mathfrak{w} = \mathfrak{s} + \mathfrak{p}$.

Proof. Let $s \in \mathfrak{s}$. Then $s = r_0 + jr_0' + x_1 + jy_1 + k$ where $r_0,\ r_0' \in \mathfrak{r}_0$, $x_1,\ y_1 \in \mathfrak{g}_1$, $k \in \mathfrak{k}$. We can write $r_0 = s_0 + jg_1$, $r_0' = s_0' + jg_1'$ with $s_0,\ s_0' \in \check{\mathfrak{s}}$, $g_1,\ g_1' \in \mathfrak{p}_1$. Hence $s = s_0 + js_0' + k + j(g_1 + y_1) + (x_1 - g_1')$ and $x_1 = g_1' \in \mathfrak{p}_1$, $y_1 = -g_1 \in \mathfrak{p}_1$. Therefore $s \in \mathfrak{w}$. To finish the proof it suffices to show $\mathfrak{p} \subset \mathfrak{w}$. But $\mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_1 \subset \mathfrak{g}_1 + j\mathfrak{g}_1$ where $\mathfrak{p}_j = \mathfrak{p} \cap \mathfrak{g}_j$. Hence $\mathfrak{p}_0 = j\mathfrak{p}_1$ and $\mathfrak{p} \subset \mathfrak{w}$ follows.

It is clear that we can apply the induction hypothesis to $\mathfrak{w} = \mathfrak{p} + \mathfrak{s}$.

COROLLARY 1. a) *$\mathfrak{s} = \mathfrak{s}_s + \mathfrak{k}$ where $\mathfrak{s}_s$ is a solvable subalgebra of $\mathfrak{s}$.*

b) *After an inessential change of j, which does not alter j on $\mathfrak{g}_1 + j\mathfrak{g}_1$, we can assume $j\mathfrak{s}_s \subset \mathfrak{s}_s$.*

Proof. a) Let $\mathfrak{h}$ be a maximal semisimple subalgebra of $\mathfrak{s}$ containing $[\mathfrak{k}, \mathfrak{k}]$. From the Radical Conjecture applied to $\mathfrak{w} \supset \mathfrak{s}$ it follows that a maximal compact subalgebra of $\mathfrak{h}$ is already contained in $\mathfrak{k}$. From this the claim follows.

b) follows from a) and the facts $\mathfrak{s} \cap (\mathfrak{g}_1 + j\mathfrak{g}_1) = 0$, $j\mathfrak{s} \subset \mathfrak{s}$ and $j(\mathfrak{g}_1 + j\mathfrak{g}_1) \subset \mathfrak{g}_1 + j\mathfrak{g}_1$.

COROLLARY 2. a) *$\operatorname{ad}(\mathfrak{s}_s)|\mathfrak{g}_1$ is abelian*

b) $[\mathfrak{s}_s, \mathfrak{s}_s] \subset \check{\mathfrak{s}}$.

Proof. Since $\operatorname{ad} s|\mathfrak{g}_1$, $s \in \mathfrak{s}$, is skewadjoint relative to some inner product on $\mathfrak{g}_1$ we know that $\operatorname{ad}(\mathfrak{s}_s)|\mathfrak{g}_1$ is solvable and skewadjoint, whence abelian, proving a).

b) follows immdiately from a).

Remark. The importance of Corollary 2 is, that it deals with all of $\mathfrak{s}$ (modulo the isotropy part $\mathfrak{k}$).

2.13. Let s_0 be a principal idempotent of the Kähler albebra $\mathfrak{s}_s$ satisfying $[\mathfrak{k}, s_0] = 0$. Since $[js_0, s_0] = s_0$ we know from Corollary 2.12.2 that $s_0 \in \mathfrak{s}$ holds.

Let $D = \operatorname{Re}(\operatorname{ad} js_0)$ and $\mathfrak{g} = \bigoplus_{b \in R} \mathfrak{g}^{[b]}$ be the eigenspace decomposition of $\mathfrak{g}$ relative to D. Then $\mathfrak{k} \subset \mathfrak{g}^{[0]}$.

Since $\operatorname{ad} js_0|\mathfrak{g}_1$ is skewadjoint (realtive to some inner product on $\mathfrak{g}_1$) we have

(1) $\mathfrak{g}_1 \subset \mathfrak{g}^{[0]}$.

(2) D has only the eigenvalues $0, \pm 1/2, 1$ and the eigenspaces for the eigenvalues $\pm 1/2, 1$ are contained in $\mathfrak{s}_s$.

Proof. For $g \in \mathfrak{g}_1$ we have $[js_0, jg] = j[js_0, g] \bmod \mathfrak{s}$. Hence $Djg = jDg \bmod \mathfrak{s} = 0 \bmod \mathfrak{s}$. This implies that nonzero eigenvalues of D can only occur in $\mathfrak{s}$. But $\mathfrak{k} \subset \mathfrak{g}^{[0]}$; therefore nonzero eigenvalues can only occur in $\mathfrak{s}_s$. Note that in $\mathfrak{s}_s$ only the eigenvalues $0, \pm 1/2, 1$ can occur.

2.14. We want to apply the appendix to $D = \operatorname{Re}(\operatorname{ad} js_0)$. Let $\mathfrak{q}$ be the algebraic hull of $\operatorname{ad}\mathfrak{g} \subset \operatorname{End}_R(\mathfrak{g})$. Then $D \in \mathfrak{q}$ is a semisimple endomorphism of $\mathfrak{g}$. Hence $D = D_s + D_r$ where $D_s \in \mathfrak{H}$, $\mathfrak{H}$ some maximal semisimple subalgebra of $\mathfrak{q}$, and $D_r \in \operatorname{rad}\mathfrak{q}$ satisfy $[D_s, D_r] = 0$. Since $[\mathfrak{H}, \mathfrak{H}] = \mathfrak{H}$ we have $\mathfrak{H} = \operatorname{ad}\mathfrak{h}$ for some maximal semisimple subalgebra of $\mathfrak{g}$. Let $h_0 \in \mathfrak{h}$ so that $\operatorname{ad} h_0 = D_s$. Since $\operatorname{ad} h_0$ is semisimple with only real eigenvalues, h_0 is contained in some Cartan subalgebra of $\mathfrak{h}$. Therefore, if $\operatorname{ad} h_0$ has an eigenvalue $\lambda \neq 0$, then it also has the eigenvalue $-\lambda \neq 0$. Moreover, there exist $x \in \mathfrak{h}_\lambda$ and $y \in \mathfrak{h}_{-\lambda}$ such that $[x, y]$ acts semisimply on $\mathfrak{g}$. Since the eigenvalues of $\operatorname{ad} h_0$ are also eigenvalues for D we conclude $\lambda = \pm 1/2$ But we have seen in 2.13 that the eigenvalues $\pm 1/2$ of D are only attained in the solvable Lie aglebra $\mathfrak{s}_s$. Hence $[x, y]$ is nilpotent on $\mathfrak{g}$, a contradiction. This shows

LEMMA. $\mathfrak{h} \subset \mathfrak{g}^{[0]}$.

2.15. We consider the subalgebra $\mathfrak{m} = \mathfrak{g}^{[0]} + \mathfrak{g}^{[1]}$ of $\mathfrak{g}$.

LEMMA. a) $\mathfrak{m}$ *is a Kähler subalgebra of* $\mathfrak{g}$

b) $\mathfrak{m} = (\mathfrak{r} \cap \mathfrak{m}) + j(\mathfrak{r} \cap \mathfrak{m}) + \mathfrak{k}$.

Proof. From 2.10 we know that $\check{\mathfrak{s}}$ is an ideal of $\mathfrak{g}$. We have seen above $\mathfrak{g}^{[\lambda]} \subset \mathfrak{s}$ if $\lambda \neq 0$. Replacing e by $\mathfrak{s}_0$, $\mathfrak{n}$ by $\check{\mathfrak{s}}$, $\hat{\mathfrak{n}}$ by $\check{\mathfrak{s}}^{[1/2]}$ and defining $\mathfrak{q} = \{x \in \mathfrak{g};\ [x, s_0] \in \hat{\mathfrak{n}},\ [jx, s_0] \in \hat{\mathfrak{n}}\}$ it is straightforward to check that with the exception of 4.17 the results of 4.13 through 4.22, of [6] still hold. It is easy to verify that the proofs of 4.25 and 4.26 of [6] can be applied in our situation as well and we obtain the assertion.

2.16.

LEMMA. *If* $\mathfrak{g}^{[0]} + \mathfrak{g}^{[1]} \neq \mathfrak{g}$, *then then Radical Conjecture holds.*

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Proof. By our assumption and Lemma 2.15, we can apply the induction hypothesis to $\mathfrak{g}^{[0]} + \mathfrak{g}^{[1]}$. Therefore, there exists a solvable subalgebra $\mathfrak{s}^{[0]}$ so that $\mathfrak{s}^{[0]} + \mathfrak{k} = \mathfrak{g}^{[0]}$ holds. But then $\mathfrak{q} = \mathfrak{s}^{[0]} + \operatorname{rad}\mathfrak{g}$ is a solvable subalgebra of $\mathfrak{g}$ satisfying $\mathfrak{q} + \mathfrak{k} = \mathfrak{g}$. Since $\mathfrak{r} \subset \operatorname{nil}(\mathfrak{g})$ we have $\mathfrak{r} \subset \mathfrak{q}$. Hence, after an inessential change of j we see that $\mathfrak{r} + j\mathfrak{r}$ is a solvable Kähler subalgebra of $\mathfrak{g}$ and the assertion follows.

2.17. It is clear that we have only to consider the case $\mathfrak{g} = \mathfrak{g}^{[0]} + \mathfrak{g}^{[1]}$. Here we assume —in addition to our previous assumptions— that $\mathfrak{n}$ is chosen so that the rank of the maximal idempotent e of $\mathfrak{g}'$ is maximal.

LEMMA. *If $\mathfrak{g} = \mathfrak{g}^{[0]} + \mathfrak{g}^{[1]}$, then the Radical Conjecture holds.*

Proof. (1) We can assume that $\mathfrak{g}^{[1]} + j\mathfrak{g}^{[1]}$ is a solvable Kähler algebra with principal idempotent s_0.

(2) Let $\mathfrak{u} = \{x \in \mathfrak{g};\ [x, s_0] = 0,\ [jx, s_0] = 0\}$. Then $j\mathfrak{u} \subset \mathfrak{u}$ and as in [8; sect. 2.5] one proves $\mathfrak{g} = j\mathfrak{g}^{[1]} + \mathfrak{g}^{[1]} + \mathfrak{u}$ and $[js_0, \mathfrak{u}] \subset \mathfrak{u}$. Then $\mathfrak{u} = \mathfrak{u}^{[0]} + \mathfrak{u}^{[1]}$ and $0 = [j\mathfrak{u}^{[1]}, s_0] = \mathfrak{u}^{[1]}$. Hence $\mathfrak{g}^{[0]} = j\mathfrak{g}^{[1]} + \mathfrak{u}$.

(3) $\mathfrak{g}^{[1]} \subset r \cap \check{\mathfrak{s}}$: We know $s_0 = x + jy + a + jb + k$ where $x, y \in \mathfrak{r}^{[1]}$, $a, b \in \mathfrak{r}^{[0]}$ and $k \in \mathfrak{k}$. We split $b = jd + u$, $d \in \mathfrak{g}^{[1]}$, $u \in \mathfrak{u}$. Then $s_0 = x - d$ and $[b, s_0] = [jd + u, s_0] = d \in \mathfrak{r}^{[1]}$. Therefore $s_0 \in \mathfrak{r}^{[1]}$, whence $\mathfrak{g}^{[1]} = [j\mathfrak{g}^{[1]}, s_0] \subset \mathfrak{r}^{[1]} \cap \check{\mathfrak{s}}$.

(4) From (3) and Lemma 2.2, we know $\mathfrak{g}_1 + \mathfrak{g}^{[1]} \subset \mathfrak{r}$. We consider $A = \mathfrak{g}_1 + \mathfrak{g}^{[1]} + [\mathfrak{r}, \mathfrak{r}] = \mathfrak{g}_1 + \mathfrak{g}^{[1]} + [\mathfrak{r}^{[0]}, \mathfrak{r}^{[0]}] + \mathfrak{g}_1 + \mathfrak{g}^{[1]} + [\mathfrak{r}_0^{[0]}, \mathfrak{r}_0^{[0]}]$. If $A = \mathfrak{r}$, then $\mathfrak{g}_1 + [\mathfrak{r}_0^{[0]}, \mathfrak{r}_0^{[0]}] = \mathfrak{r}^{[0]}$ and $\mathfrak{r}_0^{[0]} = 0$ follows. But then $\mathfrak{r} = \mathfrak{g}_1 + \mathfrak{g}^{[1]}$ is, by (3), an abelian ideal of $\mathfrak{g}$ and the Radical Conjecture follows. If $A \neq \mathfrak{r}$, then we can find an ideal $\tilde{\mathfrak{n}}$ of $\mathfrak{g}$ satisfying $A \subset \tilde{\mathfrak{n}} \subsetneqq \mathfrak{r}$. But then the rank of a maximal idempotent $\tilde{e}$ associated with $\tilde{\mathfrak{n}}$ is greater than the rank of e if $\mathfrak{g}^{[1]} \neq 0$. This would be a contradiction to our choice of $\mathfrak{n}$. Therefore $\mathfrak{g}^{[1]} = 0$. This implies $\check{\mathfrak{s}}_0 = 0$ and $\check{\mathfrak{s}}_s$ is a modification of an abelian Kähler algebra. The rest of this proof is a simplification of a previous version. The present version is due to K. Nakajima. We consider $\operatorname{rad}_n(\check{\mathfrak{s}}) = \{x \in \operatorname{rad}(\check{\mathfrak{s}});\ \operatorname{ad} x | \check{\mathfrak{g}} \text{ is nilpotent}\}$

(5) $\mathfrak{r}_0 \subset \operatorname{rad}_n(\check{\mathfrak{s}}) + [j\mathfrak{g}_1, j\mathfrak{g}_1]$: Let $x \in \mathfrak{r}_0$. Then $x = s + z$ where $s \in \check{\mathfrak{s}}$ and $z \in [j\mathfrak{g}_1, j\mathfrak{g}_1]$ by Lemma 2.10. Note that $B = \mathfrak{r}_0 + [j\mathfrak{g}_1, j\mathfrak{g}_1]$ is a solvable subalgebra of $\mathfrak{g}_0$ and that x and z are contained in the ideal $B_n = \{x \in B;\ \operatorname{ad} x | \mathfrak{g} \text{ is nilpotent}\}$ of B. Hence $s \in B_n$. Let $\mathfrak{h}$ be a maximal semisimple subalgebra of $\check{\mathfrak{s}}$ and decompose $s = s' + s''$, where $s \in \operatorname{rad}(\check{\mathfrak{s}})$,

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$s'' \in \mathfrak{h}$. Then $\operatorname{ad} s'' | \mathfrak{h}$ is nilpotent since $s \in B_n$. But since $\mathfrak{g}^{[1]} = 0$ we know that $\mathfrak{s}$ corresponds to a flat homogeneous Kähler manifold. Therefore $\mathfrak{h}$ is a compact semisimple Lie algebra. Hence $\operatorname{ad} s'' | \mathfrak{h} = 0$, whence $s'' = 0$, $s = s'$ and (5) follows.

(6) $\operatorname{rad}_n(\check{\mathfrak{s}})$ is an abelian ideal of $\mathfrak{g}$: Since $\check{\mathfrak{s}}$ is an ideal of $\mathfrak{g}$ we know that $\operatorname{rad}(\check{\mathfrak{s}})$ is ideal of $\mathfrak{g}$ and $[\mathfrak{g}, \operatorname{rad}(\check{\mathfrak{s}})] \subset \operatorname{nil}_0(\check{\mathfrak{s}}) \subset \operatorname{rad}_n(\check{\mathfrak{s}})$ follows by [2; § 5, Proposition 6], where $\operatorname{nil}_0(\check{\mathfrak{s}})$ denotes the maximal nilpotent ideal of $\check{\mathfrak{s}}$. Therefore $\operatorname{rad}_n(\check{\mathfrak{s}})$ is an ideal of $\mathfrak{g}$. Moreover, since $\check{\mathfrak{s}}$ corresponds to a flat homogeneous Kähler manifold, $\operatorname{rad}_n(\check{\mathfrak{s}})$ is abelian.

From (6) we obtain

(7) $\operatorname{rad}_n(\check{\mathfrak{s}}) + \mathfrak{g}_1$ is an abelian ideal of $\mathfrak{g}$. To prove that the Radical Conjecture holds for $\mathfrak{g}$ it suffices now to note that $\mathfrak{g} = (\operatorname{rad}_n(\check{\mathfrak{s}}) + \mathfrak{g}_1) + j(\operatorname{rad}_n(\check{\mathfrak{s}}) + \mathfrak{g}_1) + \mathfrak{k}$ holds.

This finishes the proof of "Case 2".

§ 3. Case 3. $\mathfrak{g} = \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1$

3.1. We use the notation of [6] as before (see 2.1). Since -1 is not a weight for $\operatorname{ad} je$, we have from [6; Lemma 4.19]

$$\rho(e, \mathfrak{q}) = 0\,. \tag{3.1}$$

Then Lemma 4.21 of [6] simplifies to

$$\rho(e^{t \operatorname{ad} je}u, e^{t \operatorname{ad} je}v) = e^t\rho(je, [u, v]_1') + \text{const.} \tag{3.2}$$

for all $u, v \in \mathfrak{g}$, $t \in \boldsymbol{R}$. In particular, we have

$$\rho(\mathfrak{g}_\lambda, \mathfrak{g}_\mu) = 0 \qquad \text{if } \lambda + \mu \neq 0, 1\,. \tag{3.3}$$

We also recall from [6; Lemma 4.26] that the Radical Conjecture holds for the Kähler subalgebra $\mathfrak{g}_0 + \mathfrak{g}_1$ of $\mathfrak{g}$. Note $\mathfrak{k} \subset \mathfrak{g}_0$. Moreover, we have $\mathfrak{g}_0 = \mathfrak{w}_0 + \mathfrak{t}_0 + j\mathfrak{g}_1 + \mathfrak{k}$ where $\mathfrak{w}_0 + \mathfrak{t}_0$ is a modification of a split solvable Kähler algebra and $\mathfrak{w}_0 + \mathfrak{t}_0 + j\mathfrak{g}_1 + \mathfrak{g}$ is a solvable Kähler subalgebra of $\mathfrak{g}_0 + \mathfrak{g}_1$.

As in Lemma 2.2 one proves

$$\mathfrak{g}_1 = \mathfrak{n}_1 = \mathfrak{r}_1\,. \tag{3.4}$$

3.2. We consider the subspace $\mathfrak{w} = \mathfrak{g}_{1/2} + \mathfrak{g}_{-1/2} + \mathfrak{k}$ of $\mathfrak{g}$. Then $j\mathfrak{w} \subset \mathfrak{w}$ since $j\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda + \mathfrak{g}'$ and $\rho(\mathfrak{w}, \mathfrak{g}_0 + \mathfrak{g}_1) = 0$ by (3.3). Therefore, after an inessential change of j, we can even assume $j(\mathfrak{g}_{1/2} + \mathfrak{g}_{-1/2}) \subset \mathfrak{g}_{1/2} + \mathfrak{g}_{-1/2}$.

3.3. Let $\mathfrak{c}$ be the vector space of real parts of $\operatorname{ad} jc$, $c \in \mathfrak{g}_1 + \mathfrak{t}_0$, $[jc, c] = c$. Let $\mathfrak{c} \subset \mathfrak{b} \subset \operatorname{End} \mathfrak{g}$ be a maximal abelian subalgebra of the algebraic hull $\overline{\operatorname{ad} \mathfrak{g}}$ which consists of semisimple endomorphisms. Since $\operatorname{Re}(\operatorname{ad} je) \in \mathfrak{c}$ it follows $\mathfrak{b} \subset \overline{\operatorname{ad} \mathfrak{g}_0}$. From the appendix it follows that there exists a maximal semisimple subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and an abelian subalgebra $\mathfrak{A} \subset \operatorname{rad} \overline{\operatorname{ad} g}$ such that $\mathfrak{b} \subset \operatorname{ad} \mathfrak{h} + \mathfrak{A}$ and $\mathfrak{A}\mathfrak{h} = 0$. The maximality of $\mathfrak{b}$ implies $\mathfrak{b} = (\mathfrak{b} \cap \operatorname{ad} \mathfrak{h}) + \mathfrak{A}$, where $\mathfrak{b} \cap \operatorname{ad} \mathfrak{h}$ is a Cartan algebra of $\operatorname{ad} \mathfrak{h} \cong \mathfrak{h}$. In particular $R = \operatorname{Re} \operatorname{ad} je = \operatorname{ad} h_0 + R_r$ where $R_r \mathfrak{h} = 0$ and $h_0 \in \mathfrak{h}$. Hence the eigenvalues of $\operatorname{ad} h_0$ in $\mathfrak{h}$ are also eigenvalues of R in $\mathfrak{g}$. Moreover, if $\lambda \neq 0$ is an eigenvalue of $\operatorname{ad} h_0$ in $\mathfrak{h}$, then also $-\lambda$ is an eigenvalue of $\operatorname{ad} h_0$ in $\mathfrak{h}$. Since R has only the eigenvalues $0, \pm 1/2, 1$, this implies $\lambda = \pm 1/2$. If $\operatorname{ad} h_0$ has only the eigenvalue 0 in $\mathfrak{h}$, then the Radical Conjecture follows by the argument of [6; Lemma 4.32].

Hence from now on we assume that $\operatorname{ad} h_0$ has a nonzero eigenvalue in $\mathfrak{h}$.

Then $\mathfrak{h} = \mathfrak{h}_{-1/2} + \mathfrak{h}_0 + \mathfrak{h}_{1/2}$ and $\mathfrak{h}_\lambda \neq 0$ for all λ. (We will show in the rest of this paper that this assumption leads to a contradiction). Moreover, we can assume that $\mathfrak{w}_0 + \mathfrak{t}_0 + j\mathfrak{g}_1$ contains a maximal split solvable subalgebra of $\mathfrak{h}_0$ and that it also contains the Cartan algebra of $\mathfrak{h}$ which corresponds to $\mathfrak{b} \cap \operatorname{ad} \mathfrak{h}$.

Let $(\cdot, \cdot)$ denote the product in the unmodified algebra underlying $\mathfrak{g}_0 + \mathfrak{g}_1$ and denote by $\widetilde{\operatorname{ad}}$ its adjoint representation.

Then $\operatorname{Re}(\operatorname{ad} jc) = \widetilde{\operatorname{ad}} jc$ in $\mathfrak{g}_0 + \mathfrak{g}_1$ for all minimal idempotents c of $\mathfrak{g}_1 + \mathfrak{t}_0$.

LEMMA. *Let $x \in \mathfrak{g}_0$ such that $\operatorname{ad} x \in \mathfrak{b}$ holds. Then there exists a linear combination y of idempotents of $\mathfrak{g}_1 + \mathfrak{t}_0$ and $u \in \mathfrak{w}_0$ such that $x = jy + u$ and*

a) $[x, \mathfrak{w}_0] \subset \mathfrak{w}_0$,
b) $[x, u] = 0$, $[x, ju] = 0$, $[ju, u] = 0$,
c) $(jc, j] = 0$ *for all idempotents* $c \in \mathfrak{g}_1 + \mathfrak{t}_0$,
d) $[\operatorname{ad} x, \operatorname{Re}(\operatorname{ad} jc)] = 0$ *for all idempotents* $c \in \mathfrak{g}_1 + \mathfrak{t}_0$,

Proof. Let $x \in \mathfrak{g}_0$ and $\operatorname{ad} x \in \mathfrak{b}$, then $\operatorname{ad} x$ lies in the span of all $\operatorname{Re}(\operatorname{ad} jc)$, c a minimal idempotent of $\mathfrak{g}_1 + \mathfrak{t}_0$. This implies $x = jy + u$ where y is a linear combination of minimal idempotents and $u \in \mathfrak{g}_0 + \mathfrak{g}_1$ such that $(jc, u) = 0$ for all idempotents c. Moreover, we can assume that u is perpendicular to all jc. Then $u \in \mathfrak{w}_0$. Since the modification

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derivations $D(v)$ of $\mathfrak{g}_0+\mathfrak{g}_1$ annihilate all idempotents we already get a). We also note that c) and d) are clear as well. In particular $(x, u)=0$. To see that also $[x, u]=0$ holds we note $D(x)=0$, since $\operatorname{ad} x \in \mathfrak{b}$, whence $[x, u]=(x, u)-D(u)x=-D(u)u$. Since $\operatorname{ad} x$ is selfadjoint and $D(u)$ skew adjoint relative to the inner product $\rho(a, jb)$ on $\mathfrak{w}_0$ we obtain $D(u)u=0$ and $[x, u]=0$. Let $\mathfrak{w}_0=\hat{\mathfrak{w}}_0+\hat{\mathfrak{w}}_1$ as in [5; 3.3]. Then $\operatorname{ad} jc$ leaves $\hat{\mathfrak{w}}_0$ and $\hat{\mathfrak{w}}_1$ invariant. Hence $[x, u_1]=0$ and $(jc, u_i)=0$ where $u=u_0+u_1$, $u_i \in \hat{\mathfrak{w}}_i$. But then $(jc, ju_i)=0$ and $[x, ju_0]=(x, ju_0)=0$ follows. Finally, $[x, ju_1]=(x, ju_1)-D(ju_1, x)=-D(ju_1)u_0$. But $[x, \hat{\mathfrak{w}}] \subset \hat{\mathfrak{w}}_1$ implies $[x, w_1]=(x, w_1)-D(w_1)x=(jy, w_1)+(u, w_1)-D(w_1)u \in \hat{\mathfrak{w}}_1$ for all $w_1 \in \hat{\mathfrak{w}}_1$. Since $(jy, w_1) \in \hat{\mathfrak{w}}_1$, $(u, w_1)=0$ and $D(w_1)u \in \hat{\mathfrak{w}}_0$ we obtain $D(w_1)u=0$. From this we derive $[x, ju_1]=0$ and $[ju, u]=0$.

Remark. In what follows we will use frequently the representation theory of $\mathrm{sl}(2, \boldsymbol{R})$. We will only consider such copies of $\mathrm{sl}(2, \boldsymbol{R})$ which are of type $\mathrm{sl}(2, \boldsymbol{R}) \cong \boldsymbol{R}f_{-1/2}+\boldsymbol{R}f_0+\boldsymbol{R}f_{1/2}$, $\operatorname{ad} f_0 \in \mathfrak{b} \cap \operatorname{ad} \mathfrak{h}$ and $f_0 \in \mathfrak{w}_0+\mathfrak{t}_0+j\mathfrak{g}$.

It is clear that we can apply the above lemma to f_0.

We would like to point out that we can actually find $f_\lambda \in \mathfrak{g}_\lambda$, $\lambda=\pm 1/2$, 0, so that in addition to the above properties f_λ is a simultaneous eigenvector for all $b \in \mathfrak{b}$.

We will make it explicitly clear where we use f_λ's with this additional property. The other properties will always tacitely be assumed.

3.4. In this section we consider the action of $sl(2, \boldsymbol{R}) \cong \boldsymbol{R}f_{-1/2}+\boldsymbol{R}f_0+\boldsymbol{R}f_{1/2}$ on $\mathfrak{g}$. We know that $\operatorname{ad} f_0$ has only integral eigenvalues and in an irreducible representation all integers $m, m-2, \cdots, -m$ occur. Moreover, starting from an appropriately chosen eigenvector x_1 in $\mathfrak{g}_1$ we get a basis of an irreducible representation of $sl(2, \boldsymbol{R})$ in $\mathfrak{g}$ by applying $\operatorname{ad} f_{-1/2}$ to x_1. The eigenvalues of $\operatorname{ad} f_0$ in $\mathfrak{g}_1$ are therefore all non-negative or all non-positive (depending on the sign in $[f_0, f_{-1/2}]=\pm 2f_{-1/2}$) and only the integers 0, 1, 2, 3 can occur (for simplicity we assume that only non-negative integers occur in $\mathfrak{g}_1$; the other case follows by the same arguments). Thus we get the following chart indicating the chains of eigenvalues that can possibly occur in some irreducible representation of $sl(2, \boldsymbol{R})$ in $\mathfrak{g}$. Note that the vector space corresponding to the various integers in the same row all have the same dimension.

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$\mathfrak{g}_{-1/2}$	$\mathfrak{g}_0$	$\mathfrak{g}_{1/2}$	$\mathfrak{g}_1$
-3	-1	1	3
	-2	0	2
		-1	1
			0
-2	0	2	
	-1	1	
		0	
-1	1		
	0		
0			

3.5. We write $f_0 = jd + t_0 + w_0$ where $d \in \mathfrak{g}_1$, $t_0 \in \mathfrak{t}_0$, $w_0 \in \mathfrak{w}_0$. We know that d is a linear combination of idempotents, $d = 3d_3 + 2d_2 + d_1$, $[jd_\lambda, d_\lambda] = d_\lambda$, $\lambda = 1, 2, 3$. We set $d_0 = e - d_1 - d_2 - d_3$. Here some of the d_λ may be 0. In what follows we use the algebra $\mathscr{A}$ on $\mathfrak{g}_1$ associated with $e \in \mathfrak{g}_1$ and the tube domain $\mathfrak{g}_1 + j\mathfrak{g}_1$ in [4].

LEMMA. *$\mathscr{A} = \mathscr{A}_1(d_3 + d_1) \oplus \mathscr{A}_1(d_2 + d_0)$ as product of algebras.*

Proof. (1) $d_3 \neq 0$, $d_2 \neq 0$ implies $\mathscr{A}_{1/2}(d_3, d_2) = \{x \in \mathscr{A}; (jd_3, x) = (1/2)x = (jd_2, x)\} = 0$ since $[f_0, x] = (f_0, x) = 3(jd_3. x) + 2(jd_2, x) = (5/2)x$ and $5/2$ is not an eigenvalue for $\operatorname{ad} f_0$. Similarly one proves

(2) $\mathscr{A}_{1/2}(d_2, d_1) = 0$, $\mathscr{A}_{1/2}(d_1, d_0) = 0$, $\mathscr{A}_{1/2}(d_3, d_0) = 0$.

From (1) and (2) we get the claim.

3.6. We will need some information on the eigenvalues of $\operatorname{ad} jd_\iota$.

LEMMA. *Let $c \in \mathfrak{g}_1$ satisfy $[jc, c] = c$ and $[\mathfrak{k}, c] = 0$. Then $\operatorname{ad} jc$ has only the weights $0, \pm 1/2, 1$ or the weights $0, +1$ in $\mathfrak{g}$.*

Proof. Let $x \in \mathfrak{g}$. Then $x = x' + q$ where $x' \in \mathfrak{g}'$ and $q \in \mathfrak{q}$. Since $\mathfrak{g}_1 \subset \mathfrak{g}'$ we can assume $q = q_0 + q_{1/2} + q_{-1/2}$ where $q_\lambda \in \mathfrak{q} \cap \mathfrak{g}_\lambda$. We also write $x' = \sum x'_i$.

$$\rho(jc, e^{t \operatorname{ad} jc}x) = \rho(jc, e^{t \operatorname{ad} jc}x'_1)\,. \tag{1}$$

Proof. From [6; Corollary 4.22] we know $\rho(jc, \mathfrak{g}_{1/2} + \mathfrak{g}_{-1/2}) = 0$. We also have $\rho(jc, \mathfrak{g}_0) = 0$. Hence $\rho(jc, Wx'_1 + Wx'_{1/2} + Wx'_{-1/2} + Wx'_0) = \rho(jc, Wx'_1)$ where $W = e^{t \operatorname{ad} jc}$.

Decomposing x'_1 further into weight vectors of $\operatorname{ad} jc$ we get $x'_1 = y^{[0]}$

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$y^{[1/2]} + y^{[1]}$ and

$$\rho(jc, e^{t\,\mathrm{ad}\,jc}x) = e^t\rho(jc, y^{[1]}). \tag{2}$$

From this we derive, using [7; chap. III, Lemma 9]

$$\rho(e^{t\,\mathrm{ad}\,jc}u, e^{t\,\mathrm{ad}\,jc}v) = ae^t + b. \tag{3}$$

In particular, we obtain from this for the weight spaces $\mathfrak{g}^{[\lambda]}$ of $L = \mathrm{Re}(\mathrm{ad}\,jc)$ in $\mathfrak{g}$:

$$\rho(\mathfrak{g}^{[\lambda]}, \mathfrak{g}^{[\mu]}) = 0 \qquad \text{if } \lambda + \mu \neq 0, 1. \tag{4}$$

Next we prove (that after an inessential change of j)

$$j\mathfrak{g}^{[\lambda]} \subset \mathfrak{g}^{[\lambda]} \qquad \text{if } \lambda \neq 0,\ \pm 1/2,\ \pm 1,\ 3/2. \tag{5}$$

Proof. From the integrability condition and $\mathfrak{g}_1 = \mathfrak{n}_1$ we get as usual $j\mathfrak{g}^{[\lambda]} \subset \mathfrak{g}^{[\lambda]} + \mathfrak{g}'$. Hence for $x \in \mathfrak{g}^{[\lambda]}$ we have $jx = y + z$ where $y \in \mathfrak{g}^{[\lambda]}$ and $z \in \mathfrak{g}'$. We note that $\lambda + \{0, \pm 1/2, 1\} \in \{0, 1\}$ implies $\lambda \in \{0, \pm 1, \pm 1/2, 3/2\}$. But we have excluded these values for λ, whence $\rho(\mathfrak{g}^{[\lambda]}, \mathfrak{g}') = 0$. In particular $0 = \rho(x, z) = \rho(jx, jz) = \rho(y + z, jz) = \rho(z, jz)$. Hence $z \in \mathfrak{k}$. But then $j\mathfrak{g}^{[\lambda]} \subset \mathfrak{g}^{[\lambda]} + \mathfrak{k}$ and the assertion follows.

Since $2\lambda \neq 0, 1$ if $\lambda \neq 0,\ \pm 1/2,\ \pm 1,\ 3/2$ and $k \subset \mathfrak{g}^{[0]}$ we obtain from (5):

$$\mathfrak{g}^{[\lambda]} = 0 \qquad \text{if } \lambda \neq 0,\ \pm 1/2,\ \pm 1,\ 3/2. \tag{6}$$

Using (4) we prove as in [6; Lemma 4.25]

$$j\mathfrak{g}^{[n]} \subset \mathfrak{g}^{[n]} + \mathfrak{g}'^{[1]} + \mathfrak{g}'^{[0]} \qquad \text{for all } n \in \boldsymbol{Z}. \tag{7}$$

Now we can repeat the proof of [6; Lemma 4.30] and obtain

$$\mathfrak{g}^{[3/2]} = 0. \tag{8}$$

Finally, the argument of [6; Lemma 4.26] is applicable in our situation and yields

(9) $\hat{\mathfrak{g}} = \mathfrak{g}^{[-1]} + \mathfrak{g}^{[0]} + \mathfrak{g}^{[1]}$ is a j-invariant subalgebra and
$\hat{\mathfrak{g}} = (\mathfrak{r} \cap \mathfrak{g}) + j(\mathfrak{r} \cap \mathfrak{g}) + \mathfrak{k}$.

We consider the two possibilities $\hat{\mathfrak{g}} = \mathfrak{g}$ or $\hat{\mathfrak{g}} \neq \mathfrak{g}$. In the latter case we can apply the induction hypothesis and obtain $\mathfrak{g}^{[-1]} = 0$ (and from this the assertion). If $\hat{\mathfrak{g}} = \mathfrak{g}$, then we have again two subcases. The first, $\mathfrak{g}^{[-1]} = 0$, is exactly what we want. The second case, $\mathfrak{g}^{[-1]} \neq 0$, allows us to argue as in [6; Lemma 4.32] so that the Radical Conjecture holds in this case. But then jc does not have the eigenvalue -1 in $\mathfrak{g}$, so that this case actually does not occur. This proves the claim.

3.7. By the result of the last section we can assume that $\operatorname{ad} j(d_3 + d_1)$ and $\operatorname{ad} j(d_2 + d_0)$ have only the (real) eigenvalues 0, $\pm 1/2$, 1 in $\mathfrak{g}$. Moreover, these weights occur in the eigenspaces of $\operatorname{ad} f_0$ in the spaces $\mathfrak{g}_\lambda$. In the proof of the last section we have also seen that for $\mathfrak{g}^{[0]} + \mathfrak{g}^{[1]}$ the Radical Conjecture holds, where $\mathfrak{g}^{[1]}$ is defined for jc as in 3.6. We can assume $\mathfrak{g}^{[1]} + \mathfrak{g}^{[-1/2]} \neq 0$.

The following argument is a simplification of a previous version of the proof. We use ideas of K. Nikajima.

First, it is easy to see that $j\mathfrak{g}_{1/2}$ is invariant under je. Hence $j\mathfrak{g}_{1/2} = (j\mathfrak{g}_{1/2}) \cap \mathfrak{g}_{1/2} + (j\mathfrak{g}_{1/2}) \cap \mathfrak{g}_{-1/2}$ and $\mathfrak{g}_{1/2} = \mathfrak{u}_{1/2} + \mathfrak{w}_{1/2}$, where $\mathfrak{u}_{1/2} = \{x \in \mathfrak{g}_{1/2}, jx \in \mathfrak{g}_{-1/2}\}$ and $\mathfrak{w}_{1/2} = \{x \in \mathfrak{g}_{1/2}, jx \in \mathfrak{g}_{1/2}\}$.

A direct computation shows that $\mathfrak{w}_{1/2}$ is invariant under $j\mathfrak{g}_1$. Therefore $j\mathfrak{g}_1 + \mathfrak{w}_{1/2} + \mathfrak{g}_1$ is a Kähler algebra of domain type. In particular for $c_1 = d_3 + d_1$ and $c_2 = d_2 + d_0$ we know that jc_1, jc_2 have only the eigenvalues 0 or 1/2 on $\mathfrak{w}_{1/2}$ and the sum of their eigenvalues adds up to 1/2.

Next we consider $\mathfrak{u}_{1/2}$. We know $j\mathfrak{u}_{1/2} \subset \mathfrak{g}_{-1/2}$ and $j\mathfrak{g}_{-1/2} \subset \mathfrak{g}_{-1/2} + \mathfrak{g}'$. From this it follows $\mathfrak{u}_{1/2} \subset \mathfrak{g}'_{1/2}$, whence $\mathfrak{u}_{1/2} = [e, j\mathfrak{u}_{1/2}] \subset \mathfrak{u}_{1/2}$.

Since $\mathfrak{u}_{1/2} \subset \mathfrak{n}_{1/2}$ we know that every $h_{1/2} \in \mathfrak{h}_{1/2} = \mathfrak{g}_{1/2} \cap \mathfrak{h}$ has a non-zero component in $\mathfrak{w}_{1/2}$. This implies that $\mathfrak{h} \cap \mathfrak{g}_{1/2}^{[1/2]}(c_i) \not\equiv 0 \bmod \mathfrak{n}$ for c_1 or for c_2. If $\mathfrak{h}_{1/2} \cap \mathfrak{g}^{[1/2]}(c_i) \not\equiv 0 \bmod \mathfrak{n}$ only for one of the idempotents c_1, c_2, then denote by c the other idempotent. If this space is nontrivial for c_1 and for c_2, then choose $c = c_1$.

We consider the Kähler subalgebra $\tilde{\mathfrak{g}} = \mathfrak{g}^{[0]}(c) + \mathfrak{g}^{[1]}(c)$; as mentioned above we know that for this algebra the Radical Conjecture holds. Since $e - c \in \tilde{\mathfrak{g}}$ we can form the weight space decomposition of $\tilde{\mathfrak{g}}$ relative to $j(e-c)$. From our construction it follows that there exists a semisimple subalgebra $\tilde{\mathfrak{h}}$ of $\tilde{\mathfrak{g}}$ such that $\mathfrak{k} \subset \tilde{\mathfrak{g}}^{[0]}(e-c)$ and $\tilde{\mathfrak{h}} \cap \tilde{\mathfrak{g}}^{[1/2]}(e-c) \not\equiv 0 \bmod \operatorname{rad} \tilde{\mathfrak{g}}$ holds. But by the Radical Conjecture this is not possible. Hence we have shown

$$d_3 + d_1 = 0 \qquad \text{or } d_2 + d_0 = 0.$$

3.8. We refine the description of $f_0 = jd + t_0 + w_0$. We know that t_0 is a linear combination of elements of type jq_i, where q_i is an idempotent of $\mathfrak{t}_0$, $t_0 = \sum a_i jq_i$. Since $[f_0, q_i]$ $a_i q_i$, we have $a_i \in \{0, \pm 1, -2\}$ by 3.4. If $a_i = -2$, then there exists $x \in \mathfrak{g}_1$ such that $q_i = (\operatorname{ad} f_{-1/2})^2 x$. Then $q_i \in \mathfrak{n}$.

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Since $e \in \mathfrak{g}_1$ is the maximal idempotent in $\mathfrak{n}$, this case cannot occur. Hence

LEMMA. *$t_0 = jq_1 - jq_2$ where q_i are idempotents in $\mathfrak{t}_0$.*

3.9. In this section we want to prove $d_3 = 0$. Otherwise $(\mathrm{ad} f_{-1/2})^3 d_3 \in \mathfrak{n}_{-1/2}$ is an eigenvector of f_0 for the eigenvalue -3. We note that as in 2.9 we get $\mathfrak{g}_0 = \mathfrak{s} + j\mathfrak{g}_1$ where $\mathfrak{s} = \{x \in \mathfrak{g}_0; [x, e] = 0\}$ is j-invariant. Since $q_1, q_2, w_0 \in \mathfrak{s}$ it is easy to see that the Kähler algebra generated by q_1, q_2, w_0 acts symplectic on the abelian Kähler algebra $\mathfrak{v} = \mathfrak{n}_{-1/2} + j\mathfrak{n}_{-1/2}$. Note $j\mathfrak{n}_{-1/2} = [e, \mathfrak{n}_{-1/2}]$. Therefore, jq_1 and jq_2 have only the eigenvalues 0, $\pm 1/2$ on $\mathfrak{v}$ and w_0 has no real eigenvalues on $\mathfrak{v}$. Next we consider the elements jd_3 and $j(d_3 + d_1)$. We know that they leave the flat part of $\mathfrak{g}'$ invariant and have only the eigenvalues 0, $\pm 1/2$ there. Hence, $f_0 = 2jd_3 + j(d_3 + d_1) + jq_1 - jq_2 + w_0$ cannot have the eigenvalue -3 on $\mathfrak{n}_{-1/2}$.

3.10. In this section we show $d_2 = 0$. Suppose not, then $(\mathrm{ad} f_{-1/2})^2 d_2 \in \mathfrak{n}_0$ is an eigenvector of f_0 for the eigenvalue -2. Let $x \in \mathfrak{n}_0$ and write $x = a_0 + t_0 + jx_1$ where a_0 is in the flat part $\mathfrak{w}_0$ of $\tilde{\mathfrak{g}} = \mathfrak{g}_0 + \mathfrak{g}_1$ and t_0 is in the domain part $\mathfrak{t}_0$ of $\tilde{\mathfrak{g}}$. Note $\mathfrak{w}_0 + \mathfrak{t}_0 \subset \mathfrak{s}$ (see 3.9). Then $[f_0, x] = -2x$ implies $D(x) = 0$ and $(jq_1 - jq_2, t_0) = -2t_0$, $(j(2d_2), jx_1) = -2x_1$ follows, where $(\cdot, \cdot)$ denotes the product in the underlying unmodified algebra. But $j(2d_2)$ has only the eigenvalues ± 1, 0 in $j\mathfrak{g}_1$, whence $x_1 = 0$. A similar argument shows that $jq_1 - jq_2$ does not have the eigenvalue -2 in $\mathfrak{t}_0$. Hence x is contained in the flat part $\mathfrak{w}_0 \subset \mathfrak{g}_0$ of $\tilde{\mathfrak{g}}$. But there the idempotents jd_2, $j(d_2 + q_1)$ and jq_2 have only the eigenvalues 0, $\pm 1/2$. Hence -2 cannot be obtained.

3.11. By the last sections we have only to consider the cases $d_1 = e$ and $d_0 = e$. In this section we consider the case $d_0 = e$ and $f_0 = w_0 \in \mathfrak{w}_0$. This is impossible as follows from

LEMMA. *Let $w_0 \in \mathfrak{w}_0$ such that* $\mathrm{ad}\,\mathfrak{w}_0$ *is semisimple and has only real eigenvalues. Then* $\mathrm{ad}\, w_0 = 0$.

Proof. From [7; Chap. III, Lemma 9] we know

$$(1) \qquad \frac{d}{dt} \rho(e^{t\,\mathrm{ad}\, w_0} u, e^{t\,\mathrm{ad}\, w_0} v) = \rho(w_0, e^{t\,\mathrm{ad}\, w_0}[u, v]) .$$

Since $w_0 \in \mathfrak{g}_0$, only the component of $[u, v]$ in $\mathfrak{g}_0 + \mathfrak{g}_1$ wwill contribute to the right hand side by [6; Corollary 4.22]. Using the induction hypothesis

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shows that only the component in $\mathfrak{g}_0$ can contribute. But we know $[w_0, a_0 + jx_1] = (w_0, jx_1) - D(a_0 + jx_1)w_0$, whence $(\operatorname{ad} w_0)^2 | \mathfrak{g}_0 = 0$. Since $\operatorname{ad} w_0$ is semisimple we obtain

$$(2) \qquad \operatorname{ad} w_0 | \mathfrak{g}_0 = 0\,.$$

Therefore the right hand side of (1) is just $\rho(w_0, [u, v])$. An integration yields

$$(3) \qquad \rho(e^{t\operatorname{ad} w_0}u, e^{t\operatorname{ad} w_0}v) = t\rho(w_0, [u, v]) + \rho(u, v)\,.$$

By assumption $\operatorname{ad} w_0$ is semisimple with only real eigenvalues. Let $u = u_\lambda$, $v = v_\mu$ be eigenvectors for $\operatorname{ad} w_0$. Then (3) yields $e^{t(\lambda+\mu)}\rho(u, v) = t\rho(w_0, [u, v]) + \rho(u, v)$. This implies

$$(4) \qquad \rho(w_0, [u, v]) = 0 \qquad \text{for all } u, v \in \mathfrak{g}\,.$$

This shows that $\operatorname{ad} w_0$ is symplectic on $\mathfrak{g}$. Moreover, $[jw_0, w_0] = 0$ by (2). Now we apply the proof of [7; chap. II, Lemma 3]. Let $A(x)$ denote the j-linear part of $\operatorname{ad} x$, $x \in \boldsymbol{R}w_0 + \boldsymbol{R}jw_0$, and $B(x)$ the j-antilinear part. As in loc. cit. one shows

$$(5) \qquad B(jw_0) = jB(w_0) \quad \text{and} \quad 2B(w_0)^2 = [jA(jw_0), A(w_0)]\,.$$

This yields trace $B(w_0)^2 = 0$. Finally, since $\operatorname{ad} w_0$ is symplectic, it is easy to see that $B(w_0)$ is selfadjoint and $A(w_0)$ is skewadjoint relative to $\langle u, v\rangle = \rho(ju, v)$ modulo $\mathfrak{k}$. Altogether this implies $B(w_0)\mathfrak{g} \subset \mathfrak{k}$. Thus $\operatorname{ad} w_0 = A(w_0)$ is skewadjoint on $\mathfrak{g}/\mathfrak{k}$, whence $\operatorname{ad} w_0\mathfrak{g} \subset \mathfrak{k}$, since the eigenvalues of $\operatorname{ad} w_0$ are assumed to be real. From this the lemma follows

3.12. In this section we exclude the case $e = d_1$ and $f_0 = je + w_0$.

LEMMA. *The case $f_0 = je + w_0$ does not occur.*

Proof. We note first that $\operatorname{ad} w_0 | \mathfrak{g}_0$ is skewadjoint since $[w_0, je] = 0$ and $\operatorname{ad} je | \mathfrak{g}_0$ is semisimple as well as $\operatorname{ad} f_0 = \operatorname{ad} w_0 + \operatorname{ad} je$.

Let $\mathfrak{u}^{[\lambda]}$ denote eigenspace for the eigenvalue λ of the real part of $\operatorname{ad} w_0$ on $\mathfrak{u} = \mathfrak{g}_{1/2} + \mathfrak{g}_{-1/2}$.

We start again from the equation

$$(1) \qquad \frac{d}{dt}\rho(e^{t\operatorname{ad} w_0}u, e^{t\operatorname{ad} w_0}v) = \rho(w_0, e^{t\operatorname{ad} w_0}[u, v])\,.$$

As before we only have to consider the component of $[u, v]$ in $\mathfrak{g}$. But

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from above we have $\operatorname{ad} w_0 | \mathfrak{g}_0 = D(w_0) | \mathfrak{g}_0$ where $D(w_0)$ is the modification derivation of w_0 in $\mathfrak{g}_0 + \mathfrak{g}_1$. This implies that the right hand side of (1) is $\rho(w_0, [u, v])$. Therefore an integration yields

$$\rho(e^{t \operatorname{ad} w_0} u, e^{t \operatorname{ad} w_0} v) = ta + b . \tag{2}$$

For $u \in \mathfrak{u}^{[\lambda]}$, $v \in \mathfrak{u}^{[\mu]}$, $\lambda + \mu \neq 0$ the left side grows here like $e^{t(\lambda+\mu)}$ and the right side is polynomial. This is a contradiction. Hence we obtain

$$\rho(\mathfrak{u}^{[\lambda]}, \mathfrak{u}^{[\mu]}) = 0 \qquad \text{if } \lambda + u \neq 0 . \tag{3}$$

From above we know that $\operatorname{ad} f_0$ attains only the eigenvalue 0 in $\mathfrak{g}_0$. Hence, from 3.4 we derive that $\operatorname{ad} f_0$ can only have the eigenvalues -1, $2, 0$ in $\mathfrak{g}_{1/2}$ and $-2, 0$ in $\mathfrak{g}_{-1/2}$. Since $\operatorname{ad} je$ has the weight $1/2$ and $-1/2$ there respectively, the real part of $\operatorname{ad} w_0$ has the eigenvalues $-3/2$, $3/2$, $-1/2$ in $\mathfrak{g}_{1/2}$ and $-3/2$, $1/2$ in $\mathfrak{g}_{-1/2}$ (in the same order as above). Since ρ is nondegenerate on $\mathfrak{u} = \mathfrak{g}_{1/2} + \mathfrak{g}_{-1/2}$ we derive from (3) that the weight spaces with opposite signs have the same dimension. Therefore $\dim \mathfrak{g}_{1/2}^{[3/2]} = \dim \mathfrak{g}_{1/2}^{[-3/2]} + \dim \mathfrak{g}_{-1/2}^{[-3/2]}$ and $\dim \mathfrak{g}_{1/2}^{(2)} = \dim \mathfrak{g}_{1/2}^{[3/2]} = \dim \mathfrak{g}_{-1/2}^{(-2)} = \dim \mathfrak{g}_{-1/2}^{[-3/2]}$ where we have used 3.4 and the notation $\mathfrak{g}_*^{(\lambda)}$ for the eigenspaces of $\operatorname{ad} f_0$ in $\mathfrak{g}_*$. But then, again by 3.4, we have $0 = \dim \mathfrak{g}_{1/2}^{[-3/2]} = \dim \mathfrak{g}_{1/2}^{(-1)} = \dim \mathfrak{g}_1^{(1)} = \dim \mathfrak{g}_1$. This is a contradiction, proving the lemma.

3.13. In this section we start to look at $\mathfrak{g}_0$ more closely.

Using the induction hypothesis we see that $\mathfrak{w}_0 + \mathfrak{k}$ and $j\mathfrak{g}_1 + \mathfrak{t}_0 + \mathfrak{k}$ are subalgebras of $\mathfrak{g}_0$, $\mathfrak{w}_0 + \mathfrak{k}$ is j-invariant. By 1.4, we can even assume (after an inessential change of j) that $[\mathfrak{k}, \mathfrak{w}_0] \subset \mathfrak{w}_0$ holds. We can write $\mathfrak{k} = \mathfrak{k}_a + \mathfrak{k}_0 + \mathfrak{k}_1$ such that $\mathfrak{k}_1 + j\mathfrak{g}_1$ and $\mathfrak{k}_0 + \mathfrak{t}_0$ are subalgebras where $\mathfrak{k}_i$ does not contain any ideal of the corresponding algebra. Moreover, we can assume that $[\mathfrak{k}_0, \mathfrak{k}_0 + \mathfrak{t}_0 + \mathfrak{k}_1 + j\mathfrak{g}_1] = 0$ holds. This implies that $(\mathfrak{k}_1 + j\mathfrak{g}) + (\mathfrak{k}_0 + \mathfrak{t}_0)$ contains a maximal noncompact semisimple subalgebra $\mathfrak{h}_{0s}$ of $\mathfrak{g}_0$. We can assume that a Cartan algebra of $\mathfrak{h}_{0s}$ is contained in the span of the jc, c a minimal idempotent of $\mathfrak{t}_0 + \mathfrak{g}_1 + j\mathfrak{g}_1$. From this it is easy to derive that h_{0s} is contained in the subspace $\mathfrak{h}_0$ of the maximal semisimple subalgebra $\mathfrak{h} = \mathfrak{h}_{-1/2} + \mathfrak{h}_0 + \mathfrak{h}_{1/2}$ of $\mathfrak{g}$ considered in 3.3. Clearly we have $\mathfrak{h}_{0s} = \mathfrak{h}_{0s}^0 \oplus \mathfrak{h}_{0s}^1$ where $\mathfrak{h}_{0s}^0 \subset \mathfrak{k}_0 + \mathfrak{t}_0$ and $\mathfrak{h}_{0s}^1 \subset \mathfrak{k}_1 + j\mathfrak{g}_1$.

Assume $\mathfrak{d} = [\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}] \cap (\mathfrak{k}_1 + j\mathfrak{g}_1)$ contains a nontrivial, noncompact simple subalgebra $\mathfrak{h}''$. We can assume that $\mathfrak{h}''$ is maximal in $\mathfrak{d}$ and an ideal of $\mathfrak{d}$. Then there exists a simple ideal $\mathfrak{h}' \subset \mathfrak{h}$, $\mathfrak{h}' = \mathfrak{h}'_{-1/2} + \mathfrak{h}'_0 + \mathfrak{h}'_{1/2}$ satisfying $\mathfrak{h}'' \subset \mathfrak{h}'_0$ and $\mathfrak{h}'_{\pm 1/2} \neq 0$. Since $\mathfrak{h}'' \neq 0$ we can choose $f_0 \in \mathfrak{h}'_0$ so that it has a nontrivial component f_0'' in $\mathfrak{h}''$. But we have reduced the discussion before to the case $f_0 = \lambda je + jq' + w_0$ where $\lambda = 0, 1$, $q' \in \mathfrak{t}_0$, $w_0 \in \mathfrak{w}_0$. This shows that f_0 commutes with $\mathfrak{h}''$ on $\mathfrak{g}_1$, whence $f_0'' = 0$. This is a contradiction and implies that $\mathfrak{h}_{0s}^1$ commutes with $[\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}]$ and with $\mathfrak{h}_{-1/2} + \mathfrak{h}_{1/2}$. Moreover, $[\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}] \subset \mathfrak{k}_0 + \mathfrak{t}_0 + \mathfrak{k}$ holds.

3.14. We continue investigating $\mathfrak{g}_0$ by considering $\mathrm{rad}(\mathfrak{g}_0)$.
First we prove

LEMMA. $\mathrm{rad}([\mathfrak{g}_0, \mathfrak{g}_0]) \subset \mathrm{nil}(\mathfrak{g})$.

Proof. The maximal semisimple subalgebra $\mathfrak{h}$ under consideration can be written as sum of ideals $\mathfrak{h} = \mathfrak{h}^* \oplus \tilde{\mathfrak{h}}$ where $\mathfrak{h}^* = \mathfrak{h}_{-1/2} + [\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}] + \mathfrak{h}_{1/2}$. By construction, $[\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}] + \tilde{\mathfrak{h}} \subset \mathfrak{g}_0$ and $[\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}] = \mathfrak{h}_0' + \mathfrak{z}$ is a reductive Lie algebra with center $\mathfrak{z}$ and semisimple part $\mathfrak{h}_0'$. It is clear that $\mathfrak{g}_0 = \mathfrak{h} \cap \mathfrak{g}_0 + (\mathrm{rad}(\mathfrak{g}))_0$ holds, where $(\mathrm{rad}(\mathfrak{g}))_0 = \mathfrak{g}_0 \cap \mathrm{rad}(\mathfrak{g})$. Therefore $\mathrm{rad}(\mathfrak{g}_0) = \mathfrak{z} + (\mathrm{rad}(\mathfrak{g}))_0$ and $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{h}_0' + \tilde{\mathfrak{h}} + [\mathfrak{g}_0, (\mathrm{rad}(\mathfrak{g}))_0]$. Since $\mathfrak{h}_0' + \tilde{\mathfrak{h}}$ is semisimple and $\mathfrak{v} = [\mathfrak{g}_0, (\mathrm{rad}\,\mathfrak{g})_0]$ is a solvable ideal, we have $\mathfrak{v} = \mathrm{rad}([\mathfrak{g}_0, \mathfrak{g}_0])$. Clearly, $\mathfrak{v} \subset \mathrm{nil}(\mathfrak{g})$. Hence the claim.

3.15. We had chosen $\mathfrak{n}$ to be maximal in $\mathfrak{r}$. Therefore, since $[\mathrm{nil}(\mathfrak{g}), \mathfrak{r}] \subsetneq \mathfrak{r}$ is an ideal of $\mathfrak{g}$, we can — and will — assume that $\mathfrak{n} \supset [\mathrm{nil}(\mathfrak{g}), \mathfrak{r}]$ holds.

Moreover, if x is an element or a subspace of $\mathfrak{r}$ which is invariant under the family $\mathfrak{b}$ of endomorphisms chosen in 3.3, and if $\mathfrak{u}(x)$ is the $\mathfrak{h}$-module generated by x, then $\mathfrak{u}(x) + \mathfrak{n} \subset \mathfrak{r}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{r} = \mathfrak{u}(x) + \mathfrak{n}$ follows.

3.16. To complete the proof of this "Case 3" we need detailed information on $\mathfrak{t}_0 + \mathfrak{k}$. We recall that, by the induction hypothesis, $\mathfrak{t}_0 + \mathfrak{k}$ is a j-invariant subalgebra of $\mathfrak{g}_0$ which corresponds to a homogeneous Siegel domain.

LEMMA. $\mathrm{nil}(\mathfrak{t}_0 + \mathfrak{k}) \subset \mathrm{rad}([\mathfrak{g}_0, \mathfrak{g}_0]) \subset \mathrm{nil}(\mathfrak{g})$.

Proof. It is clear by 3.14 that we only have to prove the first inclusion. Since $\mathrm{nil}(\mathfrak{t}_0 + \mathfrak{k})$ is invariant under modification derivations and since the nilradical does not change when considering the algebraic hull of $\mathfrak{g}_0 + \mathfrak{g}_1$, we can assume that $\mathfrak{g}_0 + \mathfrak{g}_1$ is algebraic and $\mathfrak{w}_0 + \mathfrak{t}_0$ is split solvable. But then $[\mathfrak{k}, \mathfrak{w}_0] \subset \mathfrak{w}_0$ and $\mathrm{nil}(\mathfrak{t}_0 + \mathfrak{k})$ is a solvable ideal of $\mathfrak{g}_0$. Therefore $\mathrm{nil}(\mathfrak{t}_0 + \mathfrak{k}) \subset \mathrm{rad}([\mathfrak{g}_0, \mathfrak{g}_0])$ as claimed.

3.17. An application of the last section yields

LEMMA. $\mathfrak{t}_0$ *corresponds to a symmetric tube domain.*

Proof. Let q be the maximal idempotent of $\mathfrak{t}_0$. Then $\mathfrak{u} = \mathfrak{g}_0 + \mathfrak{g}_1$ splits into $\mathfrak{u}_1 + \mathfrak{u}_{1/2} + \mathfrak{u}_{-1/2} + \mathfrak{u}_0$ relative to jq where $\mathfrak{g}_1 + j\mathfrak{g}_1 \subset \mathfrak{u}_0$. Set $\mathfrak{r}_u = \mathfrak{r} \cap \mathfrak{u}$. We have seen in [6; Lemma 4.26] that $\mathfrak{u} = \mathfrak{r}_u + j\mathfrak{r}_u + \mathfrak{k}$ holds. Clearly $\mathfrak{r}_u = \oplus(\mathfrak{r}_u)_\lambda$ where $(\mathfrak{r}_u)_\lambda = \mathfrak{r}_u \cap \mathfrak{u}_\lambda$. We can use the proof of [6; 4.10] and obtain $\mathfrak{u}_1 = (\mathfrak{r}_u)_1$. Moreover, since $j\mathfrak{u}_{-1/2} = [q, \mathfrak{u}_{-1/2}]$, $j\mathfrak{u}_{-1/2} \subset (\mathfrak{r}_u)_{1/2}$ holds. Therefore, if $\mathfrak{t}_0 \cap \mathfrak{u}_{1/2} \neq 0$, then $\mathfrak{t}_0 \cap (\mathfrak{r}_u)_{1/2} \neq 0$. But since $\mathfrak{t}_0 + \mathfrak{u}_{1/2} \subset \mathrm{nil}(\mathfrak{t}_0 + \mathfrak{k})$ this implies $[j(\mathfrak{t}_0 \cap (\mathfrak{r}_u)_{1/2}), \mathfrak{t}_0 \cap (\mathfrak{r}_u)_{1/2}] \subset [\mathrm{nil}(\mathfrak{t}_0 + k), \mathfrak{r} \cap \mathfrak{g}_0] \subset [\mathrm{nil}(\mathfrak{g}), \mathfrak{r}] \subset \mathfrak{n}$ by 3.15 and 3.16. Therefore $\mathfrak{n} \cap \mathfrak{u}_1 \neq 0$. But then $\mathfrak{u}_1$ contains an idempotent which is already contained in $\mathfrak{n}$. This is a contradiction to the choice of e. Hence $\mathfrak{t}_0 \cap \mathfrak{u}_{1/2} = 0$. This proves that $\mathfrak{t}_0$ corresponds to a tube domain.

To finish the proof it suffices to show that $(\mathfrak{t}_0 + \mathfrak{k}) \cap \mathfrak{u}_0$ is reductive. But otherwise $\mathfrak{v} = \mathrm{nil}(\mathfrak{t}_0 + \mathfrak{k}) \neq 0$. Since $\mathfrak{v}$ is invariant under all jv_i, v_i a minimal idempotent of $\mathfrak{t}_0$, we obtain $\mathfrak{v} = \oplus \mathfrak{v}_{ij}$ where $\mathfrak{v}_{ij} = (\mathfrak{t}_0)_{ij}$. Note $[\mathfrak{v}_{ij}, \mathfrak{u}_1] \subset [\mathrm{nil}(\mathfrak{g}), \mathfrak{r}] \subset \mathfrak{n}$ by 3.15 and 3.16. Finally, if $\mathfrak{v}_{ij} \neq 0$, then it is easy to see that $[\mathfrak{v}_{ij}, \mathfrak{u}_1] \subset \mathfrak{n}$ contains an idempotent of $\mathfrak{n}$, again a contradiction. This proves the lemma.

3.18. Let $\mathfrak{h} = \mathfrak{h}^* \oplus \tilde{\mathfrak{h}}$ where $\mathfrak{h}^* = \mathfrak{h}_{-1/2} + [\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}] + \mathfrak{h}_{1/2}$ and $\tilde{\mathfrak{h}} \subset \mathfrak{g}_0$ is an ideal of $\mathfrak{h}$. We set $\mathfrak{h}^*_\lambda = \mathfrak{h}^* \cap \mathfrak{g}_\lambda$.

LEMMA. *Let* $\tilde{\mathfrak{h}}_i$ *be a simple noncompact summand of* $\tilde{\mathfrak{h}}$ *satisfying* $\tilde{\mathfrak{h}}_i \subset \mathfrak{t}_0 + \mathfrak{k}$. *Then*

a) $\tilde{\mathfrak{h}}_i$ *is the noncompact part of* $\tilde{\mathfrak{h}} \cap (\mathfrak{t}_0 + \mathfrak{k})$.

b) $\mathfrak{h}^* \cap \mathfrak{g}_0$ *is reductive with compact semisimple part.*

c) $\mathfrak{t}_0$ *corresponds to the irreducible symmetric tube domain associated with* $\tilde{\mathfrak{h}}_i$.

Proof. Denote by $\mathfrak{p}$ the ideal of $\mathfrak{t}_0 + \mathfrak{k}$ associated with $\tilde{\mathfrak{h}}_i$. Then $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_0$ where $\mathfrak{p}_1 \subset \mathfrak{r}$, $\mathfrak{p}_0 \supset j\mathfrak{p}_1$ and $\mathfrak{p}_1$ is an $\tilde{\mathfrak{h}}_i$-module and invariant under $\mathfrak{b}$. Moreover, $\tilde{\mathfrak{h}}_i$ acts irreducibly on $\mathfrak{p}_1$ and trivially on all other ideals of $\mathfrak{t}_0 + \mathfrak{k}$. Finally, $\tilde{\mathfrak{h}}$ commutes with $\mathfrak{h}^*_\lambda$ for $\lambda = 0, \pm 1/2$ and $[\mathfrak{h}^*_0, \mathfrak{p}_1] \subset \mathfrak{p}_1$.

Set $X = \sum [\mathfrak{h}_{\varepsilon_1}[\mathfrak{h}_{\varepsilon_2}, \cdots, [\mathfrak{h}_{\varepsilon_r}, \mathfrak{p}_1] \cdots]$ where $\varepsilon_i \in \{\pm 1/2\}$ and $r \geq 0$. Then X is an $\mathfrak{h}$-module and $X \subset \mathfrak{r}$. Since $\mathfrak{p}_1 \subset X$ we have $X \not\subset \mathfrak{n}$ and $\mathfrak{r} = \mathfrak{n} + X$ follows. Clearly, $X = X_{-1/2} + X_0 + X_{1/2} + X_1$ where $X_\lambda = X \cap \mathfrak{g}_\lambda$ and X_0 corresponds to the summands satisfying $\varepsilon_1 + \cdots + \varepsilon_r = 0$. Since $\tilde{\mathfrak{h}}_i$ commutes with $\mathfrak{h}_{\pm 1/2} = \mathfrak{h}^*_{\pm 1/2}$, X_0 is a sum of irreducible $\tilde{\mathfrak{h}}_i$-modules isomorphic

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to $\mathfrak{p}_1$.

Next we note that $\mathfrak{n} \cap \mathfrak{g}_0 \subset \mathfrak{w}_0 + j\mathfrak{g}_1$ holds. Otherwise there exists some $n \in \mathfrak{n} \cap \mathfrak{g}_0$, $n = a + t + jx$, where $a \in \mathfrak{w}_0$, $x \in \mathfrak{g}_1$ and $0 \neq t \in \mathfrak{t}_0$. It is easy to see that we can even assume $t \in (\mathfrak{t}_0)_1$ where $(\mathfrak{t}_0)_k$, $k = 0, 1/2, 1$, are the weight spaces in $\mathfrak{t}_0$ of a maximal idempotent of $\mathfrak{t}_0$. Since $\mathfrak{n}$ is invariant under all $\mathrm{Re}(\mathrm{ad}\,jc)$, c a minimal idempotent in $\mathfrak{t}_0$, we can even assume $\mathfrak{n} \cap (\mathfrak{t}_0)_1 = 0$. But now it is easy to derive that $\mathfrak{n}$ contains a minimal idempotent of $\mathfrak{t}_0$. This is a contradiction since e was chosen maximal in $\mathfrak{g}'$.

Now suppose $R = \mathrm{Re}(\mathrm{ad}\,jc)$ where c is the maximal idempotent of some summand $\mathfrak{v}$ of $\mathfrak{t}_0 + \mathfrak{k}$ which is different from the one associated with $\tilde{\mathfrak{h}}_i$. Then $R\mathfrak{h}_{\pm 1/2} \subset \mathfrak{h}_{\pm 1/2}$ by our choice of $\mathfrak{h}$ (see 3.3) and $R\mathfrak{p}_1 = 0$. Hence $RX \subset X$. Since R and $\mathrm{ad}\,\tilde{\mathfrak{h}}_i$ commute, RX is also an $\tilde{\mathfrak{h}}_i$-module. Moreover, the eigenspaces $X_0^{(r)}$ of R in X_0 are $\tilde{\mathfrak{h}}_i$-modules. By the remark above they are even a sum of modules isomorphic to $\mathfrak{p}_1$. Therefore, since $X_0^{(1)} \subset \mathfrak{v}$, $X_0^{(1)} = 0$. This shows that $\mathfrak{r} = X + \mathfrak{n}$ has no component in $R\mathfrak{v}$. But we have seen in the proof of 3.17 that $\mathfrak{r} \cap R\mathfrak{v} \neq 0$ holds. This is a contradiction, proving the lemma.

3.19. We consider $\mathfrak{p} = \mathfrak{t}_0 + \mathfrak{k}$ more closely. First we split $\mathfrak{p} = \oplus_i \mathfrak{p}_0 + \mathfrak{k}_0$ where each ${}_i\mathfrak{p}$ corresponds to an irreducible symmetric tube domain and $\mathfrak{k}_0$ is an ideal of $\mathfrak{p}$ contained in $\mathfrak{k}$. Hence ${}_i\mathfrak{p} = {}_i\mathfrak{p}_0 + {}_i\mathfrak{p}_1$ and ${}_i\mathfrak{p}_1$ contains a maximal idempotent p_i of ${}_i\mathfrak{p}$.

Let ${}_i\mathfrak{h}^* = {}_i\mathfrak{h}_{-1/2} + {}_i\mathfrak{h}_0^* + {}_i\mathfrak{h}_{1/2}$ be a simple summand of $\mathfrak{h}^*$. Then $({}_i\mathfrak{h}_{1/2}, {}_i\mathfrak{h}_{-1/2})$ carries naturally the structure of a simple Jordan pair [11; chapter II]. Using a Cartan involution of this Jordan pair [12; § 5], we even get the structure of a compact Jordan triple system on $V = {}_i\mathfrak{h}_{1/2}$ [12; § 5]. Then ${}_i\mathfrak{h}_0^*$ is the "structure algebra" of V.

LEMMA. *The Jordan triple V has rank $V \leq 1$.*

Proof. Suppose V has rank ≥ 2, then there exist at least two minimal orthogonal idempotents v_1, v_2 of V. Using the Peirce decomposition of V relative to v_1, v_2 it is easy to see that $gl(2, \boldsymbol{R}) \subset {}_i\mathfrak{h}_0^*$ such that its off-diagonal parts are contained in some rootspace of ${}_i\mathfrak{h}_0^*$ (relative to some maximal $\boldsymbol{R}$-split toral subalgebra). We can choose (different) subalgebras $sl(2, \boldsymbol{R})$ of $\mathfrak{h}$ so that the corresponding Cartan algebras are spanned by f_0, f_0' and $f_0 + f_0'$ which corresponds to the matrices $E_{11} = \mathrm{diag}\{1, 0\}$, $E_{22} = \mathrm{diag}\{0, 1\}$ and $E = \mathrm{diag}\{1, 1\}$. These facts can be derived easily from

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[15; IV, § 2]. Since f_0 and f'_0 have only the eigenvalues 0, ± 1, on ${}_i\mathfrak{p}$ and since $gl(2, \boldsymbol{R})$ splits into rootspaces of ${}_i\mathfrak{h}_0^*$ it is easy to see that there exist minimal idempotents c_1, c_2 in ${}_i\mathfrak{p}_1$ such that $f_0 = \lambda je + jc_1 - jc_2 - jq + r$, $f'_0 = \lambda je + jc_2 - jc_1 - jq + r'$, where $\lambda \in \{0, 1\}$, $q = p_i - c_1 - c_2$ and r' acts trivially on ${}_i\mathfrak{p}_1$. Since $f_0 + f'_0$ has also only the eigenvalues 0, ± 1 on ${}_i\mathfrak{p}_1$, $\lambda = 0$ follows. Similarly we get $q = 0$. This implies in particular, that the cone corresponding to ${}_i\mathfrak{p}$ has rank ≤ 2. But then $f_0 + f'_0$ centralizes, ${}_i\mathfrak{h}_0^*$ whence $f_0 + f'_0$ has only the eigenvalues ± 2 on ${}_i\mathfrak{h}_{\pm 1/2}$. Moreover, $\mathrm{ad}(f_0 + f'_0)\,|\,{}_i\mathfrak{p}_1 = 0$ and there exists a k such that $\mathrm{ad}(f_0 + f'_0)\,|\,{}_k\mathfrak{p}_1 \neq 0$. We recall that $\mathrm{ad}(f_0 + f'_0)$ has only the eigenvalues 0, ± 1 in $\mathfrak{g}_0$.

Consider the vector space U spanned by "monomials" of type $[{}_{r_1}\mathfrak{h}_{\varepsilon_1}, [{}_{r_2}\mathfrak{h}_{\varepsilon_2}, \cdots, {}_i\mathfrak{p}_1]\cdots]$, where r_k is arbitrary, $\varepsilon_t = \pm 1/2$ and i is fixed. It is easy to see that U is an $\mathfrak{h}$-module and invariant under $\mathfrak{b}$. Moreover, $f_0 + f'_0$ has only the eigenvalues 0, ± 2 on U. Hence $\mathrm{ad}(f_0 + f'_0)\,|\,U \cap \mathfrak{g}_0 = 0$. This implies that U has no component in ${}_k\mathfrak{p}_1$, if $\mathrm{ad}(f_0 + f'_0)\,|\,{}_k\mathfrak{p}_1 \neq 0$. As in the proof of 3.18 we consider $\mathfrak{r} = U + \mathfrak{n}$ and obtain a contradiction, since ${}_k\mathfrak{p}_1 \subset \mathfrak{r}$. This proves the lemma.

3.20. We continue the investigations of the last section. We assume that $\mathfrak{h}$ has a simple subalgebra ${}_i\mathfrak{h} = {}_i\mathfrak{h}_{-1/2} + {}_i\mathfrak{h}_0 + {}_i\mathfrak{h}_{1/2}$ such that ${}_i\mathfrak{h}_0$ has a noncompact simple subalgebra. We have seen in the last section that $V = {}_i\mathfrak{h}_{1/2}$, considered as Jordan triple, has rank $V = 1$.

V is said to be of "algebra type" if there exists some subalgebra $sl(2, \boldsymbol{R})$ of ${}_i\mathfrak{h}$ such that the corresponding f_0 has only the eigenvalue 2 on ${}_i\mathfrak{h}_{1/2}$, -2 or ${}_i\mathfrak{h}_{-1/2}$ and 0 or ${}_i\mathfrak{h}_0$.

LEMMA. *$V = {}_i\mathfrak{h}_{1/2}$ is of algebra type.*

Proof. Suppose this is wrong, then our assumptions imply that there exists a subalgebra $sl(3, \boldsymbol{R}) \cong \mathfrak{h}' \subset {}_i\mathfrak{h}$ such that $\mathfrak{h}' = \mathfrak{h}'_{-1/2} + \mathfrak{h}'_0 + \mathfrak{h}'_{1/2}$ where (we may assume w.r.g) $\mathfrak{h}'_{-1/2} \cong \left\{\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix};\ a \in \boldsymbol{R}^2\right\}$, $\mathfrak{h}'_0 \cong \left\{\begin{pmatrix} -\mathrm{tr}(A) & 0 \\ 0 & A \end{pmatrix},\ A \in gl(2, \boldsymbol{R})\right\}$, $\mathfrak{h}'_{1/2} \cong \left\{\begin{pmatrix} 0 & b^t \\ 0 & 0 \end{pmatrix};\ b \in \boldsymbol{R}^2\right\}$. Moreover, we can assume that the rootspaces of $sl(3, \boldsymbol{R})$ are contained in rootspaces of ${}_i\mathfrak{h}$ (relative to some maximal $\boldsymbol{R}$-split toral subalgebra). This follows from [15] and [12; § 3.2]. We consider the two copies of $sl(2, \boldsymbol{R})$ inside $\mathfrak{h}' = sl(3, \boldsymbol{R})$ spanned by $f_{-1/2}$, f_0, $f_{1/2}$ and $\tilde{f}_{-1/2}$, $\tilde{f}_0$, $\tilde{f}_{1/2}$ respectively, where one has the following correspondences: $f_{1/2} \leftrightarrow (1, 0)$, $\tilde{f}_{1/2} \leftrightarrow (0, 1)$, $f_{-1/2} \leftrightarrow (1, 0)^t$, $\tilde{f}_{-1/2} \leftrightarrow (0, 1)^t$, $f_0 = [f_{1/2}, f_{-1/2}]$

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$\leftrightarrow \operatorname{diag}\{1, -1, 0\}$, $\tilde{f}_0 = [\tilde{f}_{1/2}, \tilde{f}_{-1/2}] \leftrightarrow \operatorname{diag}\{1, 0, -1\}$. Moreover, we know that the subalgebra $sl(2, \boldsymbol{R}) \subset \mathfrak{h}_0'$ acts on a selfdual cone in ${}_i\mathfrak{p}$. We may assume that it acts in the natural way on a three dimensional subspace H of ${}_i\mathfrak{p}$ realized as 2×2 symmetric matrices. We denote $c_1 = \operatorname{diag}\{1, 0\}$, $c_2 = \operatorname{diag}\{0, 1\}$, $x_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in H$. Since f_0 and $\tilde{f}_0$ are selfadjoint (with integral eigenvalues) we can label elements in $\mathfrak{g}$ by the pair of eigenvalues corresponding to f_0 and $\tilde{f}_0$ respectively. We also note that f_0 and $\tilde{f}_0$ have only the eigenvalues $0, \pm 1$, in $\mathfrak{g}_0$. Thus we may assume that c_1 belongs to $(1, -1)$, c_2 to $(-1, 1)$ and x_{12} to $(0, 0)$.

A straightforward computation in $sl(3, \boldsymbol{R})$ shows $[f_0, f_{\pm 1/2}] = \pm 2 f_{\pm 1/2}$, $[f_0, \tilde{f}_{\pm 1/2}] = \pm \tilde{f}_{1/2}$, and $[\tilde{f}_0, f_{\pm 1/2}] = \pm f_{\pm 1/2}$, $[\tilde{f}_0, \tilde{f}_{\pm 1/2}] = \pm 2 \tilde{f}_{\pm 1/2}$. Moreover, for $x = [\tilde{f}_{-1/2}, f_{1/2}]$ we have $[x, \tilde{f}_{-1/2}] = 0$ and $[x, c_2] = 0$. The eigenvalues of $y = [f_{1/2}, c_2]$ are $(1, 2)$. Since c_2 has eigenvalue -1 relative to f_0, $y \neq 0$. Also note that y has eigenvalue 2 for $\tilde{f}_0$, whence $[\tilde{f}_{-1/2}, y] \neq 0$ and $[\tilde{f}_{-1/2}, [\tilde{f}_{-1/2}, y]] \neq 0$. But $[\tilde{f}_{-1/2}, y] = [\tilde{f}_{-1/2}, [f_{1/2}, c_2]] = [x, c_2] + [f_{1/2}, [\tilde{f}_{-1/2}, c_2]] = [f_{1/2}, [\tilde{f}_{-1/2}, c_2]]$, hence $[\tilde{f}_{-1/2}, [\tilde{f}_{-1/2}, y]] = [\tilde{f}_{-1/2}, [f_{1/2}, [\tilde{f}_{-1/2}, c_2]]] = [x, [\tilde{f}_{-1/2}, c_2]] + [f_{1/2}, [\tilde{f}_{-1/2}, [\tilde{f}_{-1/2}, c_2]]] = 0$, a contradiction. This proves the lemma.

3.21. By the results of the last sections we know that for each simple summand ${}_i\mathfrak{h} = {}_i\mathfrak{h}_{-1/2} + {}_i\mathfrak{h}_0 + {}_i\mathfrak{h}_{1/2}$ of $\mathfrak{h}$, ${}_i\mathfrak{h}_{\pm 1/2} \neq 0$, on the space $V = {}_i\mathfrak{h}_{1/2}$ we obtain naturally the structure of a simple Jordan triple of algebra type and of rank 1. This implies [14; Lemma 2.1] that V is isomorphic to a Jordan triple of a quadratic form $[\boldsymbol{R}^n; \mathrm{Id}]$, $n \geq 1$. Moreover, in all these cases the "structure algebra" ${}_i\mathfrak{h}_0 = [{}_i\mathfrak{h}_{-1/2}, {}_i\mathfrak{h}_{1/2}]$ is isomorphic to $\boldsymbol{R} \oplus {}_i\mathfrak{k}$ where ${}_i\mathfrak{k} = \mathfrak{k} \cap {}_i\mathfrak{h}_0$ [13; § 5]. It is easy to see that ${}_i\mathfrak{k} = 0$ if and only if ${}_i\mathfrak{h} \cong sl(2, \boldsymbol{R})$.

Finally, in all the cases except $[\boldsymbol{R}; \mathrm{Id}]$ above, the Jordan triple $\boldsymbol{C}$ is naturally a subtriple of V.

3.22. As a corollary of the last section we see that the noncompact part of $[\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}]$ is also the center of this Lie algebra. In particular, a "f_0" as considered before, contained in $[\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}]$, commutes with $\mathfrak{h} \cap \mathfrak{g}_0$. Therefore such an f_0 has only the eigenvalues $0, \pm 2$ on $\mathfrak{h}$ (see also Lemma 3.20). Moreover, by 3.11 and 3.12 we can assume that f_0 has a nonvanishing "q-part".

LEMMA. *$f_0 = \lambda je + jq_1 - jq_2 + w_0$, where $\lambda \in \{0, 1\}$, q_i is a sum of maximal tripotents of irreducible factors of $\mathfrak{t}_0$ and $q_1 + q_2$ is the maximal*

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idempotent of $\mathfrak{t}_0$.

Proof. Suppose there exists an idempotent $0 \neq c \in \mathfrak{t}_0$, $[f_0, c] = 0$, where $f_0 \in sl(2, \boldsymbol{R}) \cong \boldsymbol{R}f_{-1/2} + \boldsymbol{R}f_0 + \boldsymbol{R}f_{1/2}$. Let U be the $\mathfrak{h}$-module generated by c. It is easy to see that f_0 has only even integral eigenvalues on $U \cap \mathfrak{g}_0$. Hence $[f_0, U \cap \mathfrak{g}_0] = 0$. As before we consider $\mathfrak{r} = \mathfrak{n} + U$ and see that no element of $\mathfrak{r}$ has a component in $\mathrm{Re}(\mathrm{ad} jq)\mathfrak{t}_0$. This is a contradiction since $[\mathrm{Re}(\mathrm{ad} jq)\mathfrak{t}_0] \cap \mathfrak{r} \neq 0$ as shown in 3.17 and the assertion follows.

3.23. In this section we reduce further the possibilities for $\mathfrak{h}$.

LEMMA. $\mathfrak{h}^* = \mathfrak{h}_{-1/2} + [\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}] + \mathfrak{h}_{1/2}$ *is simple.*

Proof. Suppose there exist different simple summands ${}_1\mathfrak{h}$ and ${}_2\mathfrak{h}$ of $\mathfrak{h}^*$. By 3.20 we know that these Lie algebras have real rank one. Let f_1, f_2 be corresponding elements "of type f_0". Then, by Lemma 3.22, $f_i = \lambda_i je + jq'_i + w_{oi}$ for $i = 1, 2$. Since we can assume that also $f_1 + f_2$ is "of type f_0", $\mathrm{ad}(f_1 + f_2)$ has only the eigenvalues 0, ± 1 in $\mathfrak{g}_0$ and 0 or 1 in $\mathfrak{g}_1$. But $f_1 + f_2 = (\lambda_1 + \lambda_2)je + j(q'_1 + q'_2) + (w_{01} + w_{02})$ and $q'_1 + q'_2 = 0$ follows. The remaining case $f_1 + f_2 = \lambda' je + w'_0$ was already excluded in 3.11 and 3.12. This proves the lemma.

3.24. From 3.17 we know that $\mathfrak{t}_0$ corresponds to a tube domain. Hence $\mathfrak{t}_0 = \mathfrak{u} + j\mathfrak{u}$ where $\mathfrak{u} \subset \mathfrak{r}$. By 3.18, if $\mathfrak{t}_0 + \mathfrak{k}$ contains a noncompact semisimple ideal of $\mathfrak{h}$, then $\mathfrak{u}$ corresponds to an irreducible cone. Otherwise all occuring cones are one dimensional by 3.22.

LEMMA. $\mathfrak{t}_0$ *corresponds to an irreducible symmetric tube domain.*

Proof. Let c be the maximal idempotent of an irreducible summand of $\mathfrak{t}_0$. By the remarks above we can assume that the corresponding cone is one dimensional. Since $[\mathfrak{h}_0, c] \subset \boldsymbol{R}c$ we see that $U = \sum [\mathfrak{h}_{\varepsilon_1}[\cdots[\mathfrak{h}_{\varepsilon}, c]\cdots]$, $\varepsilon_i = \pm 1/2$, is an $\mathfrak{h}$-module and invariant under $\mathfrak{b}$. We recall $\mathfrak{h} = \mathfrak{h}^* + \tilde{\mathfrak{h}}$ where $\mathfrak{h}^* = \mathfrak{h}_{-1/2} + [\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}] + \mathfrak{h}_{1/2}$ and $\tilde{\mathfrak{h}} \subset \mathfrak{g}_0$. Moreover, by 3.23, $\mathfrak{h}^*$ is simple. Since $R = \mathrm{Re}(\mathrm{ad} jc')$ leaves $\mathfrak{h}$ invariant by construction 3.3, we see that for every idempotent c' of $\mathfrak{u}$ the derivation R of $\mathfrak{h}$ is sI for some value s on $\mathfrak{h}_{1/2}$ and then $-sI$ on $\mathfrak{h}_{-1/2}$. Suppose that also $[jc', c] = 0$ holds, then $[jc', U \cap \mathfrak{g}_0] = 0$. Therefore $\mathfrak{r} = \mathfrak{n} + U$ has no component in $\boldsymbol{R}c'$. But $c' \in \mathfrak{u} \subset \mathfrak{r}$, a contradiction.

3.25. The above considerations restrict the possibilities for $\mathfrak{h}^* = \mathfrak{h}_{1/2} + [\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}] + \mathfrak{h}_{1/2}$ quite a bit. But we can even show

LEMMA. $\mathfrak{h}^* \cong sl(2, \boldsymbol{R})$.

Proof. Suppose this is wrong. Then from 3.21 it follows that $sl(2, \boldsymbol{C}) \subset \mathfrak{h}^*$. (Since $\boldsymbol{C}$ is a subtriple of V this follows from [12; § 3.2].) Moreover, the rootspaces of $sl(2, \boldsymbol{C})$ are contained in rootspaces of $\mathfrak{h}^*$ (relative to some maximal $\boldsymbol{R}$-split toral subalgebra). We choose f_0 for the canonically embedded $sl(2, \boldsymbol{R}) \subset sl(2, \boldsymbol{C})$. From 3.22 we know that f_0 is of type $f_0 = \lambda je + \varepsilon jq + w_0$, $\varepsilon = \pm 1$. Therefore, f_0 has the eigenvalue 0 on $\mathfrak{w}_0 + j\mathfrak{u} + \mathfrak{k} + j\mathfrak{g}_1$ and the eigenvalue ε on $\mathfrak{u}$. We also note, that f_0 has on $\mathfrak{h}_{\pm 1/2}$ the eigenvalue ± 2. Let U denote the $\mathfrak{h}$-module generated by $\mathfrak{u}$. It is easy to see that $[\mathfrak{h}_\alpha, [\mathfrak{h}_\alpha, \mathfrak{u}]] = 0$ and $[\mathfrak{h}_\alpha, [\mathfrak{h}_{-\alpha}, \mathfrak{u}]] \subset \mathfrak{u}$ for $\alpha = \pm 1/2$. This shows that $U = \mathfrak{u} + [\mathfrak{h}^*, \mathfrak{u}]$ holds. Let $[\mathfrak{h}_{-1/2}, \mathfrak{h}_{1/2}] \cong \boldsymbol{R} \oplus \mathfrak{k}^*$, where $\mathfrak{k}^* \subset \mathfrak{k}$. Then $[\mathfrak{k}^*, \mathfrak{u}] = 0$ since $\mathfrak{u}$ is either one-dimensional or it is associated with some ideal $\tilde{\mathfrak{h}} \subset \mathfrak{g}_0$ of $\mathfrak{h}$. Let $W \subset U$ be an irreducible $sl(2, \boldsymbol{C})$-submodule of U. Then $W = W_0 + W_\alpha$ where $W_\beta = W \cap \mathfrak{g}_\beta$. Since $f_0 \in sl(2, \boldsymbol{C})$ and $W_0 \subset \mathfrak{u}$, W is not a trivial representation. Hence $W \cong \boldsymbol{C}^2$. The subalgebra $\mathfrak{v} = \boldsymbol{C}\,\mathrm{diag}\{i, -i\}$ of $sl(2, \boldsymbol{C})$ corresponds to a subalgebra of $\mathfrak{k}^*$. Considered as subalgebra of $sl(2, \boldsymbol{C})$ it acts non-trivially on W_0, but as subalgebra of $\mathfrak{k}^*$ it acts trivially on $W_0 \subset \mathfrak{u}$. This is a contradiction.

3.26. From 3.24 we know that the cone C corresponding to $\mathfrak{t}_0$ is irreducible. Moreover, C is not one-dimensional only if it is associated with an ideal $\tilde{\mathfrak{h}} \subset \mathfrak{g}_0$ of $\mathfrak{h}$. In this case we set $\tilde{\mathfrak{k}} = \tilde{\mathfrak{h}} \cap \mathfrak{k}$. Then $\tilde{\mathfrak{k}}$ is maximal compact in $\tilde{\mathfrak{h}}$.

LEMMA. *It suffices to consider the case where C is one-dimensional.*

Proof. Since $\mathfrak{h}^* \cong sl(2, \boldsymbol{R})$, $\mathfrak{g}_\alpha = \boldsymbol{R} f_\alpha + \mathfrak{v}_\alpha$ where $\mathfrak{v}_\alpha \subset \mathrm{rad}\,\mathfrak{g}$, $\alpha = \pm 1/2$. Therefore $[\mathfrak{g}_{-1/2}, \mathfrak{g}_{1/2}] \subset \boldsymbol{R} f_0 + \mathrm{nil}(\mathfrak{g})$. We also note that $\tilde{\mathfrak{g}}_0 = \mathfrak{w}_0 + \mathfrak{t}_0 + j\mathfrak{g}_1$ is a solvable subalgebra of $\mathfrak{g}_0$, $\tilde{\mathfrak{g}}_0 + \mathfrak{k} = \mathfrak{g}_0$. It is easy to verify that $\tilde{\mathfrak{g}} = \mathfrak{g}_{-1/2} + \tilde{\mathfrak{g}}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1$ is a Kähler subalgebra of $\mathfrak{g}$ satisfying $\tilde{\mathfrak{g}} + \mathfrak{k} = \mathfrak{g}$. Since $\tilde{\mathfrak{g}}_0$ is solvable, the maximal semisimple subalgebra of $\tilde{\mathfrak{g}}$ is $\mathfrak{h}^* \cong sl(2, \boldsymbol{R})$. In particular there is no ideal of $\mathfrak{h}$ contained in $\tilde{\mathfrak{g}}_0$. Therefore, applying the previous sections to $\tilde{\mathfrak{g}}$ shows that we can assume that $\mathfrak{t}_0$ corresponds to a one dimensional cone.

3.27. From 3.25 we know $\mathfrak{h}^* \cong sl(2, \boldsymbol{R})$. Let f_0 denote the canonical generator of the Cartan subalgebra of $\mathfrak{h}^*$. Then $f_0 = \lambda je + \varepsilon q + w_0$. From 3.26 it follows that we can assume $\mathfrak{u} = \boldsymbol{R} q$. Moreover, as in the proof of 3.25 we see that for the $\mathfrak{h}$-module U, generated by $\mathfrak{u}$ we have $U =$

$\boldsymbol{R}q + \boldsymbol{R}[f_{1/2}, q]$, if $\varepsilon = -1$ and $U = \boldsymbol{R}q + \boldsymbol{R}[f_{-1/2}, q]$, if $\varepsilon = 1$. We also know $\mathfrak{r} = U + \mathfrak{n}$ and $\mathfrak{g} = \mathfrak{g}' + U + jU$. Note that $(\mathfrak{g}_{1/2} + \mathfrak{g}_{-1/2})$ modulo $(\mathfrak{g}'_{1/2} + j\mathfrak{g}'_{1/2})$ is at most two-dimensional.

In the following sections we will exclude the four possibilities for f_0: $\lambda = 0, 1$, $\varepsilon = \pm 1$.

3.28. Since $\mathfrak{g}/\mathfrak{g}'$ is of low dimension it is natural to consider some of the cases $\mathfrak{g}_\lambda = \mathfrak{g}'_\lambda$.

LEMMA. *The case* $\mathfrak{g}_{1/2} = \mathfrak{g}'_\lambda$ *does not occur.*

Proof. Suppose $\mathfrak{g}_{1/2} = \mathfrak{g}'_{1/2}$. Then $\mathfrak{g}_{1/2} = \mathfrak{u}_{1/2} + \mathfrak{w}_{1/2}$, where $\mathfrak{u}_{1/2}$ and $\mathfrak{w}_{1/2}$ are defined as in 3.7. Recall $\mathfrak{u}_{1/2} = j\mathfrak{n}_{-1/2} = [e, \mathfrak{n}_{-1/2}] \subset \mathfrak{n}$, whence —in $\mathfrak{g}'$— $[\mathfrak{w}_{1/2}, \mathfrak{u}_{1/2}] = 0$. Under our assumptions we also know that $\mathfrak{u}_{1/2}$ is the bilinear kernel of ρ restricted to $\mathfrak{g}_{1/2}$. The closedness condition for ρ implies that $\mathfrak{w}_0 + \mathfrak{t}_0$ leaves $\mathfrak{u}_{1/2}$ invariant. From this and the integrability condition it follows that $\mathfrak{w}_0 + \mathfrak{t}_0$ acts symplectically (in the sense of [7; § 6]) on $\mathfrak{g}'_{1/2}/\mathfrak{u}_{1/2}$. Therefore $\mathfrak{w}_0$ has only the (real) eigenvalue 0 and jq has only the (real) eigenvalues 0, $\pm 1/2$. Hence, the eigenvalues of f_0 on $\mathfrak{g}_{1/2}$ are $\lambda/2$ and $\lambda/2 \pm \varepsilon/2$. It is easy to see that for $\lambda = 0, 1$ and $\varepsilon = \pm 1$ the eigenvalue 2 does not occur. But then $\mathfrak{h}_{1/2} = 0$, a contradiction.

3.29.

LEMMA. *The case* $f_0 = \lambda je + jq + w_0$ *does not occur.*

Proof. By 3.28 we can assume $\mathfrak{g}_{1/2} \neq \mathfrak{g}'_{1/2}$. Since, by 3.27, $(\mathfrak{g}_{1/2} + \mathfrak{g}_{-1/2})$ modulo $(\mathfrak{g}'_{1/2} + \mathfrak{g}'_{-1/2})$ is at most two-dimensional, we know $\mathfrak{g}_{-1/2} = \mathfrak{g}'_{-1/2} + \boldsymbol{R}v_{-1/2}$. In particular $f_{-1/2} = ju_{1/2} + bv_{-1/2}$ for some $u_{1/2} \in \mathfrak{u}_{1/2} \subset \mathfrak{n}$, $b \in \boldsymbol{R}$. This implies $\rho(ju_{1/2}, u_{1/2}) = \rho(f_{-1/2} - bv_{-1/2}, u_{1/2})$. We want to show that this expression vanishes. Then $u_{1/2} = 0$ and $f_{-1/2} \in \boldsymbol{R}\mathfrak{v}_{-1/2} \subset \mathfrak{r} \subset \mathrm{nil}(\mathfrak{g})$, a contradiction. To see that $\rho(ju_{1/2}, u_{1/2})$ vanishes we note that f_0 has the eigenvalue -2 on $f_{-1/2}$ and -1 on $\mathfrak{v}_{-1/2}$. In the situation under consideration f_0 can only have the eigenvalues 2, -1, 0 on $\mathfrak{g}_{1/2}$. We note that $(\nu + \mu)\rho(x^{(\nu)}_{-1/2}, n^{(\mu)}_{1/2}) = \rho([f_0, x^{(\nu)}_{-1/2}], n^{(\mu)}_{1/2}) + \rho(x^{(\nu)}_{-1/2}, [f_0, n^{(\mu)}_{1/2}]) = \rho(f_0, [x^{(\nu)}_{-1/2}, n^{(\mu)}_{1/2}])$ holds. If $n^{(\mu)}_{1/2} \in \mathfrak{n}$, then $[x^{(\nu)}_{-1/2}, n^{(\mu)}_{1/2}] \in \mathfrak{n}^{(\nu+\mu)}_0$. This expression vanishes if $\nu + \mu \neq 0$ as shown in 3.18. Therefore $\rho(\mathfrak{g}^{(\nu)}_{-1/2}, \mathfrak{n}^{(\mu)}_{1/2}) = 0$ if $\nu + \mu \neq 0$. This applies in particular to $\nu = -2, -1$ and $\mu = 0, -1$. To finish the proof of this lemma it suffices to show $\mathfrak{n}^{(2)}_{1/2} = 0$. But if $\mathfrak{n}^{(2)}_{1/2} \neq 0$, then also $0 \neq [f_{-1/2}, [f_{-1/2}, \mathfrak{n}^{(2)}_{1/2}]] \subset \mathfrak{n}^{(-2)}_{-1/2}$. To see that this is impossible we consider

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the space $\mathfrak{v} = \mathfrak{n}_{-1/2} = j\mathfrak{n}_{-1/2}$. Since $j\mathfrak{n}_{-1/2} = [e, \mathfrak{n}_{-1/2}]$ we see that $\mathfrak{v}$ is invariant under $\mathfrak{w}_0 + \mathfrak{t}_0$. It is easy to see that the representation of $\mathfrak{w}_0 + \mathfrak{t}_0$ on $\mathfrak{v}$ is symplectic. Therefore jq has only the (real) eigenvalues 0, $\pm 1/2$ and w_0 only the (real) eigenvalue 0 on $\mathfrak{v}$. Thus f_0 cannot have the eigenvalue -2 on $\mathfrak{v}$. This contradiction finishes the proof of the lemma.

3.30. In this section we finish the proof of "Case 3" by showing

LEMMA. *The case $f_0 = \lambda je - jq + w_0$ does not occur.*

Proof. By 3.28 we can assume $\mathfrak{g}_{1/2} \neq \mathfrak{g}'_{1/2}$. We also know $U = \boldsymbol{R}q + \boldsymbol{R}v_{1/2}$ and $\mathfrak{g} = \mathfrak{g}' + U + jU$. In particular $\mathfrak{g}_{-1/2} = \mathfrak{g}'_{-1/2} + \boldsymbol{R}(jv_{1/2})_{-1/2}$. Splitting $v_{1/2} = u_{1/2} + w_{1/2}$ where $u_{1/2} \in \mathfrak{u}_{1/2} \subset \mathfrak{n}$ and $w_{1/2} \in \mathfrak{w}_{1/2}$ (see 3.7) we see $jv_{1/2} = ju_{1/2} + jw_{1/2}$, hence $(jv_{1/2})_{-1/2} = ju_{1/2} \in \mathfrak{g}'$. Therefore $\mathfrak{g}_{-1/2} = \mathfrak{g}'_{-1/2}$. But then $j\mathfrak{g}_{-1/2} = [e, \mathfrak{g}_{-1/2}]$ and $\mathfrak{v} = \mathfrak{g}_{-1/2} + j\mathfrak{g}_{-1/2}$ is left invariant by $\mathfrak{w}_0 + \mathfrak{t}_0$. From this it follows that jq has only the (real) eigenvalues 0, $\pm 1/2$ and w_0 has only the (real) eigenvalue 0 on $\mathfrak{v}$. Therefore f_0 does not have the eigenvalue -2 on $\mathfrak{g}_{-1/2}$ and the lemma is proven.

APPENDIX. We want to prove the following general result.

LEMMA. *Let $\mathfrak{q}$ be an algebraic Lie algebra of endomorphisms of some vector space V and $\mathfrak{b} \subset \mathfrak{q}$ an abelian sabspace such that every $b \in \mathfrak{b}$ is a semisimple endomorphism of V.*

Then there exists a maximal semisimple subalgebra $\mathfrak{h}$ of $\mathfrak{q}$ and an algebraic abelian subalgebra $\mathfrak{a} \subset \mathrm{rad}(\mathfrak{g})$ that consists of semisimple endomorphisms such that

a) $[\mathfrak{h}, \mathfrak{a}] = 0$,

b) $\mathfrak{b} \subset \mathfrak{h} + \mathfrak{a}$.

Proof. We prove the assertion by induction on $m = \dim \mathfrak{b}$. If $m = 1$ it suffices to consider some $0 \neq q \in \mathfrak{b}$. From [3, chap. VI, § 4, Proposition 18] we know that there exists a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{q}$ containing q. Hence by loc. cit. Proposition 20 there exists a maximal semisimple subalgebra $\mathfrak{h}'$ of $\mathfrak{q}$ such that $c'_h = \mathfrak{h}' \cap \mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{h}'$ and $\mathfrak{c} = \mathfrak{c}'_h + (\mathfrak{c} \cap \mathrm{rad}(\mathfrak{q}))$. We note that $\mathfrak{c}'_h$ consists of semisimple endomorphisms of V. From [3, chap. V, § 4, Proposition 5] we derive that we can write $\mathrm{rad}(\mathfrak{q}) = \mathfrak{a}' + \mathfrak{n}$ where $\mathfrak{a}'$ is abelian, algebraic and commutes with $\mathfrak{h}'$ and where $\mathfrak{n} \subset \mathrm{rad}(\mathfrak{q})$ is the greatest ideal of $\mathfrak{q}$ consisting of nilpotent endomorphisms. Then $q = h + q + n$ and $h + q$ is semisimple. We write

$n = n_0 + n_1$ where n_0 is in the kernel of $H = \mathrm{ad}(h + a)$ and n_1 is in the sum $\mathfrak{w}$ of the eigenspaces of H for eigenvalues $\lambda \neq 0$. Clearly, H is invertible on $\mathfrak{w}$. We denote by $\mathfrak{n}^{(k)}$ the space of k-fold commutators of elements from $\mathfrak{n}$, $\mathfrak{n}^{(1)} = \mathfrak{n}$. Then $\mathfrak{n}^{(k)}$ is left invariant by H. Since $\mathfrak{n}^{(k)} \supset \mathfrak{n}^{(k+1)}$ there exists some H-invariant complement $\mathfrak{u}^{(k)}$ of $\mathfrak{n}^{(k+1)}$ in $\mathfrak{n}^{(k)}$. We note that for every eigenspace $\mathfrak{v}_\lambda$ of H we have $\mathfrak{v}_\lambda \cap \mathfrak{n}^{(k)} = \mathfrak{v}_\lambda \cap \mathfrak{u}^{(k)} + \mathfrak{v}_\lambda \cap \mathfrak{n}^{(k+1)}$. We write $n_0 = u_0^{(1)} + n_0^{(2)}$ and $n_1 = u_1^{(1)} + n_1^{(2)}$; then $u^{(1)}$, $n_0^{(2)}$ both are in the kernel of H and H is invertible on $\mathfrak{u}^{(1)} \cap \mathfrak{w}$. Let $A_1 = \exp \mathrm{ad}(H^{-1}u_1^{(1)})$. Then $A_1(h + a + n) = h + a + n - [h + a + n, H^{-1}u_1^{(1)}] \bmod \mathfrak{n}^{(2)} = h + a + u_0^{(1)} + u_1^{(1)} - u^{(1)} \bmod \mathfrak{n}^{(2)} = h + a + u_0^{(1)} \bmod \mathfrak{n}^{(2)}$. We iterate this procedure and assume that we have found already inner automorphisms $A_1, \cdots, A_{r-1}$ so that $A_{r-1}, \cdots, A_1(h + a + n) = h + a + u_0^{(1)} + u_0^{(2)} + \cdots + u_0^{(r-1)} + n^{(r)}$ for some $n^{(r)} \in \mathfrak{n}^{(r)}$. We write $n^{(r)} = n_0^{(r)} + n_1^{(r)}$ and $n_j^{(r)} = u_j^{(r)} + n_j^{(r+1)}$, $j = 0, 1$, with $n_0^{(r)} \in \ker H$ and $n_j^{(r)} \in \mathfrak{w}$. Set $A_r = \exp \mathrm{ad}(H^{-1}u_1^{(r)})$. Then $A_r, \cdots, A_1(h + a + n) = A_r(h + a + u_0^{(1)} + \cdots + u_0^{(r-1)} + n^{(r)}) = h + a + u_0^{(1)} + \cdots + u_0^{(r)} + u_1^{(r)} - [h + a + u_0^{(1)} + \cdots, H^{-1}u_1^{(r)}] \bmod \mathfrak{n}^{(r+1)} = h + a + u_0^{(1)} + \cdots + u_0^{(r)} \bmod \mathfrak{n}^{(r+1)}$. Hence we find an inner automorphism of $\mathfrak{q}$ such that $W(h + a + n) = h + a + x$ where $[h + a, x] = 0$. But $W(h + a + n)$ and $h + a$ are semisimple endomorphisms and x is nilpotent. Therefore $x = 0$. We set $\mathfrak{h} = W^{-1}\mathfrak{h}'$ and $\mathfrak{a} = W^{-1}\mathfrak{a}'$. Then $q = W^{-1}h + W^{-1}a \in \mathfrak{h} + \mathfrak{a}$ and the assertion follows.

Assume now $\dim \mathfrak{b} = m$ and the assertion holds for dimensions less than m. We write $\mathfrak{b} = \mathfrak{b}' \oplus \boldsymbol{R}q$ and apply the induction hypothesis to $\mathfrak{b}'$. The corresponding subalgebras will be denoted by $\mathfrak{h}'$ and $\mathfrak{a}'$. Hence $b' = h' + a'$, where $h' \in \mathfrak{h}'$, $a' \in \mathfrak{a}'$, for all $b' \in \mathfrak{b}'$ and $q = h + a + n$ where $\mathfrak{n}$ is as above. Since $[b', q] = 0$ we have

(1) $$[b', h + a] = 0\,,$$

(2) $$[b', n] = 0\,.$$

Now we repeat the proof above and note that in every step the inner automorphisms A_i fix $\mathfrak{b}'$. From this the assertion follows.

Added in proof. The following paper builds on the present article: J. Dorfmeister, K. Nakajima. The fundamental conjecture for homogeneous Kähler manifolds, Acta Math., 161 (1988), 23–70.

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