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THE G-HILBERT SCHEME FOR $\frac{1}{r}(1, a, r-a)$

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Abstract. Following Craw, Maclagan, Thomas and Nakamura's works [**2, 7**] on Hilbert schemes for abelian groups, we give an explicit description of the $\mathrm{Hilb}^G\,\mathbb{C}^3$ scheme for $G = \langle \mathrm{diag}(\varepsilon, \varepsilon^a, \varepsilon^{r-a})\rangle$ by a classification of all G-sets. We describe how the combinatorial properties of the fan of $\mathrm{Hilb}^G\,\mathbb{C}^3$ relates to the Euclidean algorithm.

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1. Introduction. For any finite, abelian subgroup G of $\mathrm{GL}(n, \mathbb{C})$ of order r, Nakamura defines the G-Hilbert scheme $\mathrm{Hilb}^G\mathbb{C}^n$ as the irreducible component of the G-fixed set of the scheme $\mathrm{Hilb}^r\,\mathbb{C}^n$ which contains free orbits.

For such groups, the normalisation of $\mathrm{Hilb}^G\,\mathbb{C}^n$ is a toric variety. The scheme $\mathrm{Hilb}^G\,\mathbb{C}^n$ is described in [**7**] in terms of G-sets. In fact, the description is carried by a classification of G-sets.

There are several known cases when $\mathrm{Hilb}^G\,\mathbb{C}^n$ itself is a toric variety (i.e. it is normal): for $n = 2$ and $G \subset \mathrm{GL}(2, \mathbb{C})$ by Kidoh [**5**], for $n = 3$ and $G \subset \mathrm{SL}(3, \mathbb{C})$ by Craw and Reid [**3**], for any $n \geq 2$ and $G = \langle \mathrm{diag}(\varepsilon, \varepsilon^2, \varepsilon^4, \ldots, \varepsilon^{2^n})\rangle$ by Sebestean [**8**]. In all these cases, if $n \geq 3$ the quotient $\mathbb{C}^n/G$ has canonical, non-terminal singularities.

Craw, Maclagan and Thomas in [**2**] describe $\mathrm{Hilb}^G\,\mathbb{C}^n$ for any finite, abelian group $G \subset \mathrm{GL}(n, \mathbb{C})$ in terms of initial ideals of some fixed monomial ideal by varying weight order. This gives a numerical method for finding the fan of $\mathrm{Hilb}^G\,\mathbb{C}^n$.

In this paper, we use [**2, 7**] to give a conceptual description of $\mathrm{Hilb}^G\,\mathbb{C}^3$ scheme for any cyclic subgroup $G \subset \mathrm{GL}(3, \mathbb{C})$ for which the quotient $\mathbb{C}^3/G$ is a terminal singularity (see Theorem 6.2). By Morrison and Stevens [**6**], any such group is conjugated to a group generated by a diagonal matrix $\mathrm{diag}(\varepsilon, \varepsilon^a, \varepsilon^{r-a})$, where a and r are any coprime natural numbers and ε is an rth primitive root of unity.

The description is carried out by classification of all possible G-sets in families, called triangles of transformations. These families correspond to steps in the Euclidean algorithm for b and $r - b$, where b is an inverse of a modulo r (see Main Theorem 6.2). We prove that there are $\frac{1}{2}(3r + b(r - b) - 1)$ different G-sets (see Theorem 6.4).

We show that for $a, r - a > 1$ the $\mathrm{Hilb}^G\,\mathbb{C}^3$ scheme is a normal variety with quadratic singularities. Note that $\mathrm{Hilb}^G\,\mathbb{C}^3$ for $a = 1$ or $r - a = 1$ is isomorphic to the Danilov resolution of $\mathbb{C}^3/G$ singularity by [**4**].

The paper is organised as follows. Section 2 recalls basic definitions from [**7**]. Section 3 contains classification of the G-sets by the number of valleys. It is used to show that the Hilb^G is normal. Section 4 contains definition of a primitive G-set. Every such G-set gives rise to a family of G-sets. The union of toric cones corresponding to G-sets in such family is called a triangle of transformations. In Section 5, we show how to obtain a new primitive G-set from another one. In Sections 6, the combinatoric properties of primitive G-sets and the triangles of transformations are related to the Euclidean algorithm. We show that all subcones of cones in all triangles of transformations form the fan of Hilb^G scheme. The formula counting the number of G-sets is given at the end of Section 6. Section 7 contains a concrete example of Hilb^G scheme for $G \cong \mathbb{Z}_{14}$.

I would like to thank Professor Miles Reid for introducing me to this subject.

2. Basic definitions. Let us fix two coprime integers $r, a \geq 2$. Without loss of generality we may assume that $a < r - a < r$. Denote by G the cyclic group $\mathbb{Z}_r$, considered as a subgroup of $GL(3, \mathbb{C})$, generated by matrix $\mathrm{diag}(\varepsilon, \varepsilon^a, \varepsilon^{r-a})$, where $\varepsilon = e^{\frac{2\pi i}{r}}$. The group G has r characters, which may be identified with $1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{r-1}$.

We follow the notation of [7]. Let $N_0 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$ denote a free $\mathbb{Z}$-module with $\mathbb{Z}$-basis e_i. The lattice dual to N_0 will be denoted $M_0 = \mathrm{Hom}_{\mathbb{Z}}(N_0, \mathbb{Z}) = \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^* \oplus \mathbb{Z}e_3^*$, where $e_i^*(e_j) = \delta_{ij}$. In this paper, the variables x, y, z will be identified with e_1^*, e_2^*, e_3^* and a multiplicative notation will be used in the lattice M_0. For example, vector $2e_1^* - e_3^*$ will be identified with the Laurent monomial x^2z^{-1}.

Let M_0^0 be the positive octant in M_0, identified with monomials in the ring $\mathbb{C}[x, y, z]$. Set $N = N_0 + \mathbb{Z}\frac{1}{r}(e_1 + ae_2 + (r - a)e_3)$ and let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be a dual lattice. Lattice M will be identified with a sublattice of M_0 consisting of G-invariant Laurent monomials. When no confusion arise, vector $a_1e_1 + a_2e_2 + a_3e_3$ will be denoted (a_1, a_2, a_3). For example, $\frac{1}{5}(1, 2, 3)$ stands for $\frac{1}{5}e_1 + \frac{2}{5}e_2 + \frac{3}{5}e_3$.

Let $G^\vee$ denote the character group of G. The group G acts on the left on regular functions on $\mathbb{C}^3$ by setting $(g \cdot f)(p) = f(g^{-1}p)$, where $g \in G$, $p \in \mathbb{C}^3$ and f is a regular function on $\mathbb{C}^3$. This action can be extended to the lattice M_0 (by identifying M_0 with the lattice of exponents of Laurent monomials in x, y, z). Thus, we have the natural grading:

$$M_0 = \bigoplus_{\chi \in G^\vee} M_0^\chi.$$

Definition 2.1. Let $\mathrm{wt} : M_0 \longrightarrow G^\vee$ denote group homomorphism sending an element of the lattice M_0 to its grade.

We will denote by $m \mod n$ an integer $k \in 0, \ldots, n - 1$ such that $n | (m - k)$.

Definition 2.2 (Nakamura). A subset Γ of monomials in $\mathbb{C}[x, y, z]$ is called a *G-set* if

(1) it contains the constant monomial 1,
(2) if $vw \in \Gamma$ then $v \in \Gamma$ and $w \in \Gamma$,
(3) the restriction of the function wt to Γ is a bijection.

Remark. Since $\mathrm{wt}(1) = \mathrm{wt}(yz)$, it follows that $yz \notin \Gamma$ for any G-set Γ. Hence the monomials in Γ are of the form x^*y^* and x^*z^*, where $*$ stands for any non-negative integer.

DEFINITION 2.3. For any G-set Γ define $i(\Gamma), j(\Gamma), k(\Gamma)$ to be the unique non-negative integers such that

$$\begin{aligned} x^{i(\Gamma)} &\in \Gamma, & x^{i(\Gamma)+1} &\notin \Gamma, \\ y^{j(\Gamma)} &\in \Gamma, & y^{j(\Gamma)+1} &\notin \Gamma, \\ z^{k(\Gamma)} &\in \Gamma, & z^{k(\Gamma)+1} &\notin \Gamma. \end{aligned}$$

When no confusion arise we write for short:

$$\begin{aligned} i &= i(\Gamma), \\ j &= j(\Gamma), \\ k &= k(\Gamma). \end{aligned}$$

DEFINITION 2.4 (Nakamura). A monomial $x^m y^n$ (resp. $x^m z^n$) for $m, n \geq 0$ is called a *y-valley* (resp. *z-valley*) for Γ, if

$$x^m y^n, x^{m+1} y^n, x^m y^{n+1} \in \Gamma \quad \text{but} \quad x^{m+1} y^{n+1} \notin \Gamma$$
$$(\text{resp. } x^m z^n, x^{m+1} z^n, x^m z^{n+1} \in \Gamma \ \text{ but } \ x^{m+1} z^{n+1} \notin \Gamma).$$

We call a y-valley or z-valley a valley for brevity.

DEFINITION 2.5. For any $v \in M_0^0$ let $\mathrm{wt}_\Gamma(v)$ denote the unique $w \in \Gamma$ such that $\mathrm{wt}(v) = \mathrm{wt}(w)$.

3. Classification of G-sets. In this section, we show that any G-set has at most one y-valley and at most one z-valley. Following Nakamura, for every G-set we construct a semigroup $\mathrm{S}(\Gamma)$ in the lattice M and prove that it is saturated. It turns out that the G-sets correspond to the cones of maximal dimension in the fan of $\mathrm{Hilb}^G \mathbb{C}^3$.

REMARK 3.1. The following statements are immediate from the definitions:
(1) if $\mathrm{wt}_\Gamma(v) = w,\ v \notin \Gamma$ and $u \cdot w \in \Gamma$, then $u \cdot v \notin \Gamma$,
(2) if $\mathrm{wt}_\Gamma(v) = w$, then $\mathrm{wt}_\Gamma(u \cdot v) = u \cdot w$ for any $u \in M_0$ such that $u \cdot w \in \Gamma$,
(3) if $\mathrm{wt}_\Gamma(v) = w,\ u \in M$ then $\mathrm{wt}_\Gamma(u \cdot v) = w$.

COROLLARY 3.2. *Let Γ be a G-set and $v \in M_0^0 - \Gamma$. If $x^{-1} \cdot v \in \Gamma$ (resp. $y^{-1} \cdot v \in \Gamma, z^{-1} \cdot v \in \Gamma$) then $\mathrm{wt}_\Gamma(v) = w$, where $w \in \Gamma$ but $x^{-1} \cdot w \notin \Gamma$ (resp. $z \cdot w \notin \Gamma, y \cdot w \notin \Gamma$).*

Proof. Use observation (1) and (3) from Remark 3.1. □

LEMMA 3.3. *A G-set can only have* 0, 1 *or* 2 *valleys.*

Proof. Suppose that $x^m y^n$ is a $y-$valley for Γ. Then $v = x^{m+1} y^{n+1}$ satisfies assumptions of Corollary (3.2). Hence, $x^{-1} \cdot \mathrm{wt}_\Gamma(v) \notin \Gamma$ and $z \cdot \mathrm{wt}_\Gamma(v) \notin \Gamma$, so $\mathrm{wt}_\Gamma(v) = z^{k(\Gamma)}$. Therefore, G-set Γ has at most one y-valley, and, analogously at most one z-valley. □

COROLLARY 3.4. *Suppose that G-set Γ has y-valley w and z-valley v. Then*

$$\begin{aligned} \mathrm{wt}_\Gamma(y^{j(\Gamma)+1}) &= x \cdot w, \\ \mathrm{wt}_\Gamma(z^{k(\Gamma)+1}) &= x \cdot v. \end{aligned}$$

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Proof. Use observation (2) from Remark 3.1. □

NOTATION 3.5. *From now on we will usually denote by i_y, j_y the exponents of the y-valley $x^{i_y}y^{j_y}$ and by i_z, k_z the exponents of the z-valley $x^{i_z}z^{k_z}$ of some fixed G-set Γ.*

LEMMA 3.6. *The only possible G-sets with no valleys are*

$$\Gamma^{x} = \{1, x, \dots, x^{r-1}\},$$
$$\Gamma_l^{yz} = \{y^{r-l-1}, \dots, y, 1, z, \dots, z^l\} \text{ for } l = 0, \dots r-1.$$

Proof. Let i, j, k be integers like in Definition 2.3. Corollary 3.2 shows that $\mathrm{wt}_\Gamma(y^{j+1}) = x^{i'}z^k$, for some $i' \geq 0$. If $i' = 0$, then $\mathrm{wt}(z^{k+1}) = \mathrm{wt}(y^j)$ and since a, r are coprime, it follows that $j = r-k-1$, hence $i = 0$. Consider the case $i' > 0$. Then $\mathrm{wt}_\Gamma(x^{i'-1}z^{k+1}) = x^{i''}y^j$ by Corollary 3.2. It follows immediately that $i'' = i = r-1$ and so $j = k = 0$. □

LEMMA 3.7. *Let Γ be a G-set with exactly one valley. If Γ has y-valley equal to $x^{i_z}z^{k_z}$, then*

$$\mathrm{wt}_\Gamma(x^{i+1}) = z^{k-k_z},$$
$$\mathrm{wt}_\Gamma(z^{k+1}) = x^{i-i_z}y^j.$$

If G-set Γ z-valley equal to $x^{i_y}y^{j_y}$, then

$$\mathrm{wt}_\Gamma(x^{i+1}) = y^{j-j_y},$$
$$\mathrm{wt}_\Gamma(y^{j+1}) = x^{i-i_y}z^k.$$

Proof. We prove the lemma in the case of z-valley $w = x^{i_z}z^{k_z}$. The monomial $\mathrm{wt}_\Gamma(z^{k+1})$ is of the form $x^l y^j$, where $0 \leq l \leq i$. Noting that $\mathrm{wt}_\Gamma(xz \cdot w) = y^j$ we get $l = i - i_z$. It follows that the monomials $x^{i-i_z}y^j$ and z^{k+1} are of the same weight, therefore $\mathrm{wt}_\Gamma(z^{k+1}) = x^{i-i_z}y^j$. □

LEMMA 3.8. *Let Γ be a G-set with two valleys v, w, where*

$$v = x^{i_y}y^{j_y},$$
$$w = x^{i_z}z^{k_z}.$$

Then $i_y + i_z + 1 = i$, and

$$\mathrm{wt}_\Gamma(x^{i+1}) = \begin{cases} y^{(j-j_y)-(k-k_z)} & \text{if } (j-j_y)-(k-k_z) \geq 0, \\ z^{(k-k_z)-(j-j_y)} & \text{otherwise.} \end{cases}$$

Proof. Let u be a monomial such that $u \notin \Gamma$ and $x^{-1}u \in \Gamma$. Then $\mathrm{wt}_\Gamma(u) = z^l$ for some $0 \leq l \leq k$ or $\mathrm{wt}_\Gamma(u) = y^l$ for $0 \leq l \leq j$. We know already that $\mathrm{wt}_\Gamma(xz \cdot w) = y^j$ and $\mathrm{wt}_\Gamma(xy \cdot v) = z^k$, which implies that $\mathrm{wt}_\Gamma(x^{i+1}) = y^{(j-j_y)-(k-k_z)}$ if $(j-j_y)-(k-k_z) \geq 0$ and $\mathrm{wt}_\Gamma(x^{i+1}) = z^{(k-k_z)-(j-j_y)}$ otherwise. The monomial $x^{i_y+i_z+1}$ has the same weight as x^{i+1} hence they are equal. □

DEFINITION 3.9 (Nakamura). For any $v \in M_0$ and a G-set Γ define (using a multiplicative notation in the lattice M_0)

$$s_\Gamma(v) = v\,\mathrm{wt}_\Gamma^{-1}(v).$$

We will write it simply $s(v)$ when no confusion can arise. Define the cones

$$\sigma(\Gamma) = \{\alpha \in N_0 \otimes_{\mathbb{Z}} \mathbb{R} | \langle \alpha, s_\Gamma(v)\rangle \geq 0, \quad \forall v \in M_0^0\},$$
$$\sigma^\vee(\Gamma) = \{v \in M_0 \otimes_{\mathbb{Z}} \mathbb{R} | \langle \alpha, v\rangle \geq 0, \quad \forall \alpha \in \sigma(\Gamma)\},$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between N_0 and M_0.

Let $\mathrm{S}(\Gamma)$ be a sub-semigroup of the lattice M, generated by the set $\{s_\Gamma(v) \in M \mid v \in M_0^0\}$ as a semigroup. Set

$$\mathrm{V}(\Gamma) = \mathrm{Spec}\, \mathbb{C}[\mathrm{S}(\Gamma)].$$

Note that

$$\mathbb{C}[\mathrm{S}(\Gamma)] \subset \mathbb{C}[\sigma^\vee(\Gamma) \cap M].$$

Moreover, the cones $\sigma(\Gamma)$, $\sigma^\vee(\Gamma)$ are dual to each other and the cone $\sigma^\vee(\Gamma) \cap M$ is the saturation of the semigroup $\mathrm{S}(\Gamma)$ in the lattice M. It will follow from Lemma (3.11) that $S(\Gamma)$ is finitely generated as a semigroup.

THEOREM 3.10 (Nakamura). *Let G be a finite abelian subgroup of* $\mathrm{GL}(3, \mathbb{C})$. *When Γ varies through all G-sets the set of all faces of all three-dimensional cones $\sigma(\Gamma)$ forms a fan in lattice $N \otimes \mathbb{R}$ supported on the positive octant. Toric variety defined by this fan is isomorphic to the normalisation of the* $\mathrm{Hilb}^G\,\mathbb{C}^3$ *scheme (see* [**7**, Theorem 2.11] *and* [**1**, Section 5]*). Moreover, the affine varieties* $\mathrm{V}(\Gamma)$ *form an open covering of the* $\mathrm{Hilb}^G\,\mathbb{C}^3$ *scheme when Γ varies through all G-sets.*

LEMMA 3.11 (Nakamura). *Let $A \subset M_0^0 - \Gamma$ be a finite set such that $M_0^0 - \Gamma = A \cdot M_0^0$. If $\sigma(\Gamma)$ is a three-dimensional cone then* $\mathrm{S}(\Gamma)$ *is generated by the finite set $\{\, s_\Gamma(v) \mid v \in A \,\}$ as a semigroup (see* [**7**, Lemma 1.8]*).*

REMARK 3.12. Note that Theorem 3.10 and Lemma 3.11 are stated in [**7**] without the assumption on dimension of $\sigma(\Gamma)$ in which case they are false. A counter-example and a correction can be found in [**2**, Example 4.12 and Theorem 5.2].

LEMMA 3.13. *Suppose that Γ is a G-set in the case of $\frac{1}{r}(1, a, r-a)$ action. Then the cone $\sigma(\Gamma)$ is three-dimensional. Moreover, if Γ has* 0 *or* 1 *valley then* $\mathrm{S}(\Gamma) \cong \mathbb{C}[x, y, z]$. *If Γ has* 2 *valleys then* $\mathrm{S}(\Gamma) \cong \mathbb{C}[x, y, z, w]/(xy - zw)$.

Proof. The lemma will be proven only in the case of a G-set with 2 valleys as the method carries over to the other cases.

Suppose that Γ is a G-set with 2 valleys, $v = x^{i_y}y^{j_y}$, $w = x^{i_z}z^{k_z}$ and set

$$\begin{aligned}\alpha &= x^{i+1},\\ \beta &= y^{j+1},\\ \gamma &= z^{k+1},\\ \delta_y &= xy\cdot v,\\ \delta_z &= xz\cdot w,\end{aligned}$$

where i, j, k are the largest exponents such that x^i, y^j, z^k belong to Γ. We will start by showing that $s(\beta), s(\gamma), s(\delta_y)$ and $s(\delta_z)$ generate semigroup $\mathrm{S}(\Gamma)$. Assume that $u \in M_0^0$, $t = x, y$ or z and note that

$$s(t\cdot u) = s(u)s(t\cdot \mathrm{wt}_\Gamma(u)).$$

By the above formula it suffices to show that for any $u \in \Gamma$ such that $t\cdot u \notin \Gamma$ the Laurent monomial $s(t\cdot u)$ can be expressed as a product of $s(\beta), s(\gamma), s(\delta_y)$ and $s(\delta_z)$ with non-negative exponents. By Lemma 3.8,

$$s(\alpha) = \begin{cases} x^{i+1}y^{-(j-j_y)+(k-k_z)} & \text{if } (j-j_y)\geq (k-k_z),\\ x^{i+1}z^{(j-j_y)-(k-k_z)} & \text{otherwise,}\end{cases}$$

$$\begin{aligned}s(\beta) &= xy^{-(j+1)}\cdot w,\\ s(\gamma) &= xz^{-(k+1)}\cdot v,\\ s(\delta_y) &= xyz^{-k}\cdot v,\\ s(\delta_z) &= xy^{-j}z\cdot w,\end{aligned}$$

hence

$$s(\beta)s(\delta_z) = s(\gamma)s(\delta_y) = s(yz),$$

$$s(\alpha) = \begin{cases} s(\delta_y)s(\delta_z)(yz)^{j-j_y-1} & \text{if } (j-j_y)\geq (k-k_z),\\ s(\delta_y)s(\delta_z)(yz)^{k-k_z-1} & \text{otherwise.}\end{cases}$$

Let $u \in \Gamma$ and $y\cdot u \notin \Gamma$. If $u = x^l y^j$, where $l = 0, \ldots, i_y$ then $s(y\cdot u) = s(\beta)$. If $u = x^l y_y^j$, where $l = i_y + 1, \ldots, i$ then $s(y\cdot u) = s(\delta_y)$. Analogously $s(z\cdot u)$ is equal to $s(\gamma)$ or to $s(\delta_y)$ for any $u \in \Gamma$, $z\cdot u \notin \Gamma$.

It remains to consider $u \in \Gamma$ such that $x\cdot u \notin \Gamma$. Observe that $\mathrm{wt}_\Gamma(x\cdot u)$ is of the form y^l or z^l for some positive l ($l = 0$ can happen only if $\Gamma = \Gamma^{\mathrm{x}}$). If $u' = y^{-1}u \in \Gamma$ then $x\cdot u' \notin \Gamma$ and

$$s(x\cdot u) = s(y\cdot xu') = s(xu')s(y\,\mathrm{wt}_\Gamma(x\cdot u')) = s(xu')(yz)^n, \text{ where } n = 0, 1.$$

By induction for any such $u \in \Gamma$ the monomial $s(x\cdot u)$ is equal to $p\cdot (xy)^m$, where $m > 0$ and $p = s(\alpha), s(\delta_y)$ or $s(\delta_z)$.

This shows that $\mathrm{S}(\Gamma)$ is generated by $s(\beta), s(\gamma), s(\delta_y)$ and $s(\delta_z)$. To conclude it is enough to show that some (in fact any) 3 out of 4 generators form a $\mathbb{Z}$-basis of the lattice M. This is implied by computing the following determinant, using equality from

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Lemma 3.8:

$$\begin{vmatrix} -i_z - 1 & j+1 & -k_z \\ i_y + 1 & j_y + 1 & -k \\ -i_y - 1 & -j_y & k+1 \end{vmatrix} = r.$$

□

COROLLARY 3.14. *The semigroup* $\mathrm{S}(\Gamma)$ *coincides with the semigroup algebra* $\mathbb{C}[\sigma^\vee(\Gamma') \cap M]$ *for any G-set. In particular,* $\mathrm{Hilb}^G\,\mathbb{C}^3$ *is normal.*

4. *G*-igsaw transformations. To get an effective description of the fan of the Hilb^G scheme, we introduce Nakamura's G-igsaw transformation, which will allow to organise G-sets in families and to explain how these are related to each other.

G-igsaw transformation is a method of constructing a new G-set from the other. In fact, two G-sets Γ and Γ' are related by a G-igsaw transformation if and only if the cones $\sigma(\Gamma)$ and $\sigma(\Gamma')$ share a two-dimensional face.

When reading Sections 4–6, it may be useful for a reader to consult an example provided in Section 7.

LEMMA 4.1 (Nakamura). *Let* Γ *be a G-set for the action of type* $\frac{1}{r}(1, a, r-a)$ *and let* τ *be a two-dimensional face of* $\sigma(\Gamma)$*. There exist two monomials* $u \in M_0^0$ *and* $v \in \Gamma$ *such that*

(1) $v = \mathrm{wt}_\Gamma(u)$,
(2) u, v *do not have common factors in* M_0^0,
(3) uv^{-1} *is a primitive monomial,*
(4) $\tau = \sigma(\Gamma) \cap (uv^{-1})^\perp$,

Proof. This is a particular case of [**7**, Lemma 2.5] □

DEFINITION 4.2 (Nakamura). Let Γ be a G-set and let τ be a two-dimensional face of $\sigma(\Gamma)$. Suppose that monomials u, v given by Lemma 4.1 are not equal to 1 and set $c(w) = \max\{c \in \mathbb{Z} \mid wv^{-c} \in M_0^0\}$ for any $w \in \Gamma$. We define the G-igsaw transformation *of* Γ in the direction of τ to be the set

$$\Gamma' = \{w \cdot u^{c(w)} v^{-c(w)} \mid w \in \Gamma\}.$$

LEMMA 4.3 (Nakamura). *The G-igsaw transformation of a G-set is a G-set.*

Proof. See [**7**, Lemma 2.8] □

LEMMA 4.4. *Suppose that* Γ *is a G-set for the action* $\frac{1}{r}(1, a, r-a)$*. Let* $\alpha = x^{i+1}$, $\beta = y^{j+1}$, $\gamma = z^{k+1}$, *where i, j, k are the maximal exponents such that* $x^i, y^j, z^k \in \Gamma$*. Let* τ *be a two-dimensional face of* $\sigma(\Gamma)$ *and let u be the monomial given by Lemma 4.1. If* Γ *has* 0 *or* 1 *valley then* $u = \alpha, \beta$ *or* γ*. If* Γ *has* 2 *valleys then* $u = \beta, \gamma, \delta_y$ *or* δ_z*, where* δ_y *is equal to the y-valley of* Γ *multiplied by xy and* δ_z *is equal to the z-valley of* Γ *multiplied by xz.*

Proof. Suppose that Γ has one valley and τ is a face of $\sigma(\Gamma)$ dual to the ray of $\sigma^\vee(\Gamma)$ spanned by $s(\alpha)$. The one-dimensional lattice $M \cap \tau^\perp$ has 2 generators. Therefore uv^{-1} is equal either to $s(\alpha)$ or $s(\alpha)^{-1}$. Clearly, the only choice is $u = \alpha$, $v = \mathrm{wt}_\Gamma(\alpha)$. Suppose

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that $d \in M_0^0$ is a common factor of u and v. Then both ud^{-1}, vd^{-1} belong to Γ and they are of the same weight. Hence $d = 1$. □

DEFINITION 4.5. Let Γ be a G-set with 0 or 1 valley and let τ be the two-dimensional face of $\sigma(\Gamma)$. The G-igsaw transformation of Γ in the direction of τ is called *upper (resp. right, left) transformation* if $u = \alpha$ (resp. $u = \beta, u = \gamma$.), where the monomial u is as in Lemma 4.1. The upper, left and right transformations of Γ will be denoted by $T_U(\Gamma)$, $T_R(\Gamma)$ and $T_L(\Gamma)$, respectively.

By slight abuse of notation, the G-igsaw transformation of G-set Γ with 2 valleys is called *left (resp. upper left, right, left) transformation* if the corresponding monomial u is equal to β (resp. $\gamma, \delta_y, \delta_z$). The right, left, upper right and upper left G-igsaw transformations of Γ will be denoted by $T_{UR}(\Gamma)$, $T_{UL}(\Gamma)$, $T_R(\Gamma)$, $T_L(\Gamma)$, respectively.

DEFINITION 4.6. We say that a G-set Γ is *spanned* by monomials $u_1, \ldots, u_n$ if Γ consists of all monomials dividing $u_1, \ldots, u_n$. If G-set Γ is spanned by monomials $u_1, \ldots, u_n$ we write

$$\Gamma = \operatorname{span}(u_1, \ldots, u_n).$$

LEMMA 4.7. *Let* $\Gamma = \operatorname{span}(x^{i_y}y^j, x^iz^k)$, *where* $i_y < i$ *(resp. let* $\Gamma = \operatorname{span}(x^iy^j, x^{i_k}z^k)$, *where* $i_z < i$*) be a G-set with one y-valley equal to* x^{i_y} *(resp. one z-valley equal to* x^{i_z}*). Then*

$$T_U(\Gamma) = \operatorname{span}(x^{i+i_y+1}, x^{i_y}y^{j-1}, x^iz^k),$$
$$(resp. T_U(\Gamma) = \operatorname{span}(x^{i+i_z+1}, x^iy^j, x^{i_k}z^{k-1})).$$

In particular, the upper transformation of Γ *has*

- *no valleys if and only if* $j = 1, k = 0$ *(resp.* $j = 0, k = 1$*). In fact, in this case* $T_U(\Gamma) = \Gamma^{\mathrm{x}}$.
- *one z-valley (resp. one y-valley) if and only if* $j = 1, k > 0$ *(resp.* $j > 0, k = 1$*). In both cases the valley is equal to* x^i.
- *two valleys: the y-valley equal to* x^{i_y} *and the z-valley equal to* x^i *(resp. the y-valley equal to* x^i *and the z-valley equal to* x^{i_z}*) in the remaining cases.*

Proof. The upper transformation is obtained by replacing each monomial $w \in \Gamma$, divisible by y^j (resp. by z^k) by the monomial $x^{n(i+1)}y^{-nj} \cdot w$ for some $n \geq 1$. The proof is straightforward. □

LEMMA 4.8. *Let* Γ *be a G-set with 2 valleys: y-valley equal to* $v = x^{i_y}y^{j_y}$ *and z-valley equal to* $w = x^{i_z}z^{k_z}$. *Assume that* Γ *is spanned by* $x^iy^{j_y}, x^iz^{k_z}, x^{i_y}y^j, x^{i_z}z^k$. *Let T stand for right, left, upper right or upper left transformation.*

Then $T(\Gamma)$ *is spanned by*

$$\begin{array}{lllll} x^iy^{j_y}, & x^iz^{k_z-1}, & x^{i_y}y^{j+1}, & x^{i_z}z^k & T = T_R, k_z \geq 1, \\ x^iy^{j_y-1}, & x^iz^{k_z}, & x^{i_y}y^j, & x^{i_z}z^{k+1} & T = T_L, j_y \geq 1, \\ x^iy^{j_y+1}, & x^iz^{k_z}, & x^{i_y}y^j, & x^{i_z}z^{k-1} \quad \text{if} & T = T_{UR}, \\ x^iy^{j_y}, & x^iz^{k_z+1}, & x^{i_y}y^{j-1}, & x^{i_z}z^k & T = T_{UL}. \end{array}$$

Proof. The proof is a matter of straightforward computation. It follows directly by considering each case separately cf. Lemma 4.4). □

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Note that the G-igsaw transformation of a G-set with two valleys may have only one valley.

COROLLARY 4.9. *Let Γ be a G-set spanned by $x^i y^{j_y}$, $x^i z^{k_z}$, $x^{i_y} y^j$, $x^{i_z} z^k$ with 2 valleys: y-valley equal to $v = x^{i_y} y^{j_y}$ and z-valley equal to $w = x^{i_z} z^{k_z}$. If $j_y, k_z \geq 1$ then*

$$T_R(T_{UL}(\Gamma)) = \Gamma,\ T_{UL}(T_R(\Gamma)) = \Gamma,$$
$$T_L(T_{UR}(\Gamma)) = \Gamma,\ T_{UR}(T_L(\Gamma)) = \Gamma,$$

that is right and upper left (resp. left and upper right) transformations are inverse operations. Moreover, if $j, k, j - j_y, k - k_z \geq 2$ then

$$T_{UL}(T_{UR}(\Gamma)) = T_{UR}(T_{UL}(\Gamma)),$$

that is, upper left and upper right transformations commute.

COROLLARY 4.10. *Let Γ be a G-set spanned by $x^i y^{j_y}$, $x^i z^{k_z}$, $x^{i_y} y^j$, $x^{i_z} z^k$, with 2 valleys: y-valley equal to $v = x^{i_y} y^{j_y}$ and z-valley equal to $w = x^{i_z} z^{k_z}$. Let $\Gamma' = T^m_{UR}(T^n_{UL}(\Gamma))$, where $m + n \leq \min\{j, k, j - j_y, k - k_z, \}$. Then Γ' is spanned by $x^i y^{j_y+m}$, $x^i z^{k_z+n}$, $x^{i_y} y^{j-n}$, $x^{i_z} z^{k-m}$. If $m + n < \min\{j, k, j - j_y, k - k_z\}$ then Γ' has two valleys. If $m + n = \min\{j, k, j - j_y, k - k_z\}$ then Γ' has one valley (one of the monomials $x^i y^{j_y+m}$, $x^i z^{k_z+n}$, $x^{i_y} y^{j-n}$, $x^{i_z} z^{k-m}$ spanning Γ' is redundant).*

5. Triangles of transformations and primitive G-sets.

In this section, we introduce primitive G-sets, which have a particular shape. Every primitive G-set such gives rise to a family of G-sets, called here a triangle of transformations. It will turn out that most G-sets belong to some triangle of transformations. We define a sequence of primitive G-sets containing every primitive G-set for fixed integers r and a.

DEFINITION 5.1. Let Γ be a G-set with two valleys, spanned by $x^i y^{j_y}$, $x^i z^{k_z}$, $x^{i_y} y^j$, $x^{i_z} z^k$. The set

$$\Theta(\Gamma) = \{T^m_{UR}(T^n_{UL}(\Gamma)) \mid m + n \leq \min\{j, k, j - j_y, k - k_z\}\}$$

will be called *triangle of transformations* of Γ.

The union of the supports of G-sets belonging to the set $\Theta(\Gamma)$ is a simplicial cone (see Corollary 5.13), hence we call $\Theta(\Gamma)$ a triangle of transformations.

DEFINITION 5.2. A G-set Γ is called *primitive* if it has a y-valley equal to x^{i_y} and a z-valley equal to x^{i_z} for some non-negative i_y, i_z.

The name primitive is justified by the fact that every G-set with two valleys belong to a triangle of transformations of some primitive G-set. This fact will follow from the Main Theorem.

DEFINITION 5.3. For fixed coprime integers r, a define let Γ_1 be a G-set spanned by x, y^{b-1}, z^{r-b-1}, where $b \in \{1, \ldots, r-1\}$ is as an inverse of a modulo r.

The G-set Γ_1 is primitive and the monomial x is simultaneously its y-valley and z-valley.

LEMMA 5.4. *Let Γ be a primitive G-set spanned by x^i, $x^{i_y} y^j$, $x^{i_z} z^k$. Then $\Theta(\Gamma)$ consists of $\binom{\min\{j,k\}+2}{2}$ G-sets.*

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Proof. It is clear from definition of $\Theta(\Gamma)$. □

LEMMA 5.5. *Let Γ be a primitive G-set spanned by x^i, $x^{i_y}y^j$, $x^{i_z}z^k$. Suppose that $j < k$ (resp. $k < j$). The G-set $T_U(T^j_{UR}(\Gamma))$ (resp. $T_U(T^k_{UL}(\Gamma))$) is spanned by x^{i+i_z+1}, x^iy^j, $x^{i_z}z^{k-(j+1)}$ (resp. x^{i+i_y+1}, $x^iy^{j-(k+1)}$, x^iz^k). Moreover, if $j < k-1$ (resp. $k < j-1$) it is primitive.*

Proof. Assume that $j < k$. The G-set $T^j_{UR}(\Gamma)$ is spanned by x^iy^j, $x^{i_z}z^{k-j}$ and it has one z-valley equal to x^{i_z} by Lemma 4.8. To finish the proof apply Lemma 4.7 to the G-set $T^j_{UR}(\Gamma)$. □

The preceding lemma allows us to define a sequence of primitive G-sets.

DEFINITION 5.6. If Γ_n is a primitive G-set we set:

$$\Gamma_{n+1} = \begin{cases} T_U(T^{j_n}_{UR}(\Gamma_n)) & \text{if} \quad j_n < k_n, \\ T_U(T^{k_n}_{UL}(\Gamma_n)) & \text{if} \quad j_n > k_n, \end{cases}$$

where j_n, k_n denote the non-negative numbers such that Γ_n is spanned by the monomials x^{i_n}, $x^{i_{y,n}}y^{j_n}$, $x^{i_{z,n}}z^{k_n}$ for some $i_n, i_{y,n}, i_{z,n} \geq 0$.

Observe that if $j_n - k_n = \pm 1$ for some n then Γ_{n+1} is not primitive and the recursion stops.

COROLLARY 5.7. *The numbers j_n, k_n satisfy the following formulas:*

$$\begin{aligned} j_1 + 1 &= b, \\ k_1 + 1 &= r - b, \\ j_{n+1} + 1 &= \begin{cases} j_n + 1 & \textit{if} \quad j_n < k_n, \\ j_n + 1 - (k_n + 1) & \textit{if} \quad j_n > k_n, \end{cases} \\ k_{n+1} + 1 &= \begin{cases} k_n + 1 - (j_n + 1) & \textit{if} \quad j_n < k_n, \\ k_n + 1 & \textit{if} \quad j_n > k_n. \end{cases} \end{aligned}$$

Clearly, there is a direct link between the numbers $j_n + 1$, $k_n + 1$ and the numbers appearing in the Euclidean algorithm for b and $r - b$. This relationship will be exploited later.

DEFINITION 5.8. Let $\Theta(\Gamma)$ be a triangle of transformations of a G-set Γ. We define

$$\widetilde{\Theta}(\Gamma) = \bigcup_{\Gamma' \in \Theta(\Gamma)} \sigma(\Gamma')$$

to be the union of supports of the cones $\sigma(\Gamma')$, where Γ' runs through the G-sets in $\Theta(\Gamma)$.

To study the location of various cones in the fan $\operatorname{Hilb}^G \mathbb{C}^3$ it is convenient to give names to their rays.

DEFINITION 5.9. Let Γ_n be the primitive G-set as defined in (5.6). Denote by ρ_n the common ray of the cones $\widetilde{\Theta}(\Gamma_n)$ and $\sigma(\Gamma_n)$.

Let Γ be any G-set. A ray of $\sigma^\vee(\Gamma)$ will be called *upper, (upper) left or right ray* if it dual to the wall of $\sigma(\Gamma)$ corresponding to the upper,(upper) left or right transformation, respectively.

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REMARK 5.10. Let Γ, Γ' be any two G-sets. Suppose that the cones $\sigma(\Gamma)$ and $\sigma(\Gamma')$ intersect either in a two-dimensional face or in a ray. If the cones $\sigma^\vee(\Gamma)$, $\sigma^\vee(\Gamma')$ have a common ray ρ then there exists a two-dimensional linear subspace of $N \otimes \mathbb{R}$ containing a two-dimensional face of $\sigma(\Gamma)$ and of $\sigma(\Gamma')$, both of these dual to the ray ρ.

LEMMA 5.11. *For any G-set Γ with two valleys the set $\widetilde{\Theta}(\Gamma)$ is a rational simplicial cone.*

Proof. Assume that G-set is spanned by the monomials $x^i y^{j_y}$, $x^i z^{k_z}$, $x^{i_y} y^j$, $x^{i_z} z^k$ and let $l = \min j, k, j - j_y, k - k_z$. Because the upper right and upper left transformation commute (see Corollary 4.9), by Remark 5.10 it is enough to establish the three following facts:

- the right rays of the cones $\sigma^\vee(T^n_{UR}(\Gamma))$ for $n = 0, \ldots, l$ are the same,
- the left rays of the cones $\sigma^\vee(T^n_{UL}(\Gamma))$ for $n = 0, \ldots, l$ are the same,
- the upper rays of the cones $\sigma^\vee(T^m_{UR}(T^n_{UL}(\Gamma)))$ for $m + n = l$ are the same.

These follow from Corollary 4.10. □

LEMMA 5.12. *Let Γ be a primitive G-set spanned by x^i, $x^{i_y} y^j$, $x^{i_z} z^k$. If $j < k$ (resp. $k < j$) then $\mathbb{R}_+ e_2$ (resp. $\mathbb{R}_+ e_3$) is a ray of $\widetilde{\Theta}(\Gamma)$.*

Proof. Suppose that $j < k$. The G-set $\Gamma' = T^j_{UL}(\Gamma)$ is spanned by the monomials $x^i z^{k_z + j}$, $x^{i_z} z^k$ and it has one valley (see Corollary 4.10). The upper and left ray of $\sigma(\Gamma')$ are equal to $x^{i+1} z^{-k+k_z}$ and $x^{-i+i_z} z^{k+1}$, respectively. Evidently, the ray of $\sigma \vee (\Gamma')$, dual to the two-dimensional face of $\sigma^\vee(\Gamma)$ spanned by the upper and left ray, is equal to $\mathbb{R}_+ e_2$. □

Note that the cone $\widetilde{\Theta}(\Gamma_i)$ has, besides the ray common with $\sigma(\Gamma_i)$, two other rays: one equal to either e_2 or e_3 and the second which belongs to $\sigma(\Gamma_{i+1})$. We will investigate how the cones $\widetilde{\Theta}(\Gamma_i)$, $\widetilde{\Theta}(\Gamma_{i+1})$ fit together depending on the sign of $(j_i - k_i)(j_{i+1} - k_{i+1})$.

COROLLARY 5.13. *Let Γ_n and Γ_{n+1} be two primitive G-sets. If $(j_n - k_n)(j_{n+1} - k_{n+1}) > 0$ then the union of the supports of the cones $\widetilde{\Theta}(\Gamma_n)$, $\widetilde{\Theta}(\Gamma_{n+1})$ is a rational simplicial cone.*

LEMMA 5.14. *Let Γ_n and Γ_{n+1} be two primitive G-sets. Then $\widetilde{\Theta}(\Gamma_n) \cup \widetilde{\Theta}(\Gamma_{n+1})$ is equal to the cone spanned by ρ_n, e_2, e_3 minus (set-theoretical) the cone spanned by ρ_{n+1}, e_2, e_3.*

Proof. If $(j_n - k_n)(j_{n+1} - k_{n+1}) > 0$ this follows from Corollary 5.13. Otherwise, the cones $\widetilde{\Theta}(\Gamma_n)$, $\widetilde{\Theta}(\Gamma_{n+1})$ have a common ray and a two-dimensional face of $\widetilde{\Theta}(\Gamma_{n+1})$ is contained in a two-dimensional face of $\widetilde{\Theta}(\Gamma_n)$. To finish, note that e_2 and e_3 generate rays of $\widetilde{\Theta}(\Gamma_n)$ and $\widetilde{\Theta}(\Gamma_{n+1})$ (up to the order). □

Recall that $\Gamma^{yz}_l = \operatorname{span}(y^l, z^{r-l-1})$. We will prove that the cones $\sigma(\Gamma^{yz}_l)$ fit nicely together with the cones $\widetilde{\Theta}(\Gamma_j)$ into the fan of $\operatorname{Hilb}^G \mathbb{C}^3$.

LEMMA 5.15. *The upper transformations of Γ^{yz}_{b-1} and Γ^{yz}_b coincide, where $b \in \{1, \ldots, r-1\}$ is an inverse of a modulo r. In fact, they are equal to Γ_1.*

Proof. By definition, the upper transformation of Γ^{yz}_{b-1} and Γ^{yz}_b replaces the monomial z^{r-b} and y^b with the monomial x, respectively. □

LEMMA 5.16. *The upper rays of the cones $\sigma^\vee(\Gamma^{yz}_0), \ldots \sigma^\vee(\Gamma^{yz}_{b-1})$ (resp. $\sigma^\vee(\Gamma^{yz}_b), \ldots \sigma^\vee(\Gamma^{yz}_{r-1})$) are equal. The one-dimensional cone $\mathbb{R}_{\geq 0} e_1$ is a ray of each the cones $\sigma(\Gamma^{yz}_i)$, for $i = 0, \ldots, r-1$.*

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Proof. The upper ray of the cones $\sigma^\vee(\Gamma_0^{yz}), \dots \sigma^\vee(\Gamma_{b-1}^{yz})$ is spanned by xz^{-r+b} and the upper ray of $\sigma^\vee(\Gamma_b^{yz}), \dots \sigma^\vee(\Gamma_{r-1}^{yz})$ is spanned by xy^b. The right and left rays of $\sigma^\vee(\Gamma_l^{yz})$ are equal to $y^{-l}z^{r-l}$, $y^{l+1}z^{r-l-1}$, therefore $\mathbb{R}_+e_1$ is a ray of $\sigma(\Gamma_l^{yz})$. □

COROLLARY 5.17. *The sets*

$$\bigcup_{l=0}^{b-1} \sigma(\Gamma_l^{yz}), \quad \bigcup_{l=b}^{r-1} \sigma(\Gamma_l^{yz})$$

are rational cones in $N \otimes \mathbb{R}$ *spanned by* e_1, e_2, ρ_1 *and* e_1, e_3, ρ_1, *respectively.*

Proof. This follows from Remark 5.10 and Lemma 5.16. □

6. Main theorem and the Euclidean algorithm. By Theorem 3.10, when Γ varies through all G-sets, the cones $\sigma(\Gamma)$ form a fan supported on the cone spanned by e_1, e_2, e_3. Therefore, it is enough to find G-sets different from the G-set Γ_l^{yz} which does not belong to any triangle of transformation. By looking at the supports of triangle transformations, it will turn out that those missing G-sets are exactly the upper transformations of the last G-set Γ_n defined in (5.6). With the help of the Euclidean algorithm we will be able to give a formula for a total number of G-set for fixed r and a.

DEFINITION 6.1. Let m be an integer such that Γ_{m+1} is not primitive (i.e. Γ_m is the last primitive G-set in the sequence defined in (5.6)).

THEOREM 6.2 (Main Theorem). *Let r, a be coprime natural numbers and let b be an inverse of a modulo r. Let G be a cyclic group of order r, acting on $\mathbb{C}^3$ with weights* $1, a, r-a$.

If $\Gamma_1, \dots, \Gamma_{m+1}$ *is the sequence from Definition 5.6 (that is,* Γ_n *is a primitive G-set unless $n = m+1$) and if* $\Gamma_l^{yz} = \operatorname{span}(y^{r-l-1}, z^l)$ *then every G-set either*

- *belongs to a triangle of transformation of some* Γ_n *for* $n \leq m$, *or*
- *is equal to a G-set* Γ_l^{yz} *for some* $l = 1, \dots, n$, *or*
- *is equal to an iterated upper transformation of the G-set* Γ_{m+1}.

Proof. The proof uses Nakamura's Theorem 3.10, which asserts that the union of the supports of the cones $\sigma(\Gamma)$ is equal to the positive octant in $N \otimes \mathbb{R}$. Lemma 5.14 and Corollary 5.17 combined imply that if a G-set Γ neither belongs to some triangle of transformation nor is equal to Γ_l^{yz} for some l then the cone $\sigma(\Gamma)$ is supported in the cone spanned by e_2, e_3, ρ_{m+1}. On the other hand, the G-set Γ_{m+1} is equal either to $\operatorname{span}(x^{i_{m+1}}, x^{i_{y_{m+1}}}y^{j_{m+1}})$ or to $\operatorname{span}(x^{i_{m+1}}, x^{i_{z,m+1}}z^{k_{m+1}})$, cf. Lemma 5.5. Therefore the j_{m+1}th or k_{m+1}th iterated upper transformation of Γ_{m+1} is equal to $\Gamma^x = \operatorname{span}(x^{r-1})$. Moreover, the G-sets $T_U^l(\Gamma_{m+1})$ and $T_U^{l+1}(\Gamma_{m+1})$ satisfy assumptions of the Remark 5.10. This shows that the set

$$\bigcup_{l=0}^{\max\{j_{m+1}, k_{m+1}\}} \sigma(T_U^l(\Gamma_{m+1}))$$

is a cone generated by e_2, e_3, ρ_{m+1} which concludes the proof. □

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REMARK. The above theorem can be restated in a form of an algorithm computing the fan of the $\mathrm{Hilb}^G\,\mathbb{C}^3$ for fixed a and r (recall that the $\mathrm{Hilb}^G\,\mathbb{C}^3$ is normal, cf. Corollary 3.14).

REMARK. The two-stage construction of the $\mathrm{Hilb}^G\,\mathbb{C}^3$ for abelian subgroups in $\mathrm{SL}(3,\mathbb{C})$ by Craw and Reid in [**3**] appears to provide a coarse subdivision of the fan of the $\mathrm{Hilb}^G\,\mathbb{C}^3$ for the subgroup G in $\mathrm{GL}(3,\mathbb{C})$ of type $\frac{1}{r}(1, a, r-a)$. The coarse subdivision (i.e. with all interior lines of all triangles of transformations removed) is provided by the continued fraction expansions.

LEMMA 6.3. *Let p_l, q_l be the data of the Euclidean algorithm for the non-negative integer numbers p_1, p_2 with* $\mathrm{GCD}(p_1, p_2) = p_{n+1}$*, that is,*

$$p_i = q_i p_{i+1} + p_{i+2}, \quad 0 < p_{i+2} < p_{i+1},$$

where $p_{n+1} \neq 0$ and $p_{n+2} = 0$.

Then

$$\sum_{l=1}^{n} q_l p_{l+1} = p_1 + p_2 - p_{n+1},$$
$$\sum_{l=1}^{n} q_l p_{l+1}^2 = p_1 p_2.$$

THEOREM 6.4. *Fix some coprime numbers r and a. Let N denote the number of different G-sets for the action of type $\frac{1}{r}(1, a, r-a)$. Then*

$$N = \frac{1}{2}(3r + b(r-b) - 1).$$

Proof. Denote $\Gamma_l = \mathrm{span}(x^{i_l}, x^{i_{y,l}} y^{j_l}, x^{i_{z,l}} z^{k_l})$. The triangle of transformations of Γ_l consist of $\binom{\min\{j_l+1, k_l+1\}+1}{2}$ cones (see Lemma 5.4). Therefore

$$N = r + \max\{j_{m+1}+1, k_{m+1}+1\} + \sum_{l=1}^{m} \binom{\min\{j_l+1, k_l+1\}+1}{2},$$

where the first two terms come from the G-sets Γ_l^{yz} and the consecutive upper transformations of Γ_{m+1}.

Suppose that $b < r-b$. Let the p_l and q_l be the data of the Euclidean algorithm for the coprime numbers $p_1 = k_1 + 1 = r - b, p_2 = j_1 + 1 = b$ as in Lemma 6.3. Set $q_0 = 1$. In this notation, by the formulas from Corollary 5.7,

$$\min\{j_C + 1, k_C + 1\} = p_D$$
$$\text{for} \quad q_0 + \dots q_D \leq C < q_0 + \dots q_{D+1}.$$

Note that $p_{n+1} = 1$ and $q_n = \max\{j_{m+1}+1, k_{m+1}+1\}$, thus $N = r + q_n p_{n+1} + \frac{1}{2}\sum_{l=1}^{n-1}(q_l p_{l+1}^2 + q_l p_{l+1})$. This, by simple computation, implies the assertion. □

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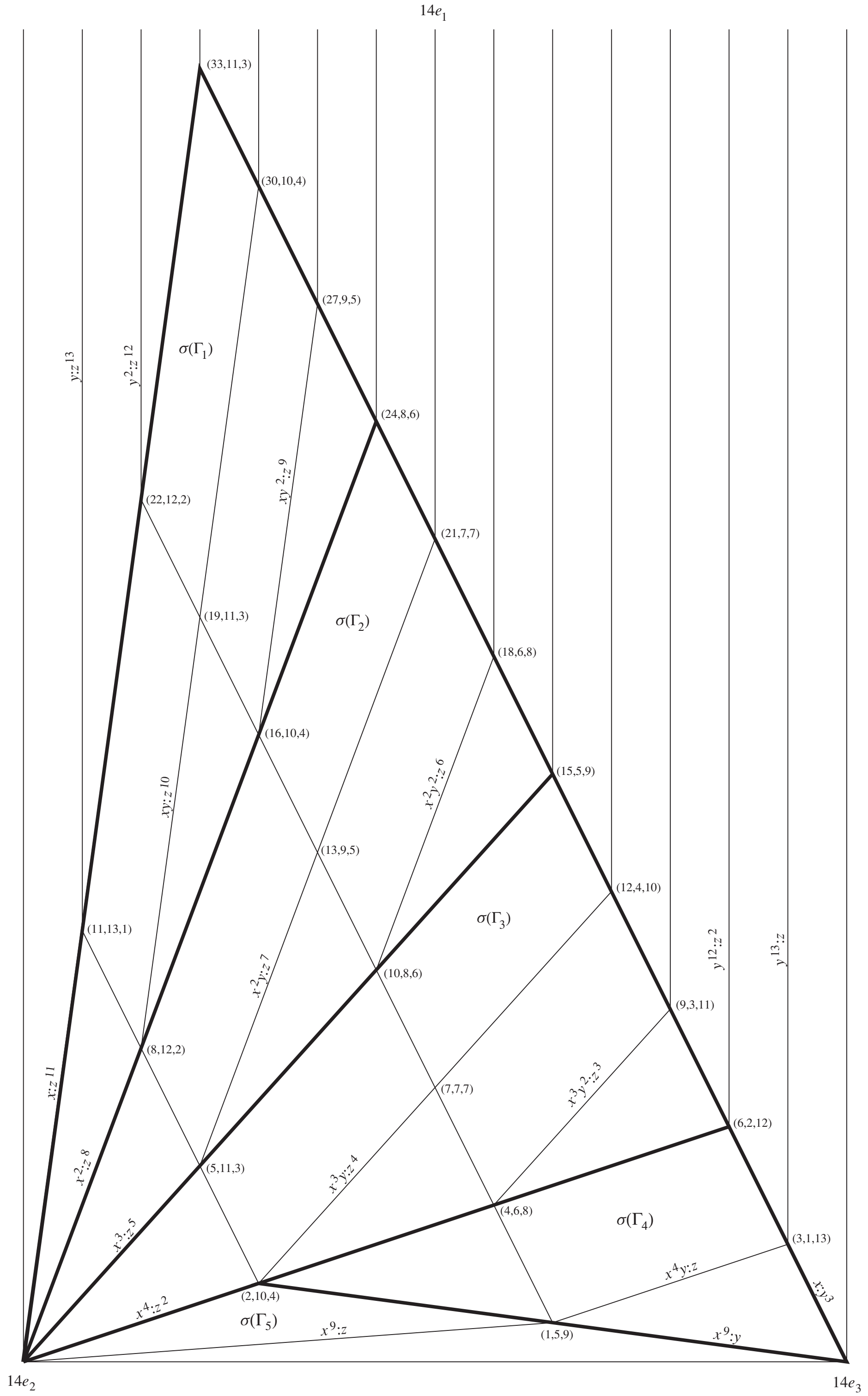

Figure 1. The fan of G- Hilb $\mathbb{C}^3$ scheme for $r = 14$, $a = 5$ intersected with hyperplane $e_2^* + e_3^* = 14$.

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7. Example. By Theorem 6.2, for $a = 5$, $r = 14$ every G-set, different from Γ_i^{yz}, belongs to a triangle of transformation of the primitive G-sets

$$\Gamma_1 = \operatorname{span}(x, y^2, z^{10}),$$
$$\Gamma_2 = \operatorname{span}(x^2, xy^2, z^7),$$
$$\Gamma_3 = \operatorname{span}(x^3, x^2y^2, z^4),$$
$$\Gamma_4 = \operatorname{span}(x^4, x^3y^2, z),$$

or is an upper transformation of

$$T_U(\Gamma_5) = \Gamma^{\mathrm{x}}.$$

There are 37 different G-sets. Figure 1 shows the fan of $G\text{-}\operatorname{Hilb}\mathbb{C}^3$, where e_1 the ray generated by e_1 is drawn at 'infinity'. The ratios along lines denote rays of the corresponding cones $\sigma^\vee(\Gamma)$ (up to an inverse in the multiplicative notation). Triangles of transformations are marked with thick line.

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