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RESEARCH ARTICLE

On split quasi-hereditary covers and Ringel duality

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Abstract
In this paper, we develop two new homological invariants called relative dominant dimension with respect to a module and relative codominant dimension with respect to a module. Among the applications are precise connections between Ringel duality, split quasi-hereditary covers and double centralizer properties, constructions of split quasi-hereditary covers of quotients of Iwahori-Hecke algebras using Ringel duality of q-Schur algebras and a new proof for Ringel self-duality of the blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. These homological invariants are studied over Noetherian algebras which are finitely generated and projective as a module over the ground ring. They are shown to behave nicely under change of rings techniques.

1. Introduction

Quasi-hereditary algebras are finite-dimensional associative algebras of finite global dimension occurring in areas like algebraic Lie theory, homological algebra and algebraic geometry. Further, all finite-dimensional algebras are centralizer subalgebras of quasi-hereditary algebras (see [DR89, Iya04]). Given a finite-dimensional algebra of infinite global dimension, we can ask to resolve it by a quasi-hereditary algebra so that their representation theories are connected by a Schur functor with nice properties. Such resolutions appear quite frequently as split quasi-hereditary covers in the sense of Rouquier [Rou08]. Well-known examples are Soergel's Struktursatz (see [Soe90]) and Schur–Weyl duality between Schur algebras $S_k(d, d)$ and symmetric groups S_d (see, for example, [Gre81]). In both cases, we are approximating a self-injective algebra with a (split) quasi-hereditary algebra. The quality of these resolutions was determined in [Fan08, FK11] and recently in [Cru24c] for the integral case using dominant dimension and relative dominant dimension as developed in [Cru22b], respectively.

Each quasi-hereditary algebra comes with a characteristic tilting module, whose endomorphism algebra is again quasi-hereditary (see [Rin91]), and it is known as the Ringel dual.

Schur algebras $S_k(d, d)$ and blocks of the BGG category $\mathcal{O}$ possess a particular symmetry; they are Ringel self-dual [Don93, Soe98], which means they are their own Ringel dual. One would like to understand the role (if it exists) of Ringel duality in Schur–Weyl duality and Soergel's Struktursatz.

A common feature in these three dualities is the existence of a double centralizer property on a summand of a characteristic tilting module. In [KSX01], Schur–Weyl duality was proved using dominant dimension and the self-injectivity of the group algebra of the symmetric group.

However, Schur algebras $S(n, d)$ with $n < d$, in general, are not Ringel self-dual. Nonetheless, a version of Schur–Weyl duality still holds between $S(n, d)$ and S_d on a module M. The object M is a summand of the characteristic tilting module of $S(n, d)$ and a right S_d-module, although it is not necessarily faithful as S_d-module. This means that the double centralizer property occurs between $S(n,d)$ and a quotient of the group algebra of the symmetric group. In [KSX01], this double centralizer property was proved using the quasi-hereditary structure of $S(n,d)$ and a generalization of dominant dimension which we shall discuss below. In addition, the quasi-hereditary structure of $S_k(n,d)$ is still deeply connected with the representation theory of S_d via the above double centralizer property. More precisely, Erdmann showed in [Erd94] that standard filtration multiplicities of summands of the characteristic tilting module are related to decomposition numbers of the symmetric group.

This situation raises the following question: *Is Schur–Weyl duality arising from the existence of a quasi-hereditary cover that extends the connection between S_d and $S_k(d,d)$ when $n \geq d$? If so, how is Ringel duality related with such a cover?*

The aim of the present paper is to give precise answers to these questions and develop new techniques on quasi-hereditary covers and Ringel self-duality continuing the approach developed in earlier papers [Cru22b, Cru24c]. In particular, we continue to use the integral setup and split quasi-hereditary covers since they are quite flexible under change of rings, and all the above examples fit in such a setup.

Motivation to generalize dominant dimension.
A module M (over a finite-dimensional algebra A) has dominant dimension at least n if there exists an exact sequence $0 \to M \to I_1 \to \cdots \to I_n$ with $I_1, \ldots, I_n$ projective and injective modules over A. In [FK11], it was observed that dominant dimension is a crucial tool to understand the quality of quasi-hereditary covers of self-injective algebras, and more importantly of quasi-hereditary covers of symmetric algebras having a simple preserving duality. This uses the Morita-Tachikawa correspondence, a theorem by Mueller [Mue68], and the fact that any generator is also a cogenerator over a self-injective algebra. Generator (resp. cogenerator) means a module whose additive closure contains all finitely generated projective (resp. injective) modules.

Recall that (A, P) is said to be a split quasi-hereditary cover of B if A is a split quasi-hereditary algebra, P is a finitely generated projective A-module, B is isomorphic to $\mathrm{End}_A(P)^{op}$ and the canonical map $A \to \mathrm{End}_B(\mathrm{Hom}_A(P,A))^{op}$ induced by the A-module structure on $\mathrm{Hom}_A(P,A)$ is an isomorphism of algebras. But, in general, $\mathrm{Hom}_A(P,A)$ is only a generator (but not a cogenerator). So, A might have dominant dimension equal to zero. So, *how to evaluate the quality of split quasi-hereditary covers, in general?* Furthermore, quotients of group algebras of the symmetric group are not in general self-injective; thus, a new generalization of dominant dimension is required.

(1) *In this paper, we provide a generalization of dominant dimension to be used as a tool to control the connection between the module category of an algebra B with the module category of the endomorphism algebra of a generator (not necessarily cogenerator) over B. In particular, it is a tool to determine the quality of (split) quasi-hereditary covers.*

Generalizations of dominant dimension have appeared several times in the literature. In [Mor70], the concept of U-dominant dimension was introduced, where the additive closure of U replaces the projective-injective modules. When U is a certain injective module, this invariant was used in [AT21] to characterize a generalization of Auslander-Gorenstein algebras in terms of the existence of tilting-cotilting modules (see also [LZ21]).

In [KSX01], this concept was applied to study Schur-Weyl duality between $S_k(n,d)$ and kS_d for a field k. That is, in [KSX01], it was proved without using invariant theory that the canonical homomorphism, induced by the right action given by place permutation of the symmetric group on d-letters S_d on $(k^n)^{\otimes d}$, $kS_d \to \mathrm{End}_{S_k(n,d)}((k^n)^{\otimes d})^{op}$ is surjective for every natural numbers n, d. Here, $S_k(n,d)$ denotes the Schur algebra $\mathrm{End}_{kS_d}((k^n)^{\otimes d})$. The case $n \geq d$ follows from $(k^n)^{\otimes d}$ being faithful over the self-injective algebra kS_d, and so it is also a generator-cogenerator. Therefore, this case can be seen as a consequence of Morita-Tachikawa correspondence and the pair $(S_k(n,d), (k^n)^{\otimes d})$ being a (split) quasi-hereditary cover. In the case $n < d$, $(k^n)^{\otimes d}$ is no longer projective, in general, over the Schur algebra but still belongs to the additive closure of the characteristic tilting module. To obtain the

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assertion then, they used the $(k^n)^{\otimes d}$-dominant dimension together with the quasi-hereditary structure of the Schur algebras to transfer the double centralizer property from the easier case $n \geq d$ to the case $n < d$. This technique also works in the quantum case replacing the Schur algebras with q-Schur algebras. The problem in both cases is the absence of a characterization of U-dominant dimension in terms of homology or cohomology like in the classical dominant dimension. Moreover, it is not strong enough to give us information if some cover is lurking around.

In [Cru22b], an integral version of dominant dimension was proposed to extend the theory of dominant dimension of finite-dimensional algebras to Noetherian algebras which are projective over the ground ring. This invariant was then used in [Cru24c] to understand the quality of the integral versions of the pair $(S_k(d,d), (k^d)^{\otimes d})$. These developments raise the following questions (to be answered in Theorem 5.3.1):

(2) *Is the above case$n < d$ a particular case of some quasi-hereditary cover? If so, does such cover admit an integral version?*

Contributions and main results.

We propose and investigate a generalization of the relative dominant dimension introduced in [Cru22b] that we call relative dominant dimension of a module M with respect to a module Q over a projective Noetherian algebra (Noetherian algebra whose regular module is also projective over the ground ring). We denote it by $Q\text{-domdim}_{(A,R)}\, M$. The projective relative injective modules are replaced by the modules in the additive closure of Q using again exact sequences which split over the ground ring and imposing an extra condition: the exact sequences considered should also remain exact under $\mathrm{Hom}_A(-, Q)$. Such a condition trivially holds for the relative dominant dimension studied in [Cru22b]. This extra condition is also trivial when U is injective, and so the concept discussed here coincides with U-dominant dimension in the cases that U is an injective module. This invariant also generalizes the concept of faithful dimension studied in [BS98]. This relative dominant dimension with respect to a module admits a relative version of a theorem by Mueller (see [Mue68, Lemma 3]) which characterizes dominant dimension in terms of cohomology:

Theorem (Theorem 3.1.1 and Theorem 3.1.3). *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that Q is a finitely generated left A-module which is projective over R so that $DQ \otimes_A Q$ is projective over R. Denote by B the endomorphism algebra* $\mathrm{End}_A(Q)^{op}$. *For any finitely generated A-module M which is projective over R, the following assertions are equivalent.*

(i) $Q\text{-domdim}_{(A,R)}\, M \geq n \geq 2$;
(ii) *The map* $\mathrm{Hom}_A(DQ, DM) \otimes_B DQ \to DM$, *given by* $f \otimes h \mapsto f(h)$, *is an A-isomorphism and* $\mathrm{Tor}_i^B(\mathrm{Hom}_A(DQ, DM), DQ) = 0$ *for all* $1 \leq i \leq n-2$.

This result extends [BS98, Proposition 2.2], [GK15, Corollary 2.16(1),(2)] and [Cru22b, Theorem 5.2].

For nice enough modules Q, M computations of relative dominant dimension can be reduced to computations over finite-dimensional algebras over algebraically closed fields (see Theorem 3.2.5).

To understand (1), we study relative codominant dimension of a characteristic tilting module with respect to a summand of a characteristic tilting module. This leads us to one of our main results:

Theorem (Theorem 5.3.1). *Let R be a commutative Noetherian ring. Let A be a split quasi-hereditary R-algebra with standard modules $\Delta(\lambda)$, $\lambda \in \Lambda$, with a characteristic tilting module T and $R(A) :=$* $\mathrm{End}_A(T)^{op}$ *the Ringel dual of A. Assume that Q is in the additive closure of T and fix $B :=$* $\mathrm{End}_A(Q)^{op}$. *If* $Q\text{-codomdim}_{(A,R)}\, T \geq 2$, *then* $(R(A), \mathrm{Hom}_A(T, Q))$ *is a split quasi-hereditary cover of B and the Schur functor* $F = \mathrm{Hom}_{R(A)}(\mathrm{Hom}_A(T,Q), -) \colon R(A)\text{-mod} \to B\text{-mod}$ *induces isomorphisms*

$$\mathrm{Ext}^j_{R(A)}(M, N) \to \mathrm{Ext}^j_B(FM, FN), \quad \forall\, 0 \leq j \leq Q\text{-codomdim}_{(A,R)}\, T - 2,$$

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for all modules M, N admitting a filtration by standard modules over the Ringel dual of A. The converse holds if R is a field.

Theorem 5.3.1 clarifies that the relative dominant (as well as codominant) dimension of a characteristic tilting module with respect to some summand Q of a characteristic tilting module measures how far Q is from being a characteristic tilting module. Equivalently, it implies that the faithful dimension of Q measures how far Q is from being a characteristic tilting module.

Recall that in Rouquier's terminology, if the Schur functor associated with some split quasi-hereditary cover (A, P) is fully faithful on standard modules, then (A, P) is called a 0-faithful (split quasi-hereditary) cover of the endomorphism algebra of P. Theorem 5.3.1 implies that any 0-faithful split quasi-hereditary cover of a finite-dimensional algebra can be detected using relative codominant dimension with respect to a summand of a characteristic tilting module and, furthermore, the quality of such a cover is completely controlled by this generalization of codominant dimension. When the quasi-hereditary cover admits a simple preserving duality, the relative dominant dimension of a characteristic tilting module with respect to Q coincides with the relative codominant dimension of a characteristic tilting module with respect to Q, where Q is a summand of a characteristic tilting module.

We answer Question (2) by picking A to be the q-Schur algebra $S_{R,q}(n, d)$ in Theorem 5.3.1 and fixing Q to be $(R^n)^{\otimes d}$.

Technically, the relative dominant dimension with respect to $(R^n)^{\otimes d}$ is different from the one used in [KSX01], but the approach taken in [KSX01] also works and is perhaps even better with the setup that we investigate here. In fact, using the Schur functor from the module category over a bigger q-Schur algebra $S_{k,q}(d, d)$ to the module category over $S_{k,q}(n, d)$, we can deduce that $(R^n)^{\otimes d}\text{-domdim}_{(S_{R,q}(n,d),R)}\, T$ is at least half of the relative dominant dimension domdim $(S_{R,q}(d, d), R)$. The computation of the relative dominant dimension of $S_{R,q}(d, d)$ is due to [FK11, FM19, Cru22b]. In particular, the relative dominant dimension of $S_{R,q}(d, d)$ is at least two, independently of R and d. Combining this with deformation techniques, we obtain the following:

Theorem (Theorem 8.1.3). *Let R be a commutative Noetherian regular ring with invertible element $u \in R$ and n, d be natural numbers. Put $q = u^{-2}$. Let T be a characteristic tilting module of $S_{R,q}(n, d)$. Denote by $R(S_{R,q}(n, d))$ the Ringel dual $\mathrm{End}_{S_{R,q}(n,d)}(T)^{op}$ of the q-Schur algebra $S_{R,q}(n, d)$ (there are no restrictions on the natural numbers n and d). Then,*

- $(R(S_{R,q}(n, d)), \mathrm{Hom}_{S_{R,q}(n,d)}(T, (R^n)^{\otimes d}))$ *is a split quasi-hereditary cover of* $\mathrm{End}_{S_{R,q}(n,d)}(V^{\otimes d})^{op}$*;*
- $(R(S_{R,q}(n, d)), \mathrm{Hom}_{S_{R,q}(n,d)}(T, (R^n)^{\otimes d}))$ *is an* $((R^n)^{\otimes d}\text{-domdim}_{(S_{R,q}(n,d),R)}\, T - 2)$*-faithful (split quasi-hereditary) cover of* $\mathrm{End}_{S_{R,q}(n,d)}(V^{\otimes d})^{op}$ *in the sense of Rouquier.*

The existence of this split quasi-hereditary cover clarifies why the quasi-hereditary structure of the Ringel dual of the Schur algebra can be used to study the decomposition numbers of the symmetric group [Erd94]. This result also explains why [KSX01] were successful in using tilting theory to establish Schur-Weyl duality (see Remark 3.1.11). If $n \geq d$, Ringel self-duality implies that the split quasi-hereditary cover constructed in Theorem 8.1.3 is equivalent to the split quasi-hereditary cover $(S_{R,q}(n, d), (R^n)^{\otimes d})$. The quality of the latter was completely determined in [Cru24c, Subsections 7.1, 7.2]. In general, the usual strategy is not sufficient for $q = -1$. In such a case, going integrally is crucial, and we make use of deformation techniques (see Corollary 5.3.5).

In [Soe98, Corollary 2.3], Soergel proved that the blocks of the BGG category $\mathcal{O}$ are Ringel self-dual by constructing the explicit functor giving Ringel self-duality. Unfortunately, such proof does not offer much information on which structural properties of $\mathcal{O}$ force its blocks to be Ringel self-dual. Later in [FKM00, Proposition 4], a different proof of Ringel self-duality of the blocks of the BGG category $\mathcal{O}$ was presented using as main tool the Enright completion functor.

In [Cru24c, Subsection 7.3], the author studied projective Noetherian algebras $A_{\mathcal{D}}$ that encode the representation theory of any block of the BGG category $\mathcal{O}$. The Ringel self-duality of blocks of the BGG category $\mathcal{O}$ is reproved in Theorem 8.2.1 by applying Theorem 5.3.1 to the algebras $A_{\mathcal{D}}$ making

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use of integral versions of Soergel's Struktursatz. In particular, we can now regard Ringel self-duality of the blocks of the BGG category $\mathcal{O}$ as an instance of uniqueness of covers from Rouquier's cover theory.

Organization

This paper is structured as follows: Section 2 sets up the notation, properties and results on covers, split quasi-hereditary algebras, relative dominant dimension over Noetherian algebras and approximation theory to use later. In Section 3, we give the definition of relative dominant (resp. codominant) dimension with respect to a module over a projective Noetherian algebra (Definition 3.0.1). In Subsection 3.1, we explore a characterization of relative dominant (resp. codominant) dimension with respect to an A-module Q in terms of homology over $\operatorname{End}_A(Q)^{op}$ (Theorems 3.1.1-3.1.4). As an application of this characterization, we see how the relative dominant dimension with respect to a module varies on exact sequences and long exact sequences in general. In Subsection 3.2, we study how computations of the relative dominant dimension $Q\text{-domdim}_{(A,R)} M$ can be reduced to computations over finite-dimensional algebras over algebraically closed fields under mild assumptions on Q and M (Theorem 3.2.5 and Lemma 3.2.3). In Section 4, we discuss the relation between the relative dominant dimension with respect to a module with the concept of reduced cograde with respect to a module. In Section 5, we investigate relative codominant dimension with respect to a module as a tool to establish double centralizer properties (Lemmas 5.1.1 and 5.1.2), to discover (Theorem 5.3.1 and Corollary 5.3.5) and to control (Corollary 5.3.1(d)) the quality of split quasi-hereditary covers. In Subsection 5.4, we discuss how under special conditions dominant dimension can be used as a tool to study Ringel self-duality. In Section 6, we clarify what the relative dominant dimension with respect to a summand of a characteristic tilting module measures. In Section 7, we explore how to obtain lower bounds to the quality of a split quasi-hereditary cover of a quotient algebra B/J using the quality of split quasi-hereditary covers of B. In Subsection 8.1, we construct a split quasi-hereditary cover of the quotient of the Iwahori-Hecke algebra involved in classical Schur-Weyl duality constituted by a Ringel dual of a q-Schur algebra using relative dominant dimension with respect to $(R^n)^{\otimes d}$ (Theorem 8.1.3). In Subsubsection 8.2.1, we use cover theory and relative dominant dimension to reprove Ringel self-duality of the blocks of the BGG category $\mathcal{O}$ (Theorem 8.2.1). In Subsubsection 8.2.2, we see how these techniques can be used for Schur algebras.

2. Preliminaries

Throughout this paper, we assume that R is a Noetherian commutative ring with identity and A is a projective Noetherian R-algebra, unless stated otherwise. Here, A is called a **projective Noetherian** R**-algebra** if A is an R-algebra so that A is finitely generated projective as R-module. We call A a **free Noetherian** R-algebra if A is a Noetherian R-algebra so that A is free of finite rank as R-module. The module category of left A-modules is denoted by A-Mod. We denote by A-mod the full subcategory of A-Mod whose modules are finitely generated and by A-Proj the subcategory of A-Mod of projective modules. Given $M \in A$-mod, we denote by $\operatorname{add}_A M$ (or just $\operatorname{add} M$) the full subcategory of A-mod whose modules are direct summands of a finite direct sum of copies of M. We write A-proj to denote $\operatorname{add} A$. By $\operatorname{End}_A(M)$ we mean the endomorphism algebra of an A-module M. By A^{op}, we mean the opposite algebra of A. We denote by D_R (or just D) the standard duality functor $\operatorname{Hom}_R(-, R)\colon A\text{-mod} \to A^{op}\text{-mod}$. We say that $M \in A\text{-mod} \cap R\text{-proj}$ is (A, R)**-injective** if $M \in \operatorname{add} DA$. By $(A, R)\text{-inj} \cap R\text{-proj}$ we mean the full subcategory of $A\text{-mod} \cap R\text{-proj}$ whose modules are (A, R)-injective. By an (A, R)**-exact sequence** we mean an exact sequence of A-modules which splits as a sequence of R-modules. By an (A, R)**-monomorphism** we mean an homomorphism $f \in \operatorname{Hom}_A(M, N)$ that fits into an (A, R)-exact sequence of the form $0 \to M \xrightarrow{f} N$. By a **generator** we mean a module $M \in A$-mod satisfying $A \in \operatorname{add} M$. Given $M \in A$-mod, we denote by $\operatorname{pdim}_A M$ (resp. $\operatorname{idim}_A M$) the projective (resp. injective) dimension of M.

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2.0.0.1. Change of rings.

We denote by $\operatorname{MaxSpec}(R)$ the set of maximal ideals of R and by $\operatorname{Spec} R$ the set of prime ideals of R. By $\dim R$ we mean the Krull dimension of R. We denote by $R_\mathfrak{p}$ the localization of R at the prime ideal $\mathfrak{p}$, and by $M_\mathfrak{p}$ the localization of M at $\mathfrak{p}$ for every $M \in A$-mod. In particular, $M_\mathfrak{p} \in A_\mathfrak{p}$-mod. We say that a local commutative Noetherian ring is **regular** if it has finite global dimension. In such a case, the global dimension coincides with the Krull dimension. We say that a commutative Noetherian ring is **regular** if for every $\mathfrak{p} \in \operatorname{Spec} R$, $R_\mathfrak{p}$ is regular. We denote by $R(\mathfrak{m})$ the residue field $R/\mathfrak{m} \simeq R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$. For each $M \in A$-mod, by $M(\mathfrak{m})$ we mean the finite-dimensional module $R(\mathfrak{m}) \otimes_R M$ over $A(\mathfrak{m}) = R(\mathfrak{m}) \otimes_R A$. By $R^\times$ we denote the set of invertible elements of R. We write $D_{(\mathfrak{m})}$ to abbreviate $D_{R(\mathfrak{m})}$ for every $\mathfrak{m} \in \operatorname{MaxSpec} R$.

2.0.0.2. The functor F_Q.

Let $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying $DQ \otimes_A Q \in R$-proj. By F_Q (or just F when no confusion arises) we mean the functor $\operatorname{Hom}_A(Q, -)\colon A\text{-mod} \to B\text{-mod}$, where B is the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. In particular, $B \in R$-proj and B-mod is an abelian category. Hence, B is a projective Noetherian R-algebra. Given two maps $\alpha \in \operatorname{Hom}_A(X, Y)$, $\beta \in \operatorname{Hom}_A(Z, W)$, we say that f and g are **equivalent** if there are A-isomorphisms $f\colon X \to Z$, $g\colon Y \to W$ satisfying $g \circ \alpha = \beta \circ f$. By $\mathbb{I}_Q$ (or just $\mathbb{I}$ when no confusion arises) we mean the left adjoint of F, $Q \otimes_B -\colon B\text{-mod} \to A\text{-mod}$. We denote by υ the unit $\mathrm{id}_{B\text{-mod}} \to F\mathbb{I}$ and χ the counit $\mathbb{I}F \to \mathrm{id}_{A\text{-mod}}$. Thus, for any $N \in B$-mod, υ_N is the B-homomorphism $\upsilon_N\colon N \to \operatorname{Hom}_A(Q, Q \otimes_B N)$, given by $\upsilon_N(n)(q) = q \otimes n$, $n \in N, q \in Q$. For any $M \in A$-mod, χ_M is the A-homomorphism $Q \otimes_B \operatorname{Hom}_A(Q, M) \to M$, given by $\chi_M(q \otimes g) = g(q)$, $g \in FM, q \in Q$. By projectivization, the restriction of F to $\operatorname{add} Q$ gives an equivalence between $\operatorname{add} Q$ and B-proj. Further, for every $X, Y \in A$-mod and every $M, N \in B$-mod, $\upsilon_{M\oplus N}$ is equivalent to $\upsilon_M \oplus \upsilon_N$ and $\chi_{X\oplus Y}$ is equivalent to $\chi_X \oplus \chi_Y$, respectively. We shall write χ^r and υ^r for the counit and unit, respectively, of the adjunction $- \otimes_B DQ \dashv \operatorname{Hom}_A(DQ, -)$. Given a left (resp. right) exact functor H between two module categories, we denote by $\mathrm{R}^i H$ (resp. $\mathrm{L}_i H$) the i-th right (resp. left) derived functor of H for $i \in \mathbb{N}$. Recall that ${}^\perp Q = \{M \in A\text{-mod} \cap R\text{-proj} \mid \operatorname{Ext}_A^{i>0}(M, Q) = 0\}$ is a resolving subcategory of $A\text{-mod} \cap R$-proj. Analogously, $Q^\perp$ is defined.

2.0.0.3. Filtrations.

Recall that for a given set (possibly infinite) of modules Θ in $A\text{-mod} \cap R$-proj, $\mathcal{F}(\Theta)$ denotes the full subcategory of $A\text{-mod} \cap R$-proj whose modules admit a finite filtration by the modules in Θ. Given a set of modules Θ in $A\text{-mod} \cap R$-proj, we denote by $\tilde{\Theta}$ the set of modules $\{\theta \otimes_R X_\theta \colon \theta \in \Theta, X_\theta \in R\text{-proj}\}$. The following well-known lemma allows us to identify the set $F_Q\tilde{\Theta} := \{F_Q X \colon X \in \tilde{\Theta}\}$ with the set $\widetilde{F_Q\Theta}$, where $F_Q\Theta := \{F_Q\theta \colon \theta \in \Theta\}$.

Lemma 2.0.1. *Let $M, N \in A$-mod and $U \in R$-proj. Then, the R-homomorphism $\varsigma_{M,N,U}\colon \operatorname{Hom}_A(M, N) \otimes_R U \to \operatorname{Hom}_A(M, N \otimes_R U)$, given by $g \otimes u \mapsto g(-) \otimes u$, is an R-isomorphism.*

2.0.0.4. Basics on approximations.

Let $T \in A$-mod. An A-homomorphism $M \to N$ is called a **left** $\operatorname{add}T$**-approximation** of M provided that N belongs to $\operatorname{add} T$ and the induced homomorphism $\operatorname{Hom}_A(N, X) \to \operatorname{Hom}_A(M, X)$ is surjective for every $X \in \operatorname{add} T$. A map $f \in \operatorname{Hom}_A(Y, M)$ is called a **right** $\operatorname{add}T$**-approximation** of M if $Y \in \operatorname{add} T$ and $\operatorname{Hom}_A(X, f)$ is surjective for every $X \in \operatorname{add} T$. So, $f \in \operatorname{Hom}_A(Y, M)$ is a right $\operatorname{add} T$-approximation of M if and only if the map $\operatorname{Hom}_A(T, f)$ is surjective and $Y \in \operatorname{add} T$.

Lemma 2.0.2. *Let $M, T \in A\text{-mod} \cap R$-proj and $N \in \operatorname{add} T$. An A-homomorphism $f\colon M \to N$ is a left $\operatorname{add} T$-approximation of M if and only if $Df\colon DN \to DM$ is a right $\operatorname{add} DT$-approximation of DM.*

Proof. Clear since $\operatorname{Hom}_A(T, f)$ and $\operatorname{Hom}_A(Df, DT)$ are equivalent (see [Cru22b, Proposition 2.2]). □

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Let $M, T \in A\text{-mod} \cap R\text{-proj}$. It is easy to check that an (A, R)-exact sequence $X_t \xrightarrow{\alpha_t} \cdots \to X_1 \xrightarrow{\alpha_1} X_0 \xrightarrow{\alpha_0} M \to 0$ remains exact under $\operatorname{Hom}_A(T, -)$ with $X_i \in \operatorname{add} T$ if and only if for every $i = 1, \ldots, t$, the induced maps $X_i \twoheadrightarrow \operatorname{im} \alpha_i$ and α_0 are right $\operatorname{add} T$-approximations. Dually, an (A, R)-exact sequence $0 \to M \xrightarrow{\alpha_0} X_0 \xrightarrow{\alpha_1} X_1 \to \cdots \to X_t$ remains exact under $\operatorname{Hom}_A(-, T)$ with $X_i \in \operatorname{add} T$ if and only if the (A, R)-monomorphisms $\operatorname{im} \alpha_{i+1} \hookrightarrow X_{i+1}$ and α_0 are left $\operatorname{add} T$-approximations with $i = 0, \ldots, t - 1$.

2.1. Split quasi-hereditary algebras

Quasi-hereditary algebras were introduced by Cline, Parshall and Scott in [CPS88], and were defined in terms of the existence of a certain idempotent ideal chain reflecting structural properties of the algebra like the finiteness of the global dimension. In [CPS90], the concept of quasi-hereditary algebra was generalized to Noetherian algebras. Among them are the split quasi-hereditary algebras which possess nicer properties with respect to change of ground rings. In particular, every quasi-hereditary algebra over an algebraically closed field is split quasi-hereditary. A module theoretical approach to split quasi-hereditary algebras over commutative Noetherian rings was considered in [Rou08]. In [Has00], a comodule theoretical approach was developed to split quasi-hereditary algebras over commutative Noetherian rings.

Definition 2.1.1. Given a projective Noetherian R-algebra A and a collection of finitely generated left A-modules $\{\Delta(\lambda) : \lambda \in \Lambda\}$ indexed by a poset Λ, we say that $(A, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ is a **split quasi-hereditary R-algebra** if the following conditions hold:

(i) The modules $\Delta(\lambda) \in A$-mod are projective over R and $\operatorname{End}_A(\Delta(\lambda)) \simeq R$, for all $\lambda \in \Lambda$.
(ii) Given $\lambda, \mu \in \Lambda$, if $\operatorname{Hom}_A(\Delta(\lambda), \Delta(\mu)) \neq 0$, then $\lambda \leq \mu$.
(iii) Given $\lambda \in \Lambda$, there is $P(\lambda) \in A$-proj and an exact sequence $0 \to C(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0$ such that $C(\lambda)$ has a finite filtration by modules of the form $\Delta(\mu) \otimes_R U_\mu$ with $U_\mu \in R$-proj and $\mu > \lambda$. Moreover, $P = \bigoplus_{\lambda \in \Lambda} P(\lambda)$ is a progenerator for A-mod.

Under these conditions, we also say that $(A\text{-mod}, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ is a **split highest weight category**. We use the terms split quasi-hereditary algebra and split highest weight category interchangeably. The modules $\Delta(\lambda)$ are known as **standard modules**. Much of the structure of a split quasi-hereditary algebra is controlled by the subcategory $\mathcal{F}(\tilde{\Delta})$, where $\tilde{\Delta}_A$ or just $\tilde{\Delta}$ (when there is no confusion on the ambient algebra) denotes the set $\{\Delta(\lambda) \otimes_R U_\lambda : \lambda \in \Lambda, U_\lambda \in R\text{-proj}\}$. This subcategory contains all projective finitely generated A-modules, it is closed under extensions, closed under kernels of epimorphisms, and closed under direct summands. Hence, $\mathcal{F}(\tilde{\Delta})$ can be viewed as the full subcategory of $A\text{-mod} \cap R\text{-proj}$ whose modules admit a finite filtration by direct summands of a direct sum of copies of standard modules.

Given another split quasi-hereditary algebra $(C, \{\Delta_C(\lambda)_{\lambda \in \Omega}\})$, A and C are **Morita equivalent as split quasi-hereditary algebras** if there exists an equivalence of categories $G : A\text{-mod} \to C\text{-mod}$ and a bijection of posets $\Phi : \Lambda \to \Omega$ such that for all $\lambda \in \Lambda$, $G\Delta_A(\lambda) \simeq \Delta_C(\Phi(\lambda)) \otimes_R U_\lambda$ for some invertible R-module U_λ.

2.1.0.1. Split heredity chains.
An alternative way to define split quasi-hereditary algebras is via split heredity chains. An ideal J is called **split heredity** of A if $A/J \in R$-proj, $J \in A$-proj, $J^2 = J$ and the R-algebra $\operatorname{End}_A(J)^{op}$ is Morita equivalent to R. A chain of ideals $0 = J_{t+1} \subset J_t \subset \cdots \subset J_1 = A$ is called **split heredity** if J_i/J_{i+1} is a split heredity ideal in A/J_{i+1} for $1 \leq i \leq t$. The algebra A is split quasi-hereditary if it admits a split heredity chain. Given an increasing bijection $\Lambda \to \{1, \ldots, t\}$, $\lambda \mapsto i_\lambda$, the standard modules and the split heredity chain are related by the following identification: $\operatorname{im}(\tau_i) \simeq J_i/J_{i+1}$, where τ_i is the map $\Delta_i \otimes_R \operatorname{Hom}_{A/J_{i+1}}(\Delta_i, A/J_{i+1}) \to A/J_{i+1}$ given by $\tau_i(l \otimes f) = f(l)$. For more details on this equivalence, we refer to [Cru24a, 3.3] and [Rou08, Theorem 4.16].

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2.1.1. Costandard modules and characteristic tilting modules

A split quasi-hereditary algebra $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ also comes equipped with a set of modules $\{\nabla(\lambda) \colon \lambda \in \Lambda\}$ known as **costandard modules**. These modules satisfy the following properties.

Proposition 2.1.2. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra. Then,*

(i) *$(A^{op}, \{D\nabla(\lambda)_{\lambda\in\Lambda}\})$ is a split quasi-hereditary algebra;*
(ii) *$\mathcal{F}(\tilde{\nabla}) = \{X \in A\text{-mod} \cap R\text{-proj} \colon \operatorname{Ext}^1_A(M, X) = 0, \ \forall M \in \mathcal{F}(\tilde{\Delta})\}$, where $\tilde{\nabla}$ denotes the set $\{\nabla(\lambda) \otimes_R U_\lambda \colon \lambda \in \Lambda, U_\lambda \in R\text{-proj}\}$;*
(iii) *For any $\mu \neq \lambda \in \Lambda$, $\operatorname{Hom}_A(\Delta(\lambda), \nabla(\lambda)) \simeq R$ and $\operatorname{Hom}_A(\Delta(\mu), \nabla(\lambda)) = 0$;*
(iv) *The choice of costandard modules satisfying the previous assertions is unique up to isomorphism;*
(v) *$\mathcal{F}(\tilde{\Delta}) = \{X \in A\text{-mod} \cap R\text{-proj} \colon \operatorname{Ext}^1_A(X, N) = 0, \ \forall N \in \mathcal{F}(\tilde{\nabla})\}$;*
(vi) *For any $M \in \mathcal{F}(\tilde{\Delta})$, the functor $- \otimes_A M \colon \mathcal{F}(D\tilde{\nabla}) \to R\text{-proj}$ is well defined and exact;*
(vii) *For any $N \in \mathcal{F}(\tilde{\nabla})$, the functor $DN \otimes_A - \colon \mathcal{F}(\tilde{\Delta}) \to R\text{-proj}$ is well defined and exact;*
(viii) *For any $M \in \mathcal{F}(\tilde{\Delta})$ and $N \in \mathcal{F}(\tilde{\nabla})$, it holds $\operatorname{Hom}_A(M, N) \in R\text{-proj}$.*

Proof. For (i), (ii), (iii), (iv), (v), we refer to [Cru24b, Proposition 3.1, Theorem 4.1] and [Rou08, Proposition 4.19, Lemma 4.21]. For (vi), (vii), (viii), we refer to [Cru24b, Proposition 4.3., Corollary 4.4.]. □

The subcategories $\mathcal{F}(\tilde{\Delta})$ and $\mathcal{F}(\tilde{\nabla})$ are determined and determine a characteristic tilting module. We call T a **characteristic tilting module** if it is a tilting module satisfying $\operatorname{add} T = \mathcal{F}(\tilde{\Delta}) \cap \mathcal{F}(\tilde{\nabla})$. By a tilting A-module we mean a module of finite projective dimension, with $\operatorname{Ext}^{i>0}_A(T, T) = 0$, and the regular module has a finite coresolution by modules in $\operatorname{add} T$. Let T be a characteristic tilting module of A. So, we have $\operatorname{add} T = \operatorname{add} \bigoplus_{\lambda\in\Lambda} T(\lambda)$ so that each $T(\lambda)$ with $\lambda \in \Lambda$ fits into exact sequences of the form

$$0 \to \Delta(\lambda) \to T(\lambda) \to X(\lambda) \to 0, \quad 0 \to Y(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0, \tag{1}$$

with $X(\lambda) \in \mathcal{F}(\tilde{\Delta}_{\mu<\lambda})$ and $Y(\lambda) \in \mathcal{F}(\tilde{\nabla}_{\mu<\lambda})$. Here, the set $\{\Delta(\mu) \otimes_R U_\mu \colon \mu \in \Lambda, \ \mu < \lambda, U_\mu \in R\text{-proj}\}$ is denoted by $\tilde{\Delta}_{\mu<\lambda}$. Analogously, the set $\tilde{\nabla}_{\mu<\lambda}$ is defined. A fundamental difference between the classical case is that, in general, we cannot choose $T(\lambda)$ to be indecomposable modules. But two distinct characteristic tilting modules have the same additive closure. These can be chosen to be indecomposable when the ground ring is a local commutative Noetherian ring. The philosophical reason is that any split quasi-hereditary algebra over a local commutative Noetherian ring is semi-perfect ([Cru24a, Theorem 3.4.1]). Let $T = \bigoplus_{\lambda\in\Lambda} T(\lambda)$ be a characteristic tilting module. It follows, by construction, that a module $M \in A\text{-mod} \cap R\text{-proj}$ belongs precisely to $\mathcal{F}(\tilde{\Delta})$ if and only if there exists a finite coresolution of M by modules in $\operatorname{add} T$. Dually, a module $M \in A\text{-mod} \cap R\text{-proj}$ belongs precisely to $\mathcal{F}(\tilde{\nabla})$ if and only if there exists a finite resolution of M by modules in $\operatorname{add} T$. For details on these statements, we refer to [Cru24b] and [Cru24a, Section 3, Appendix A and B].

2.1.1.1. Change of rings.

An important feature of split quasi-hereditary algebras is that they behave quite well under change of rings. This manifests itself in the subcategories $\mathcal{F}(\tilde{\Delta})$ and $\mathcal{F}(\tilde{\nabla})$ as follows:

Proposition 2.1.3. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra. Let $M \in A\text{-mod}$. Let Q be a commutative R-algebra and Noetherian ring. Then, the following assertions hold.*

(a) *$(Q \otimes_R A, \{Q \otimes_R \Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split quasi-hereditary algebra over Q. The costandard modules of $Q \otimes_R A$ are the form $Q \otimes_R \nabla(\lambda)$, $\lambda \in \Lambda$.*
(b) *$M \in \mathcal{F}(\tilde{\Delta})$ if and only if $M(\mathfrak{m}) \in \mathcal{F}(\Delta(\mathfrak{m}))$ for all maximal ideals $\mathfrak{m}$ of R and $M \in R\text{-proj}$.*
(c) *$M \in \mathcal{F}(\tilde{\nabla})$ if and only if $M(\mathfrak{m}) \in \mathcal{F}(\nabla(\mathfrak{m}))$ for all maximal ideals $\mathfrak{m}$ of R and $M \in R\text{-proj}$.*
(d) *Let T be a characteristic tilting module. $M \in \operatorname{add} T$ if and only if $M(\mathfrak{m}) \in \operatorname{add} T(\mathfrak{m})$ for all maximal ideals $\mathfrak{m}$ of R and $M \in R\text{-proj}$.*
(e) *Let $M \in \mathcal{F}(\tilde{\Delta})$ and let $N \in \mathcal{F}(\tilde{\nabla})$. Then, $Q \otimes_R \operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{Q\otimes_R A}(Q \otimes_R M, Q \otimes_R N)$.*

Proof. For (a), we refer to [Rou08, Proposition 4.14], [Cru24a, Proposition 3.1.1] and [Cru24b, Proposition 5.9]. For (b), (c) and (d), we refer to [Rou08, Proposition 4.30] and [Cru24b, Proposition 5.7]. For (e), we refer to [Cru24b, Corollary 5.6]. □

2.1.2. Ringel duality

Applying the methods of [AR91] to Artinian quasi-hereditary algebras, Ringel, in [Rin91], discovered that the endomorphism algebras of characteristic tilting modules admit a quasi-hereditary structure. As seen in [Cru24b, Section 7], split quasi-hereditary algebras over commutative Noetherian rings also come in pairs. The **Ringel dual of a split quasi-hereditary R-algebra** $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is, up to Morita equivalence, the endomorphism algebra $R(A) := \mathrm{End}_A(T)^{op}$ of a characteristic tilting module T of A. The standard modules of $R(A)$ are $\Delta_{R(A)}(\lambda) = \mathrm{Hom}_A(T, \nabla(\lambda))$ with $\lambda \in \Lambda^{op}$, where Λ^{op} is the opposite poset of Λ. The **Ringel dual functor** $\mathrm{Hom}_A(T, -)\colon A\text{-mod} \to R(A)\text{-mod}$ restricts to an exact equivalence $\mathcal{F}(\tilde{\nabla}) \to \mathcal{F}(\tilde{\Delta}_{R(A)})$; it sends costandard modules to standard modules, (partial) tilting modules to projective modules and modules in add DA to tilting modules.

A split quasi-hereditary R-algebra is called **Ringel self-dual** if there exists an exact equivalence between $\mathcal{F}(\tilde{\Delta})$ and $\mathcal{F}(\tilde{\nabla})$ – that is, if A and $R(A)$ are Morita equivalent as split quasi-hereditary algebras. For split quasi-hereditary algebras over local commutative Noetherian rings, it is enough to test Ringel self-duality after applying extension of scalars from the local ground ring to its residue field. For more details, we refer to [Cru24b].

2.1.3. The quasi-hereditary structure of eAe

Given a split quasi-hereditary algebra $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ over a field k, we call an idempotent e of A **cosaturated** if there exists a subset $\Gamma \subset \Lambda$ such that $e \operatorname{top} \Delta(\lambda) = 0$ precisely when $\lambda \leq \mu$ for some $\mu \in \Gamma$. Using cosaturated idempotents, we can construct new split quasi-hereditary algebras from bigger ones without using necessarily quotients by split heredity ideals.

Theorem 2.1.4. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over a field k. Then,*

- (i) *Let e be an idempotent of A and define $\Lambda' = \{\lambda \in \Lambda\colon e \operatorname{top} \Delta(\lambda) \neq 0\}$. Then, $\{e \operatorname{top} \Delta(\lambda)\colon \lambda \in \Lambda'\}$ is a full set of simple modules in eAe-*mod*.*
- (ii) *Assume that there exists a cosaturated idempotent e. Set $\Lambda' = \{\lambda \in \Lambda\colon e \operatorname{top} \Delta(\lambda) \neq 0\}$. Then,*
  - (I) *$(eAe, \{e\Delta(\lambda)_{\lambda\in\Lambda'}\})$ is a split quasi-hereditary algebra. The costandard modules of eAe are of the form $\{e\nabla(\lambda)\colon \lambda \in \Lambda'\}$. Moreover, $e\Delta(\lambda) = e\nabla(\lambda) = 0$ for $\lambda \in \Lambda \setminus \Lambda'$.*
  - (II) *The (Schur) functor $\mathrm{Hom}_A(Ae, -)\colon A\text{-mod} \to eAe\text{-mod}$ preserves (partial) tilting modules. Moreover, the partial tilting indecomposable modules of eAe are exactly $\{eT(\lambda)\colon \lambda \in \Lambda'\}$ and $eT(\lambda) = 0$ for any $\lambda \in \Lambda \setminus \Lambda'$.*
  - (III) *Let $M \in \mathcal{F}(\Delta)$ and $N \in \mathcal{F}(\nabla)$. Then, the Schur functor $\mathrm{Hom}_A(Ae, -)\colon A\text{-mod} \to eAe\text{-mod}$ induces a surjective map $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{eAe}(eM, eN)$.*

Proof. Statement (i) actually holds for finite-dimensional algebras, in general. We refer to [Gre81, Theorem 6.2g]. For (I), see [Don98, Proposition A3.11]. For (III), see, for example, [Erd94, 1.7] or [Don98, Lemma A3.12]. For (II), see [Don98, Lemma A4.5]. □

2.2. *Covers*

A module $M \in A$-mod is said to have a **double centralizer property** if the canonical homomorphism of R-algebras $A \to \mathrm{End}_{\mathrm{End}_A(M)^{op}}(M)$ is an isomorphism.

The concept of cover was introduced in [Rou08] to evaluate the quality of a module category with an approximation, in some sense, by a highest weight category. Let $P \in A$-proj. We say that the pair (A, P) is a **cover** of $\mathrm{End}_A(P)^{op}$ if there is a double centralizer property on $\mathrm{Hom}_A(P, A)$ (as a right A-module). Equivalently, (A, P) is a cover of $\mathrm{End}_A(P)^{op}$ if and only if the **Schur functor** $F_P = \mathrm{Hom}_A(P, -)\colon A\text{-mod} \to \mathrm{End}_A(P)^{op}\text{-mod}$ is fully faithful on A-proj. Fix $B = \mathrm{End}_A(P)^{op}$. Since $\mathrm{Hom}_A(P, -)\colon A\text{-mod} \to B\text{-mod}$ is isomorphic to the functor $\mathrm{Hom}_A(P, A) \otimes_A -\colon A\text{-mod} \to B\text{-mod}$, it

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follows by Tensor-Hom adjunction that $\operatorname{Hom}_B(\operatorname{Hom}_A(P, A), -)\colon B\text{-mod} \to A\text{-mod}$ is right adjoint to the Schur functor $\operatorname{Hom}_A(P, -)\colon A\text{-mod} \to B\text{-mod}$. We denote by $\eta\colon \operatorname{id}_{A\text{-mod}} \to G \circ F$ the unit of this adjunction. Observe that $\operatorname{Hom}_A(P, A)$ is a generator as a B-module and since $P \in A\text{-proj}$, the functor $\operatorname{Hom}_B(\operatorname{Hom}_A(P, A), -)\colon B\text{-mod} \to A\text{-mod}$ is fully faithful.

If $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split quasi-hereditary algebra and (A, P) is a cover of B, then we say that (A, P) is a **split quasi-hereditary cover** of B. Let $\mathcal{A}$ be a resolving subcategory of A-mod and let i be a non-negative integer. Denote by F_P the functor $\operatorname{Hom}_A(P, -)\colon A\text{-mod} \to B\text{-mod}$ and by G_P its right adjoint. We say that the pair (A, P) is an i-$\mathcal{A}$ **cover** of B if the Schur functor F_P induces isomorphisms $\operatorname{Ext}^j_A(M, N) \to \operatorname{Ext}^j_B(F_P M, F_P N), \quad \forall M, N \in \mathcal{A},\ 0 \leq j \leq i$. We say that (A, P) is an $(-\mathbf{1})$-$\mathcal{A}$ **cover** of B if (A, P) is a cover of B and the restriction of F_P to $\mathcal{A}$ is faithful. In Rouquier's terminology, when $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ is a split quasi-hereditary algebra, an i-$\mathcal{F}(\tilde{\Delta})$ cover is known as an i-faithful cover. The optimal value of i making (A, P) an i-$\mathcal{A}$ of B is called the **Hemmer-Nakano dimension** of $\mathcal{A}$ (with respect to P). In [Cru24c, Section 4], the author proved, for example, that the number of simple B-modules is an upper bound to the quality of a split quasi-hereditary cover, if the Schur functor associated with it is not fully faithful. We refer to [Cru24c] for further details. Furthermore, we refer the reader to [Cru24c, Subsection 4.3] for the definition of the concept of equivalent covers and for sufficient conditions for the uniqueness of a cover.

2.3. Relative dominant dimension over Noetherian algebras

In [FK11], it was established that dominant dimension of certain modules over finite-dimensional algebras control the quality of many A-proj-covers and $\mathcal{F}(\Delta)$-covers of a finite-dimensional algebra B. Such covers that are controlled in this way are formed by the endomorphism algebras of generator-cogenerators over B.

In [Cru22b], the author generalized the concept of dominant dimension to projective Noetherian algebras. In [Cru24c], the author showed that this invariant does not fully control the quality of a cover in the integral setup, since more factors influence the cover. But, this invariant can be exploited together with change of rings to compute the quality of a cover which is formed by an integral version of the endomorphism algebra of a generator-cogenerator. Such integral versions first appear in [Cru22b, Theorem 4.1] which we call relative Morita-Tachikawa correspondence.

Given $M \in A$-mod, the **relative dominant dimension of** M, denoted as $\operatorname{domdim}_{(A,R)} M$, is

$$\sup\{t \in \mathbb{N}\colon 0 \to M \to I_1 \to \cdots \to I_t \text{ is } (A, R)\text{-exact},\ I_1, \ldots, I_t \in A\text{-proj} \cap \operatorname{add} DA\} \in \mathbb{N} \cup \{0, +\infty\}.$$

We just write $\operatorname{domdim}_A M$ when R is a field.

We denote by $\operatorname{domdim}(A, R)$ the relative dominant dimension of the regular module A. This concept behaves well under extension of scalars. In particular, if $\operatorname{domdim}_{(A,R)} A \geq 1$, then $\operatorname{domdim}_{(A,R)} M = \inf\{\operatorname{domdim}_{A(\mathfrak{m})} M(\mathfrak{m})\colon \mathfrak{m} \in \operatorname{MaxSpec} R\}$ for any $M \in A\text{-mod} \cap R\text{-proj}$ (see [Cru22b, Theorem 6.13]).

Given $M \in A$-mod, we say that M is an (A, R)**-injective-strongly faithful module** if M is (A, R)-injective and there exists an (A, R)-monomorphism $A \hookrightarrow X$ for some $X \in \operatorname{add} M$. By a **RQF3-algebra** we mean a triple (A, P, V) formed by a projective Noetherian R-algebra A, a projective (A, R)-injective-strongly faithful left A-module P and a projective (A, R)-injective-strongly faithful right A-module V.

We recall the following properties relating relative dominant dimension with cover theory.

Proposition 2.3.1. *Let* (A, P, V) *be an RQF3-algebra over a commutative Noetherian ring R. The following assertions hold.*

(a) *If* $\operatorname{domdim}(A, R) \geq 2$, *then* $(A, \operatorname{Hom}_{A^{op}}(V, A))$ *is a cover of* $\operatorname{End}_A(V)$.
(b) *Let* $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *be a split quasi-hereditary algebra. Assume that T is a characteristic tilting module of A. Then,* $(A, \operatorname{Hom}_{A^{op}}(V, A))$ *is an* $(\operatorname{domdim}_{(A,R)} T - 2)$-$\mathcal{F}(\tilde{\Delta})$ *cover of* $\operatorname{End}_A(V)$.

Proof. For (a), see [Cru24c, Proposition 2.4.4.]. For (b), see [Cru24c, Theorem 6.0.1, Theorem 6.2.1]. ◻

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3. Relative (co-)dominant dimension with respect to a module

In this part, we study a new generalization of relative dominant dimension over projective Noetherian algebras and how it can be used to obtain information about the functor F_Q, for some $Q \in A\text{-mod} \cap R\text{-proj}$, where we do not assume Q to be necessarily projective.

Definition 3.0.1. Let $T, X \in A\text{-mod} \cap R\text{-proj}$. If X does not admit a left $\operatorname{add} T$-approximation which is an (A, R)-monomorphism, then we say that **relative dominant dimension of X with respect to T** is zero. Otherwise, the **relative dominant dimension of X with respect to T**, denoted by $T\text{-domdim}_{(A,R)} X$, is the supremum of all $n \in \mathbb{N} \subset \mathbb{N} \cup \{0, +\infty\}$ such that there exists an (A, R)-exact sequence $0 \to X \to T_1 \to \cdots \to T_n$ which remains exact under $\operatorname{Hom}_A(-, T)$ with all $T_i \in \operatorname{add} T$.

By convention, the empty direct sum is the zero module. So, the existence of a finite relative $\operatorname{add} T$-coresolutions implies that $T\text{-domdim}_{(A,R)} X$ is infinite. In the same way, we can define the relative dominant dimension of a right module with respect to a right module Q. We write $Q\text{-domdim}\,(A, R)$ instead of $Q\text{-domdim}_{(A,R)} A$. We just write $T\text{-domdim}_A X$ if R is a field.

Definition 3.0.1 generalizes the concept of relative dominant dimension introduced in [Cru22b].

Proposition 3.0.2. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra with* $\operatorname{domdim}(A, R) \geq 1$ *with projective (A, R)-injective-strongly faithful left A-module P. Then, for any $X \in A$-mod, we have* $P\text{-domdim}_{(A,R)} X = \operatorname{domdim}_{(A,R)} X$.

Proof. See [Cru22b, Proposition 3.15]. □

In order to avoid changing from left to right modules systematically in the coming sections, we can introduce the relative codominant dimension with respect to a module.

Definition 3.0.3. Let $Q, X \in A\text{-mod} \cap R\text{-proj}$. If X does not admit a surjective right $\operatorname{add} Q$-approximation, then we say that **relative codominant dimension of X with respect to Q** is zero. Otherwise, the **relative codominant dimension of X with respect to Q**, denoted by $Q\text{-codomdim}_{(A,R)} X$, is the supremum of all $n \in \mathbb{N} \subset \mathbb{N} \cup \{0, +\infty\}$ such that there exists an (A, R)-exact sequence $Q_n \to \cdots \to Q_1 \to X \to 0$ which remains exact under $\operatorname{Hom}_A(Q, -)$ with all $Q_i \in \operatorname{add} Q$.

In particular, $DQ\text{-domdim}_{(A,R)} DM = Q\text{-codomdim}_{(A,R)} M$ whenever $Q, M \in A\text{-mod} \cap R\text{-proj}$. As we will see later, these two invariants coincide in our main cases of interest.

3.1. Relative Mueller's characterisation of relative dominant dimension with respect to a module

In this section, we study a version of Mueller's theorem for the relative dominant dimension with respect to a module. The arguments for these results are inspired by [AS93, Proposition 2.1].

Theorem 3.1.1. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that* $Q \in A\text{-mod} \cap R\text{-proj}$ *satisfying in addition that* $\operatorname{Hom}_A(Q, Q) \in R\text{-proj}$. *Denote by B the endomorphism algebra* $\operatorname{End}_A(Q)^{op}$. *For* $M \in A\text{-mod} \cap R\text{-proj}$ *and $n \in \mathbb{N}$, the following assertions hold.*

(i) *The counit* $\chi_M : Q \otimes_B \operatorname{Hom}_A(Q, M) \to M$ *is surjective if and only if* $DQ\text{-domdim}_{(A,R)} DM \geq 1$.
(ii) $\chi_M : Q \otimes_B \operatorname{Hom}_A(Q, M) \to M$ *is an isomorphism if and only if* $DQ\text{-domdim}_{(A,R)} DM \geq 2$.

Proof. Assume that χ_M is surjective. Since $\operatorname{Hom}_A(Q, M) \in B$-mod, there exists $X \in \operatorname{add} Q$ and a surjective map $\operatorname{Hom}_A(Q, X) \twoheadrightarrow \operatorname{Hom}_A(Q, M)$, say g. The functor $Q \otimes_B -$ is right exact, so $Q \otimes_B g$ is surjective as well. Define $f := \chi_M \circ Q \otimes_B g \circ \chi_X^{-1} \in \operatorname{Hom}_A(X, M)$. The map f is surjective, and it satisfies $\operatorname{Hom}_A(Q, f) = g$. So Df is an (A, R)-monomorphism $DM \to DX$ which remains exact under $\operatorname{Hom}_A(-, DQ)$. Thus, $DQ\text{-domdim}_{(A,R)} DM \geq 1$. Conversely, assume that $DQ\text{-domdim}_{(A,R)} DM \geq 1$. So, there exists $X \in \operatorname{add} DQ$ and an (A, R)-monomorphism $f : DM \to X$ which is also a left $\operatorname{add} DQ$-approximation. Since χ is a natural transformation between $Q \otimes_B \operatorname{Hom}_A(Q, -)$ and $\operatorname{id}_{B\text{-mod}}$, the map $Df \circ \chi_{DX} = \chi_{DDM} \circ Q \otimes_B \operatorname{Hom}_A(Q, Df)$ is surjective. In particular, χ_{DDM} is surjective. As $DDM \simeq M$, χ_M is surjective, and (i) follows.

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Now, assume that $DQ\text{-domdim}_{(A,R)}\, DM \geq 2$. Then, there exists an (A,R)-exact sequence $0 \to DM \xrightarrow{f_0} X_0 \xrightarrow{f_1} X_1$, with $X_0, X_1 \in \operatorname{add} DQ$, which remains exact under $\operatorname{Hom}_A(-, DQ)$. As $Q \otimes_B -$ is right exact, the following diagram is commutative with exact rows:

$$\begin{array}{ccccc} Q \otimes_B \operatorname{Hom}_A(Q, DX_1) & \xrightarrow{Q\otimes_B \operatorname{Hom}_A(Q, Df_1)} & Q \otimes_B \operatorname{Hom}_A(Q, DX_0) & \xrightarrow{Q\otimes_B \operatorname{Hom}_A(Q, Df_0)} & Q \otimes_B \operatorname{Hom}_A(Q, DDM) \\ \simeq\downarrow \chi_{DX_1} & & \simeq\downarrow \chi_{DX_0} & & \downarrow \chi_{DDM} \\ DX_1 & \xrightarrow{Df_1} & DX_0 & \xrightarrow{Df_0} & DDM \end{array} \quad . \tag{2}$$

By diagram chasing, χ_{DDM} is an isomorphism. Since $DDM \simeq M$, χ_M is an isomorphism. Conversely, assume that χ_M is an isomorphism. B is a Noetherian R-algebra, so we can consider a projective B-presentation for $\operatorname{Hom}_A(Q, M)$ of the form $\operatorname{Hom}_A(Q, Q^m) \xrightarrow{g_1} \operatorname{Hom}_A(Q, Q^n) \xrightarrow{g_0} \operatorname{Hom}_A(Q, M)$ for some integers m, n. Since $\operatorname{Hom}_A(Q, -)_{|\operatorname{add} T}$ is full and faithful, there exists $f_1 \in \operatorname{Hom}_A(Q^m, Q^n)$ such that $\operatorname{Hom}_A(Q, f_1) = g_1$. Fix $f_0 = \chi_M \circ Q \otimes_B g_0 \circ \chi_{Q^n}^{-1}$. In particular, $\operatorname{Hom}_A(Q, f_0) = g_0$. Hence,

$$\begin{array}{ccccc} Q \otimes_B \operatorname{Hom}_A(Q, Q^m) & \xrightarrow{Q\otimes_B g_1} & Q \otimes_B \operatorname{Hom}_A(Q, Q^n) & \xrightarrow{Q\otimes_B g_0} & Q \otimes_B \operatorname{Hom}_A(Q, M) \\ \simeq\downarrow \chi_{Q^m} & & \simeq\downarrow \chi_{Q^n} & & \downarrow \chi_M \\ Q^m & \xrightarrow{f_1} & Q^n & \xrightarrow{f_0} & M \end{array} \tag{3}$$

is a commutative diagram. Since the vertical maps are isomorphisms and the upper row is exact, it follows that the bottom row is exact and by construction, and it remains exact under $\operatorname{Hom}_A(Q, -)$. As $M \in R$-proj, it is, also, (A,R)-exact. By applying the standard duality D, we obtain that $DQ\text{-domdim}_{(A,R)}\, DM \geq 2$. □

Remark 3.1.2. Note that, for each $M \in A\text{-mod} \cap R\text{-proj}$, the map χ_M is equivalent to the map δ_{DM} studied in [Cru22b, Lemma 3.21] when $Q = P$ is a projective (A,R)-injective-strongly faithful module.

Similarly, we can write the dual version of Theorem 3.1.1.

Theorem 3.1.3. *Let R be a commutative Noetherian ring. Let A be projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying in addition that $\operatorname{Hom}_A(Q, Q) \in R\text{-proj}$. Denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. For $M \in A\text{-mod} \cap R\text{-proj}$, the following assertions hold.*

(i) $\chi^r_{DM} \colon \operatorname{Hom}_A(DQ, DM) \otimes_B DQ \to DM$ *is surjective if and only if* $Q\text{-domdim}_{(A,R)}\, M \geq 1$.
(ii) $\chi^r_{DM} \colon \operatorname{Hom}_A(DQ, DM) \otimes_B DQ \to DM$ *is an isomorphism if and only if* $Q\text{-domdim}_{(A,R)}\, M \geq 2$.

In fact, Theorem 3.1.1 characterizes the smaller cases of relative codominant dimension with respect to a module. Now, the second part of the Mueller version for relative (co-)dominant dimension with respect to a module is as follows:

Theorem 3.1.4. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying in addition that $\operatorname{Hom}_A(Q, Q) \in R\text{-proj}$. Denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. For $M \in A\text{-mod} \cap R\text{-proj}$ and $n \in \mathbb{N}$, the following assertions hold.*

(i) $Q\text{-codomdim}_{(A,R)}\, M \geq n \geq 2$ *if and only if* $\chi_M \colon Q \otimes_B \operatorname{Hom}_A(Q, M) \to M$ *is an isomorphism of left A-modules and* $\operatorname{Tor}_i^B(Q, \operatorname{Hom}_A(Q, M)) = 0$, $1 \leq i \leq n-2$.
(ii) $Q\text{-domdim}_{(A,R)}\, M \geq n \geq 2$ *if and only if* $\chi^r_{DM} \colon \operatorname{Hom}_A(DQ, DM) \otimes_B DQ \to DM$ *is an isomorphism and* $\operatorname{Tor}_i^B(\operatorname{Hom}_A(DQ, DM), DQ) = \operatorname{Tor}_i^B(\operatorname{Hom}_A(M, Q), DQ) = 0$, $1 \leq i \leq n-2$.

Proof. We shall prove (i). Assume that $DQ\text{-domdim}_{(A,R)}\, DM \geq n \geq 2$. By Theorem 3.1.1, χ_M is an isomorphism. By definition, there exists an (A,R)-exact sequence $\delta \colon DM \hookrightarrow X_0 \to X_1 \to \cdots \to X_{n-1}$ which remains exact under $\operatorname{Hom}_A(-, DQ)$ with $X_i \in \operatorname{add} DQ$, $i = 0, \ldots, n-1$. In particular, $\operatorname{Hom}_A(X_{n-1}, DQ) \to \cdots \to \operatorname{Hom}_A(X_0, DQ) \to \operatorname{Hom}_A(DM, DQ) \to 0$ is exact and can be continued to a left projective B-resolution of $\mathrm{Hom}_A(Q, M)$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} Q \otimes_B \mathrm{Hom}_A(Q, DX_{n-1}) & \longrightarrow & \cdots & \longrightarrow & Q \otimes_B \mathrm{Hom}_A(Q, DX_0) & \twoheadrightarrow & Q \otimes_B \mathrm{Hom}_A(Q, M) \\ \simeq\downarrow \chi_{DX_{n-1}} & & & & \simeq\downarrow \chi_{DX_0} & & \simeq\downarrow \chi_{DDM} \\ DX_{n-1} & \longrightarrow & \cdots & \longrightarrow & DX_0 & \twoheadrightarrow & DDM \end{array} . \tag{4}$$

Observe that the bottom row is exact since the exact sequence δ is (A, R)-exact. Since all vertical maps are isomorphisms, it follows that the upper row is exact. Thus, $\mathrm{Tor}_i^B(Q, \mathrm{Hom}_A(Q, M)) = 0, 1 \leq i \leq n-2$.

Conversely, assume that χ_M is an isomorphism and $\mathrm{Tor}_i^B(Q, \mathrm{Hom}_A(Q, M)) = 0$ for $1 \leq i \leq n-2$. Let $\mathrm{Hom}_A(Q, X_{n-1}) \xrightarrow{g_{n-1}} \cdots \to \mathrm{Hom}_A(Q, X_0) \xrightarrow{g_0} \mathrm{Hom}_A(Q, M) \to 0$ be a truncated projective B-resolution of $\mathrm{Hom}_A(Q, M)$ and $X_i \in \mathrm{add}_A Q$. Furthermore, $\mathrm{Hom}_A(Q, -)_{|\mathrm{add}\, Q}$ is full and faithful, so each map g_i can be written as $\mathrm{Hom}_A(Q, f_i)$ including g_0 since χ_M is an isomorphism, where $f_i \in \mathrm{Hom}_A(X_i, X_{i-1})$ and $f_0 \in \mathrm{Hom}_A(X_0, M)$. So, we have a commutative diagram

$$\begin{array}{ccccccc} Q \otimes_B \mathrm{Hom}_A(Q, X_{n-1}) & \xrightarrow{Q \otimes_B \mathrm{Hom}_A(Q, f_{n-1})} & \cdots & \longrightarrow & Q \otimes_B \mathrm{Hom}_A(Q, X_0) & \xrightarrow{Q \otimes g_0} & Q \otimes_B \mathrm{Hom}_A(Q, M) \\ \chi_{X_{n-1}} \downarrow \simeq & & & & \simeq \downarrow \chi_{X_0} & & \simeq \downarrow \chi_M \\ X_{n-1} & \xrightarrow{f_{n-1}} & \cdots & \longrightarrow & X_0 & \xrightarrow{f_0} & M \end{array} . \tag{5}$$

By assumption, $\mathrm{Tor}_i^B(Q, \mathrm{Hom}_A(Q, M)) = 0, 1 \leq i \leq n-2$. So, the upper row is exact. By the exactness and the vertical maps being isomorphisms, the bottom row becomes exact. Since $M \in R$-proj, it is also (A, R)-exact, and so it remains (A, R)-exact under D. By construction, the bottom row remains exact under $\mathrm{Hom}_A(Q, -)$; thus, $DQ\text{-domdim}_{(A,R)}\, DM \geq n \geq 2$. The statement (ii) is analogous to (i). □

Remark 3.1.5. The condition $\mathrm{Hom}_A(Q, Q) \in R$-proj is used to enforce that B is a projective Noetherian R-algebra same as A. Having only the criteria in Theorems 3.1.4 and 3.1.1, we can observe that the arguments carry over if B is just a Noetherian R-algebra.

3.1.1. Identities on relative dominant and codominant dimension

An immediate consequence of Theorems 3.1.4 and 3.1.1 is the following.

Corollary 3.1.6. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that* $Q \in A\text{-mod} \cap R\text{-proj}$ *satisfying in addition that* $\mathrm{Hom}_A(Q, Q) \in R\text{-proj}$. *Then,*

$$DQ\text{-domdim}\,(A, R) = Q\text{-codomdim}_{(A,R)}\, DA = Q\text{-domdim}\,(A, R).$$

Proof. By Tensor-Hom adjunction, there are bimodule isomorphisms $\psi\colon \mathrm{Hom}_A(Q, DA) \to DQ$ and $\omega\colon Q \to \mathrm{Hom}_A(DQ, DA)$. In particular, ψ is a left B-isomorphism, while ω is a right B-isomorphism. Moreover, $\chi^r_{DA} \circ \omega \otimes_B \psi = \chi_{DA}$. By Theorems 3.1.1 and 3.1.3, $DQ\text{-domdim}\,(A, R) \geq i$ if and only if $Q\text{-domdim}\,(A, R) \geq i$ for $i = 1, 2$. By Theorem 3.1.4, $DQ\text{-domdim}\,(A, R) = DQ\text{-domdim}_{(A,R)}\, DDA_A \geq n \geq 2$ if and only if χ_{DA} is an isomorphism and $0 = \mathrm{Tor}_i^B(Q, \mathrm{Hom}_A(Q, DA)) = \mathrm{Tor}_i^B(Q, DQ) = \mathrm{Tor}_i^B(\mathrm{Hom}_A(DQ, DA), DQ)$, $1 \leq i \leq n-2$. The latter is equivalent to $Q\text{-domdim}\,(A, R) \geq n \geq 2$. □

It follows, by Theorem 3.1.4 and Corollary 3.1.6 (see also [ASS06, A.4]), that the relative dominant dimension of the regular module with respect to a module Q over a finite-dimensional algebra coincides with the **faithful dimension of** $\boldsymbol{Q}$ introduced in [BS98].

There is a version of Corollary 3.1.6 for (partial) tilting modules, if the split quasi-hereditary algebra admits a duality functor on $A\text{-mod} \cap R\text{-proj}$ interchanging $\Delta(\lambda)$ with $\nabla(\lambda)$ (or a simple preserving duality if the ground ring is a field). Here, by a **duality functor** on $A\text{-mod} \cap R\text{-proj}$ we mean an exact, involutive and contravariant autoequivalence of categories $A\text{-mod} \cap R\text{-proj} \to A\text{-mod} \cap R\text{-proj}$.

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Proposition 3.1.7. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra with a characteristic tilting module T. Let $V \in \operatorname{add}_A T$ and assume that ${}^\natural(-)\colon A\text{-mod}\cap R\text{-proj} \to A\text{-mod}\cap R\text{-proj}$ is a duality satisfying ${}^\natural\Delta(\lambda) = \nabla(\lambda)$ and ${}^\natural V \simeq V$ for all $\lambda \in \Lambda$. Then, $V\text{-domdim}_{(A,R)} T = V\text{-codomdim}_{(A,R)} T$.*

Proof. Assume that $V\text{-domdim}_{(A,R)} T \geq n \geq 1$ for $n \in \mathbb{N}$. By definition, there exists an (A, R)-exact sequence $\delta : 0 \to T \to V_0 \to V_1 \to \cdots \to V_{n-1}$, with $V_i \in \operatorname{add}_A V$, which remains exact under $\operatorname{Hom}_A(-, V)$. Applying the duality ${}^\natural(-)$ to δ, we obtain the exact sequence ${}^\natural\delta : {}^\natural V_{n-1} \to \cdots \to {}^\natural V_1 \to {}^\natural V_0 \to {}^\natural T \to 0$. Applying ${}^\natural(-)$ to the exact sequences defining $T(\lambda)$, we obtain that ${}^\natural T$ is also a characteristic tilting module and so $\operatorname{add} T = \operatorname{add} {}^\natural T$. In particular, ${}^\natural\delta$ is (A, R)-exact. It remains to show that ${}^\natural\delta$ remains exact under $\operatorname{Hom}_A(V, -) \simeq \operatorname{Hom}_A({}^\natural V, -)$. To see this, note that for every homomorphism $f \in \operatorname{Hom}_A(N, L)$, the maps $\operatorname{Hom}_A({}^\natural V, {}^\natural f)$ and $\operatorname{Hom}_A(f, V)$ are equivalent since ${}^\natural(-)$ is a duality. Hence, $\operatorname{Hom}_A({}^\natural V, {}^\natural\delta)$ is an exact complex since $\operatorname{Hom}_A(\delta, V)$ is so (see also Lemma 2.0.2). Hence, $V\text{-codomdim}_{(A,R)} T \geq n \geq 1$. Conversely, $V\text{-codomdim}_{(A,R)} T = DV\text{-domdim}_{(A,R)} DT \geq DV\text{-codomdim}_{(A,R)} DT = V\text{-domdim}_{(A,R)} T$. □

3.1.2. Behavior of relative dominant dimension on long exact sequences

Using Theorem 3.1.4, it is now clear how the relative dominant dimension with respect to a module behaves on short exact sequences.

Lemma 3.1.8. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod}\cap R\text{-proj}$ satisfying in addition that $\operatorname{Hom}_A(Q, Q) \in R\text{-proj}$. Let $M \in R\text{-proj}$ and consider the following (A, R)-exact $0 \to M_1 \to M \to M_2 \to 0$ which remains exact under $\operatorname{Hom}_A(-, Q)$. Let $n = Q\text{-domdim}_{(A,R)} M$ and $n_i = Q\text{-domdim}_{(A,R)} M_i$. Then, the following holds.*

(a) $n \geq \min\{n_1, n_2\}$.
(b) *If* $n_1 < n$, *then* $n_2 = n_1 - 1$.
(c) (i) $n_1 = n \implies n_2 \geq n - 1$.
  (ii) $n_1 = n + 1 \implies n_2 \geq n$.
  (iii) $n_1 \geq n + 2 \implies n_2 = n$.
(d) $n < n_2 \implies n_1 = n$.
(e) (i) $n = n_2 \implies n_1 \geq n_2$.
  (ii) $n = n_2 + 1 \implies n_1 \geq n_2 + 1$.
  (iii) $n \geq n_2 + 2 \implies n_1 = n_2 + 1$.

Proof. By assumption, $0 \to \operatorname{Hom}_A(DQ, DM_2) \to \operatorname{Hom}_A(DQ, DM) \to \operatorname{Hom}_A(DQ, DM_1) \to 0$ is exact. The remaining of the proof is exactly analogous to Lemma 5.12 of [Cru22b]. □

Corollary 3.1.9. *Assume that $Q \in A\text{-mod}\cap R\text{-proj}$ satisfying in addition that $\operatorname{Hom}_A(Q, Q) \in R\text{-proj}$. Let $M_i \in A\text{-mod}\cap R\text{-proj}$, $i \in I$, for some finite set I. Then,*

$$Q\text{-domdim}_{(A,R)}\left(\bigoplus_{i\in I} M_i\right) = \inf\{Q\text{-domdim}_{(A,R)} M_i : \ i \in I\}.$$

Proof. The argument given in [Cru22b, Corollary 5.10] works in this setup replacing the use of [Cru22b, Theorem 5.2, Proposition 3.23] by Theorems 3.1.1 and 3.1.4. □

Usually, proving that a certain exact sequence remains exact under a certain Hom functor might be difficult. Sometimes, we can assert that the existence of an (A, R)-exact sequence implies the existence of another whose morphisms can be factored through $\operatorname{add} Q$-approximations for some module Q.

Lemma 3.1.10. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra, $Q, M \in A\text{-mod}\cap R\text{-proj}$, $n \in \mathbb{N}$ and $X_i \in \operatorname{add} Q$ for $1 \leq i \leq n$. Assume that there exists an (A, R)-exact sequence $\delta\colon 0 \to M \to X_1 \to \cdots \to X_n$, with $X_i \in \operatorname{add} Q$, which remains exact under $\operatorname{Hom}_A(-, Q)$, that can be continued to an (A, R)-exact sequence $0 \to M \to X_1 \to \cdots \to X_n \xrightarrow{h} Y$ with $Y \in \operatorname{add} Q$. Then, $Q\text{-domdim}_{(A,R)} M \geq n + 1$.*

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Proof. Consider the exact sequence $D\delta$. Denote by α_i the maps $DX_i \to DX_{i-1}$, where we fix $X_0 := DM$. The map Dh admits a factorization through $\ker \alpha_n$, say $\nu \circ \pi$. Since B is a Noetherian R-algebra, there exists $Z \in \operatorname{add} DQ$ such that there exists a surjective map $g\colon \operatorname{Hom}_A(DQ, Z) \to \operatorname{Hom}_A(DQ, \ker \alpha_n)$. Further, by projectization, the map $\operatorname{Hom}_A(DQ, \nu) \circ g$ is equal to $\operatorname{Hom}_A(DQ, f)$ for some $f \in \operatorname{Hom}_A(Z, DX_n)$. By construction, the sequence $Z \xrightarrow{f} DX_{n-1} \to \cdots \to DX_1 \to DM \to 0$ becomes exact under $\operatorname{Hom}_A(DQ, -)$, and if exact, it is (A, R)-exact. So, it is enough to show that $\ker \alpha_n = \operatorname{im} f$. This is clear since χ^r_Z is an isomorphism, and so $\alpha_n \circ f = 0$ which, in turn, implies that $g = \operatorname{Hom}_A(DQ, s)$ for some $s \in \operatorname{Hom}_A(Z, \ker \alpha_n)$. By construction, this map is therefore surjective since π must factor through s. So, the result holds. □

Remark 3.1.11. Observe that Theorem 2.8 of [KSX01] and Theorem 2.15 of [KSX01] are particular cases of Lemma 3.1.10 (when $n = 1$) and Theorem 3.1.3.

In contrast to Lemma 3.1.10, if we know the last map in an exact sequence and its cokernel, then we can deduce the value of the relative dominant dimension with respect to a module using that exact sequence.

Proposition 3.1.12. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra and $Q \in A\text{-mod} \cap R\text{-proj}$ so that $\operatorname{Hom}_A(Q, Q) \in R\text{-proj}$ and $\operatorname{Ext}_A^{i>0}(Q, Q) = 0$. Suppose that $M \in {}^\perp Q$. An (A, R)-exact sequence $\delta\colon 0 \to M \to Q_1 \to \cdots \to Q_n$ yields $Q\text{-domdim}_{(A,R)} M \geq n$ if and only if $Q_i \in \operatorname{add} Q$ and the cokernel of $Q_{n-1} \to Q_n$ belongs to ${}^\perp Q$.*

Proof. Denote by X_i the cokernel of $Q_{i-1} \to Q_i$ and fix $Q_0 = M$. Assume that $Q_i \in \operatorname{add} Q$ and $X_n \in {}^\perp Q$. So, $X_n \in R\text{-proj}$ and δ is (A, R)-exact. It follows that $\operatorname{Ext}_A^1(X_i, Q) \simeq \operatorname{Ext}_A^{n-i+1}(X_n, Q) = 0$. This means that δ remains exact under $\operatorname{Hom}_A(-, Q)$, and so $Q\text{-domdim}_{(A,R)} M \geq n$.

Conversely, assume that δ yields $Q\text{-domdim}_{(A,R)} M \geq n$. By assumption, $Q_i \in \operatorname{add} Q$ and δ is (A, R)-exact. Hence, $X_n \in A\text{-mod} \cap R\text{-proj}$. Combining the conditions of $\operatorname{Ext}_A^{i>0}(Q_i, Q) = 0$, $\operatorname{Hom}_A(-, Q)$ being exact on δ and $M \in {}^\perp Q$, it follows by induction on i that $X_i \in {}^\perp Q$. □

We note the following application of Lemma 3.1.8 useful in examples.

Corollary 3.1.13. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$ satisfies in addition that $\operatorname{Hom}_A(Q, Q) \in R\text{-proj}$ and $\operatorname{Ext}_A^{i>0}(Q, Q) = 0$. Let $M \in A\text{-mod} \cap R\text{-proj}$ and an (A, R)-exact sequence $0 \to M \to Q_1 \to \cdots \to Q_t \to X \to 0$. If $\operatorname{Ext}_A^i(X, Q) = 0$ and $Q_i \in \operatorname{add} Q$ for $1 \leq i \leq t$, then $Q\text{-domdim}_{(A,R)} M = t + Q\text{-domdim}_{(A,R)} X$.*

Proof. Let C_i be the image of the maps $Q_i \to Q_{i+1}$, $i = 1, \ldots, t-1$. Since $Q \in {}^\perp Q$, it follows that $\operatorname{Ext}_A^1(C_i, Q) \simeq \operatorname{Ext}_A^{t-i+1}(X, Q) = 0$. So, the exact sequences $0 \to C_i \to Q_{i+1} \to C_{i+1} \to 0$ are exact under $\operatorname{Hom}_A(-, Q)$ (also with $C_0 = M$ and $C_t = X$). By Lemma 3.1.8 and induction on i, the result follows. □

3.1.2.1. Projective dimension as an upper bound for relative dominant dimension.
We should point out that under certain conditions, the relative dominant dimension of the regular module with respect to a module Q is bounded above (if finite) by the projective dimension of Q.

Proposition 3.1.14. *Let R be a commutative Noetherian ring and A be a projective Noetherian R-algebra. Let Q be a module in $Q \in A\text{-mod} \cap R\text{-proj}$ with $Q \in {}^\perp Q$ and $B := \operatorname{End}_A(Q)^{op} \in R\text{-proj}$ so that the projective dimensions $\operatorname{pdim}_A Q$ and $\operatorname{pdim}_B Q$ are finite. If $Q\text{-domdim}\,(A, R) > \max\{\operatorname{pdim}_B Q, 2\}$, then Q is a tilting A-module.*

Proof. Fix $n = \operatorname{pdim}_B Q$. By assumption, $DQ\text{-codomdim}_{(A^{op},R)} DA \geq n + 1$, and so $\operatorname{Tor}_i^B(Q, DQ) = 0$ for $i = 1, \ldots, n-1$ and $Q \otimes_B DQ \simeq DA$ by Theorem 3.1.4. Therefore, applying $- \otimes_B DQ$ to a projective resolution of Q over B yields an exact sequence $0 \to X_n \to \cdots \to X_0 \to Q \otimes_B DQ \to 0$ with $X_i \in \operatorname{add} DQ$. By applying D to this exact sequence, we obtain that Q is a tilting A-module. □

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3.2. Change of rings on relative dominant dimension with respect to a module

We now show that the relative dominant dimension with respect to a module behaves well under change of rings techniques. As usual, the following results also hold for right A-modules and consequently with codominant dimension in place of dominant dimension. For brevity, we consider only the left versions. In the following, we say that a module $Q \in A\text{-mod} \cap R\text{-proj}$ **has a base change property** if $\operatorname{Hom}_A(Q, Q) \in R\text{-proj}$ and for every commutative R-algebra Noetherian ring S, the canonical map $S \otimes_R \operatorname{Hom}_A(Q, Q) \to \operatorname{Hom}_{S\otimes_R A}(S \otimes_R Q, S \otimes_R Q)$ is an isomorphism.

Lemma 3.2.1. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$ has a base change property. Denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. Assume that $M \in A\text{-mod} \cap R\text{-proj}$ satisfies the following two conditions:*

1. $\operatorname{Hom}_A(M, Q) \in R\text{-proj}$*;*
2. *The map* $R(\mathfrak{m}) \otimes_R \operatorname{Hom}_A(M, Q) \to \operatorname{Hom}_{A(\mathfrak{m})}(M(\mathfrak{m}), Q(\mathfrak{m}))$ *is an isomorphism for all* $\mathfrak{m} \in \operatorname{MaxSpec} R$.

Then, the following assertions are equivalent.

(a) $Q\text{-domdim}_{(A,R)}\, M \geq 1$*;*
(b) $S \otimes_R Q\text{-domdim}_{(S\otimes_R A,S)}\, S \otimes_R M \geq 1$ *for every commutative R-algebra and Noetherian ring S;*
(c) $Q_\mathfrak{m}\text{-domdim}_{(A_\mathfrak{m},R_\mathfrak{m})}\, M_\mathfrak{m} \geq 1$ *for every maximal ideal $\mathfrak{m}$ of R;*
(d) $Q(\mathfrak{m})\text{-domdim}_{A(\mathfrak{m})}\, M(\mathfrak{m}) \geq 1$ *for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Let S be a commutative R-algebra. Denote by D_S the standard duality with respect to S, $\operatorname{Hom}_S(-, S)$. The result follows from the following commutative diagram:

$$\begin{array}{ccc} S \otimes_R \operatorname{Hom}_A(DQ, DM) \otimes_B DQ & \xrightarrow{S\otimes_R \chi^r_{DM}} & S \otimes_R DM \\ \simeq\downarrow \theta_{S,M} & & \\ S \otimes_R \operatorname{Hom}_A(DQ, DM) \otimes_{S\otimes_R B} S \otimes_R DQ & & \downarrow \simeq \\ \downarrow \varphi_S & & \\ \operatorname{Hom}_{S\otimes_R A}(D_S(S \otimes_R Q), D_S(S \otimes_R M)) \otimes_{S\otimes_R B} D_S(S \otimes_R Q) & \xrightarrow{\chi^r_{D_S(S\otimes_R M)}} & D_S(S \otimes_R M) \end{array}, \tag{6}$$

where the map $\theta_{S,M}$ is the isomorphism given in Proposition 2.3 of [Cru22b], while φ_S is the tensor product of the canonical map given by extension of scalars on Hom (which is not claimed at the moment to be an isomorphism) with the one providing the isomorphism $S \otimes_R DQ \simeq D_S(S \otimes_R Q)$.

The implications $(b) \Rightarrow (c) \Rightarrow (d)$ are immediate. Assume that (a) holds. Then, χ^r_{DM} is surjective. By the commutative diagram, $\chi^r_{D_S S\otimes_R M}$ is surjective, and so (b) follows. Assume that (d) holds. By condition 2, $\varphi_{R(\mathfrak{m})}$ must be an isomorphism for every $\mathfrak{m} \in \operatorname{MaxSpec} R$. Thus, by the diagram, $\chi^r_{DM}(\mathfrak{m})$ is surjective for every maximal ideal $\mathfrak{m}$ of R. By Nakayama's Lemma, χ^r_{DM} is surjective and (a) holds. □

Lemma 3.2.2. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Let $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying $\operatorname{Hom}_A(Q, Q) \in R\text{-proj}$. Denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. For $M \in A\text{-mod} \cap R\text{-proj}$, the following assertions are equivalent.*

(a) $Q\text{-domdim}_{(A,R)}\, M \geq n \geq 1$*;*
(b) $Q\text{-domdim}_{(S\otimes_R A,S)}\, S \otimes_R M \geq n \geq 1$ *for every flat commutative R-algebra and Noetherian ring S;*
(c) $Q\text{-domdim}_{(A_\mathfrak{m},R_\mathfrak{m})}\, M_\mathfrak{m} \geq n \geq 1$ *for every maximal ideal $\mathfrak{m}$ of R.*

Proof. By the flatness of S, the vertical maps of the commutative diagram (6) are isomorphisms. So, by Lemma 3.2.1, the implication $(a) \Rightarrow (b)$ is clear for $n = 1, 2$. Again, since S is flat and B is finitely generated projective over R, $S \otimes_R -$ commutes with Tor functors over B. Therefore, (b) follows by Theorem 3.1.4. Analogously, we obtain $(c) \Rightarrow (a)$. □

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It is no surprise that the relative dominant dimension with respect to a module remains stable under extension of scalars to the algebraic closure. For the sake of completeness, we give the result.

Lemma 3.2.3. *Let k be a field with algebraic closure $\overline{k}$. Let A be a finite-dimensional k-algebra and assume that $Q \in A$-mod. Then, $\overline{k} \otimes_k Q\text{-domdim}_{\overline{k}\otimes_k A}\, \overline{k} \otimes_k M = Q\text{-domdim}_A\, M$.*

Proof. Of course, $\overline{k}$ is free over k. Therefore, $\mathrm{Tor}_i^B(\mathrm{Hom}_A(DQ, DM), DQ) = 0$ if and only if $\mathrm{Tor}_i^{\overline{k}\otimes_k B}(\mathrm{Hom}_{\overline{k}\otimes_k A}(\overline{k} \otimes_k DQ, \overline{k} \otimes_k DM), \overline{k} \otimes_k DQ) = 0$. By the same reason, χ^r_{DM} is surjective (or bijective) if and only if $\chi^r_{\overline{k}\otimes_k DM}$ is surjective (or bijective). □

Lemma 3.2.4. *Let $Q \in A\text{-mod} \cap R\text{-proj}$ with a base change property. Denote by B the endomorphism algebra $\mathrm{End}_A(Q)^{op}$. Let $M \in A\text{-mod} \cap R\text{-proj}$ satisfying $\mathrm{Hom}_A(M, Q) \in R\text{-proj}$. Assume that S is a commutative R-algebra and a Noetherian ring such that the canonical map $S \otimes_R \mathrm{Hom}_A(M, Q) \to \mathrm{Hom}_{S\otimes_R A}(S \otimes_R M, S \otimes_R Q)$ is an isomorphism.*

Then, $Q\text{-domdim}_{(A,R)}\, M \leq S \otimes_R Q\text{-domdim}_{(S\otimes_R A,S)}\, S \otimes_R M$.

Proof. Assume that $Q\text{-domdim}_{(A,R)}\, M \geq n \geq 1$ for some $n \in \mathbb{N}$. Then, there exists an (A, R)-exact sequence $\delta\colon 0 \to M \to X_1 \to \cdots \to X_n$ which remains exact under $\mathrm{Hom}_A(-, Q)$, where $X_i \in \mathrm{add}_A\, Q$. The functor $S\otimes_R -$ preserves R-split exact sequences. Hence, $0 \to S\otimes_R M \to S\otimes_R X_1 \to \cdots \to S\otimes_R X_n$ is $(S \otimes_R A, S)$-exact and $S \otimes_R X_i \in \mathrm{add}_{S\otimes_R A}\, S \otimes_R Q$. Since $\mathrm{Hom}_A(M, Q) \in R$-proj, the sequence $\mathrm{Hom}_A(\delta, Q)$ splits over R, and thus, $S \otimes_R \mathrm{Hom}_A(\delta, Q)$ is an exact sequence. Using the commutative diagram

$$\begin{array}{ccccccc}
S \otimes_R \mathrm{Hom}_A(X_n, Q) & \longrightarrow & S \otimes_R \mathrm{Hom}_A(X_{n-1}, Q) & \longrightarrow & \cdots & \longrightarrow & S \otimes_R \mathrm{Hom}_A(M, Q) \\
\downarrow\simeq & & \downarrow\simeq & & & & \downarrow\simeq \\
\mathrm{Hom}_{S\otimes_R A}(S \otimes_R X_n, S \otimes_R Q) & \longrightarrow & \mathrm{Hom}_{S\otimes_R A}(S \otimes_R X_{n-1}, S \otimes_R Q) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_{S\otimes_R A}(S \otimes_R M, S \otimes_R Q),
\end{array}$$

it follows that the bottom row is exact. Hence, $S \otimes_R Q\text{-domdim}_{(A,R)}\, S \otimes_R M \geq n$. □

Finally, we reach the main result of this section.

Theorem 3.2.5. *Let R be a commutative Noetherian ring. Let A be a projective Noetherian R-algebra. Assume that $Q \in A\text{-mod} \cap R\text{-proj}$ has a base change property. Denote by B the endomorphism algebra $\mathrm{End}_A(Q)^{op}$. Assume that $M \in A\text{-mod} \cap R\text{-proj}$, satisfies the following two conditions:*

1. $\mathrm{Hom}_A(M, Q) \in R$-proj*;*
2. *The canonical map $R(\mathfrak{m}) \otimes_R \mathrm{Hom}_A(M, Q) \to \mathrm{Hom}_{A(\mathfrak{m})}(M(\mathfrak{m}), Q(\mathfrak{m}))$ is an isomorphism for every maximal ideal $\mathfrak{m}$ of R.*

Then, $Q\text{-domdim}_{(A,R)}\, M = \inf\{Q(\mathfrak{m})\text{-domdim}_{A(\mathfrak{m})}\, M(\mathfrak{m})\colon \mathfrak{m} \in \mathrm{MaxSpec}(R)\}$.

Proof. By Lemma 3.2.4, $Q(\mathfrak{m})\text{-domdim}_{A(\mathfrak{m})}\, M(\mathfrak{m}) \geq Q\text{-domdim}_{(A,R)}\, M$ for every maximal ideal $\mathfrak{m}$ of R. Conversely, assume that $Q(\mathfrak{m})\text{-domdim}_{A(\mathfrak{m})}\, M(\mathfrak{m}) \geq n$ for every maximal ideal $\mathfrak{m}$ of R. We want to show that $Q\text{-domdim}_{(A,R)}\, M \geq n$. If $n = 0$, then there is nothing to show. Using the commutative diagram (6), we obtain that if $n \geq 1$ (resp. $n \geq 2$), then $\chi^r_{DM}(\mathfrak{m})$ is surjective (resp. bijective) for every maximal ideal $\mathfrak{m}$ of R. By Nakayama's Lemma, χ^r_{DM} is surjective, and since $DM \in R$-proj, χ^r_{DM} is bijective in case $n \geq 2$. So, the inequality holds for $n = 1, 2$. Assume now that $n \geq 3$. In particular, $Q\text{-domdim}_{(A,R)}\, M \geq 2$, and therefore, $\mathrm{Hom}_A(DQ, DM) \otimes_B DQ \in R$-proj. By assumption, $\mathrm{Tor}_i^{B(\mathfrak{m})}(\mathrm{Hom}_{A(\mathfrak{m})}(M(\mathfrak{m}), Q(\mathfrak{m})), D_{(\mathfrak{m})}Q(\mathfrak{m})) = 0$ for $1 \leq i \leq n-2$ for every maximal ideal $\mathfrak{m}$ of R. Let $Q^\bullet$ be a deleted B-projective resolution of DQ. So the chain complex $P^\bullet = \mathrm{Hom}_A(M, Q) \otimes_B Q^\bullet$ is a projective complex over R since $\mathrm{Hom}_A(M, Q) \in R$-proj. Consider the Künneth spectral sequence for chain complexes (see, for example, [Wei03, Theorem 5.6.4])

$$E^2_{i,j} = \mathrm{Tor}_j^R(H_i(\mathrm{Hom}_A(M, Q) \otimes_B Q^\bullet), R(\mathfrak{m})) \Rightarrow H_{i+j}(\mathrm{Hom}_A(M, Q) \otimes_B Q^\bullet(\mathfrak{m})). \quad (7)$$

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Since $\mathrm{Hom}_A(M,Q)\otimes_B DQ \in R$-proj, $\mathrm{Hom}_A(M,Q)\otimes_B Q^\bullet(\mathfrak{m})$ becomes a deleted projective $B(\mathfrak{m})$-resolution of $DQ(\mathfrak{m})$.

We shall proceed by induction on $1\leq i\leq n-2$ to show that $\mathrm{Tor}_i^B(\mathrm{Hom}_A(M,Q),DQ)=0$. Observe that $\mathrm{Tor}_1^B(\mathrm{Hom}_A(M,Q),DQ)\otimes_R R(\mathfrak{m})=0$ for every $\mathfrak{m}\in \operatorname{MaxSpec} R$ (see for example [Cru22b, Lemma A.3]). Hence, $\mathrm{Tor}_1^B(\mathrm{Hom}_A(M,Q),DQ)=0$. Assume now that $\mathrm{Tor}_i^B(\mathrm{Hom}_A(M,Q),DQ)=0$ for all $1\leq i\leq l$ with $1\leq l\leq n-2$ for some l. Then, $E^2_{i,j}=0$, for $1\leq i\leq l$, $j\geq 0$ and $E^2_{0,j}=0$, $j>0$. It follows that $\mathrm{Tor}_{l+1}^B(\mathrm{Hom}_A(M,Q),DQ)(\mathfrak{m})=0$ for every $\mathfrak{m}\in\operatorname{MaxSpec} R$ (see [Cru22b, Lemma A.4]). Therefore, $\mathrm{Tor}_i^B(\mathrm{Hom}_A(M,Q),DQ)=0$ for $1\leq i\leq n-2$. By Theorem 3.1.4, the result follows. □

Remark 3.2.6. The condition $DQ\otimes_A M\in R$-proj implies both conditions required in Theorem 3.2.5 and $DQ\otimes_A Q\in R$-proj implies that Q has a base change property (see, for example, the first part of the proof of [Cru22b, Theorem 6.14.].)

Remark 3.2.7. We can, of course, drop the condition ${}^\natural V\simeq V$ in Proposition 3.1.7 by doing first the field case and then consider the integral case using Theorem 3.2.5.

Combining Theorem 3.2.5 with Lemma 3.2.3, we obtain that, in most applications, the computations of relative dominant dimension with respect to a module over a commutative ring can be reduced to computations of relative dominant dimension with respect to a module in the setup of finite-dimensional algebras over algebraically closed fields.

Considering the condition $DQ\otimes_A M\in R$-proj may seem unnatural, but projective modules and characteristic tilting modules of split quasi-hereditary algebras do satisfy such a condition. The following result suggests that there are many modules with such a condition (see also [CPS96, 1.5.2(e), (f)]).

Lemma 3.2.8. *Let $Q\in A$-mod $\cap R$-proj. If $\mathrm{Ext}^1_{A(\mathfrak{m})}(Q(\mathfrak{m}),Q(\mathfrak{m}))=0$ for every maximal ideal $\mathfrak{m}$ of R, then $DQ\otimes_A Q\in R$-proj.*

Proof. For each $\mathfrak{m}\in\operatorname{MaxSpec} R$,

$$\mathrm{Tor}_1^{A(\mathfrak{m})}(DQ(\mathfrak{m}),Q(\mathfrak{m}))=\mathrm{Hom}_{R(\mathfrak{m})}(\mathrm{Ext}^1_{A(\mathfrak{m})}(Q(\mathfrak{m}),Q(\mathfrak{m})),R(\mathfrak{m}))=0.$$

Let $Q^\bullet$ be a deleted projective A-resolution of Q. Since $Q\in R$-proj, $Q^\bullet(\mathfrak{m})$ is a deleted projective $A(\mathfrak{m})$-resolution of $Q(\mathfrak{m})$. Consider the Künneth Spectral sequence with $P=DQ\otimes_A Q^\bullet$, (see, for example, [Wei03, Theorem 5.6.4]) $E^2_{i,j}=\mathrm{Tor}_i^R(\mathrm{Tor}_j^A(DQ,Q),R(\mathfrak{m}))=\mathrm{Tor}_i^R(H_j(DQ\otimes_A Q^\bullet),R(\mathfrak{m}))$. It converges to $H_{i+j}(DQ\otimes_A Q^\bullet\otimes_R R(\mathfrak{m}))=\mathrm{Tor}_{i+j}^{A(\mathfrak{m})}(DQ(\mathfrak{m}),Q(\mathfrak{m}))$. Moreover,

$$E^2_{1,0}=\mathrm{Tor}_1^R(DQ\otimes_A Q,R(\mathfrak{m}))=0$$

for every maximal ideal $\mathfrak{m}$ of R (see for example [Cru22b, Lemma A.3]). Hence, $DQ\otimes_A Q\in R$-proj. □

4. The reduced grade with respect to a module

In [GK15], Gao and Koenig compared the Auslander-Bridger grade with the dominant dimension. The same method works also for relative dominant dimension over any ring with respect to a module if we replace the Ext in the notion of grade by Tor, motivating the terminology: cograde. There is, however, another modification to be considered. We are not interested in the case of grade being zero, and so we will study (the dual notion of) reduced grade (see, for example, [Hos90]).

Definition 4.0.1. Let R be a Noetherian commutative ring and A a projective Noetherian R-algebra. Let $X\in A^{op}$-mod $\cap R$-proj and $M\in A$-mod $\cap R$-proj. The **reduced cograde** of X with respect to M, written as $\operatorname{rcograde}_M X$, is defined as the value $\operatorname{rcograde}_M X=\inf\{i>0\mid \mathrm{Tor}_i^A(X,M)\neq 0\}$. Analogously, we can define the reduced cograde of a right module with respect to a left module.

The following is based on Theorem 2.3 of [GK15].

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Theorem 4.0.2. *Let A be a projective Noetherian R-algebra over a commutative Noetherian ring. Let $Q \in A\text{-mod} \cap R\text{-proj}$ with $\operatorname{Hom}_A(Q,Q) \in R\text{-proj}$ and write $B := \operatorname{End}_A(Q)^{op}$. For each $Q_0, Q_1 \in \operatorname{add} Q$ and $Y \in A\text{-mod} \cap R\text{-proj}$ with an exact sequence $Q_1 \xrightarrow{f} Q_0 \to Y \to 0$, define $X = \operatorname{coker} \operatorname{Hom}_A(f, Q) \in B^{op}\text{-mod}$. Then, $Q\text{-domdim}_{(A,R)}\, Y \geq n \geq 1$ if and only if $\operatorname{rcograde}_{DQ} X \geq n+1$.*

Proof. Write $C := \operatorname{im} \operatorname{Hom}_A(f, Q)$. Applying $-\otimes_B DQ$ to the exact sequences $C \hookrightarrow \operatorname{Hom}_A(Q_1, Q) \twoheadrightarrow X$ and $\operatorname{Hom}_A(Y, Q) \hookrightarrow \operatorname{Hom}_A(Q_0, Q) \twoheadrightarrow C$, we get that $\operatorname{Tor}_i^B(\operatorname{Hom}_A(Y,Q), DQ) \simeq \operatorname{Tor}_{i+1}^B(C, DQ) \simeq \operatorname{Tor}_{i+2}^B(X, DQ)$ for all $i \geq 1$. Moreover, $\operatorname{Tor}_2^B(X, DQ) \simeq \operatorname{Tor}_1^B(C, DQ) \simeq \ker \chi^r_{DY}$. Hence, by Theorems 3.1.3 and 3.1.4, the result follows once we prove that $\operatorname{coker} \chi^r_{DY} = 0$ if and only if $\operatorname{Tor}_1^B(X, DQ) = 0$. This is true since the following commutative diagram

$$\begin{array}{ccccc} DQ_0 & \twoheadrightarrow & \operatorname{Dim} f & \hookrightarrow & DQ_1 \\ \chi^r_{DQ_0}\uparrow\simeq & & g\uparrow & & \chi^r_{DQ_1}\uparrow\simeq \\ \operatorname{Hom}_A(DQ, DQ_0)\otimes_B DQ & \twoheadrightarrow & C\otimes_B DQ & \longrightarrow & \operatorname{Hom}_A(DQ, DQ_1)\otimes_B DQ \end{array} \tag{8}$$

yields that $\operatorname{Tor}_1^B(X, DQ) \simeq \ker g \simeq \operatorname{coker} \chi^r_{DY}$, where the last bijection follows from the use of Snake Lemma on the diagram with vertical maps $\chi^r_{DY}, \chi^r_{DQ_0}$ and g. □

5. Fully faithfulness of $\operatorname{Hom}_A(Q, -)$ on subcategories of A-mod

We will now use Theorems 3.1.1 and 3.1.4 to establish relative codominant dimension with respect to Q as a measure for the strength of the connection between A and $\operatorname{End}_A(Q)^{op}$ via the functor $\operatorname{Hom}_A(Q, -)\colon A\text{-mod} \to \operatorname{End}_A(Q)^{op}\text{-mod}$.

5.1. Q-codomdim $(A, R) \geq 2$ *as a tool for double centralizer properties*

Lemma 5.1.1. *Let $\mathcal{X}$ be the full subcategory of $A\text{-mod} \cap R\text{-proj}$ whose modules X satisfy $Q\text{-codomdim}_{(A,R)}\, X \geq 2$. Then, $\operatorname{Hom}_A(Q, -)$ is fully faithful on $\mathcal{X}$.*

Proof. By Theorem 3.1.1, χ_X is an isomorphism for every $X \in \mathcal{X}$. Fix $F_Q = \operatorname{Hom}_A(Q, -)$ and $\mathbb{I}$ its left adjoint. Then, if $F_Q f = 0$ for some $f \in \operatorname{Hom}_{\mathcal{X}}(M, N)$, we obtain $f \circ \chi_M = \chi_N \circ \mathbb{I} F_Q f = 0$. Hence, in such a case, $f = 0$. So, F_Q is faithful. To show fullness, let $g \in \operatorname{Hom}_B(F_Q M, F_Q N)$ with $M, N \in \mathcal{X}$. Fixing $h = \chi_N \circ \mathbb{I} g \circ \chi_M^{-1}$, we get $F_Q h = g$. □

The next result says that the fully faithfulness of F_Q on $\operatorname{add} DA$ is related to a double centralizer property. Such a result is exactly a version of [Rou08, Proposition 4.33] for general double centralizer properties. This result is known in the literature for Artinian algebras (see, for example, [AS93, Corollary 2.4]).

Lemma 5.1.2. *Let A be a projective Noetherian R-algebra. Let $Q \in A\text{-mod} \cap R\text{-proj}$ satisfying $\operatorname{Hom}_A(Q, Q) \in R\text{-proj}$ and denote by B the endomorphism algebra $\operatorname{End}_A(Q)^{op}$. The following assertions are equivalent.*

(i) *The canonical map of algebras $A \to \operatorname{End}_B(Q)^{op}$, given by $a \mapsto (q \mapsto aq)$, is an isomorphism.*
(ii) *$D\chi_X$ is an isomorphism of right A-modules for all $X \in (A, R)\text{-inj} \cap R\text{-proj}$.*
(iii) *The restriction of F_Q to $\operatorname{add} DA$ is full and faithful.*

Proof. Denote by ψ the canonical map of algebras $A \to \operatorname{End}_B(Q)^{op}$ and denote by ω_X the natural transformation between the identity functor and the double dual DD for $X \in A$-mod. The equivalence $(i) \Leftrightarrow (ii)$ follows from the following commutative diagram

$$\begin{array}{ccccccc} A & & & \xrightarrow[\simeq]{\omega_A} & & & DDA \\ \psi\downarrow & & & & & & \downarrow D\chi_{DA} \\ \operatorname{End}_B(Q) & \xrightarrow[\operatorname{Hom}_A(Q,\omega_Q)]{\simeq} & \operatorname{Hom}_B(Q,DDQ) & \xrightarrow[\operatorname{Hom}_B(Q,D\sigma_Q)]{\simeq} & \operatorname{Hom}_B(Q,D\operatorname{Hom}_A(Q,DA)) & \xleftarrow[\theta]{\simeq} & D(Q\otimes_B \operatorname{Hom}_A(Q,DA)) \end{array}$$

with horizontal maps being isomorphisms, where $\sigma_Q\colon \operatorname{Hom}_A(Q, DA) \to DQ$ and θ are isomorphisms given by Tensor-Hom adjunction.

For each $M, N \in A$-mod, consider the map $F_{M,N}\colon \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_B(F_Q M, F_Q N)$, given by $f \mapsto F_Q f = \operatorname{Hom}_A(Q, f)$. Observe that the maps ψ and $F_{DA,DA}$ are equivalent since the isomorphisms $A \simeq \operatorname{Hom}_A(A, A) \simeq \operatorname{Hom}_A(DA, DA)$ and $\operatorname{Hom}_B(Q, Q) \simeq \operatorname{Hom}_B(DQ, DQ) \simeq \operatorname{Hom}_B(F_Q DA, F_Q DA)$ are functorial (see, for example, [Cru22b, Proposition 2.2]). As $F_{M\oplus X,N}$ is equivalent to $F_{M,X} \oplus F_{X,N}$ and $F_{M,X\oplus N}$ is equivalent to $F_{M,X} \oplus F_{M,N}$ for every $M, X, N \in A$-mod it follows that $(iii) \Leftrightarrow (i)$. □

Remark 5.1.3. Let (A, P, V) be an RQF3-algebra over a commutative Noetherian ring R. If $\operatorname{domdim}_{(A,R)} A \geq 2$, then the Schur functor $\operatorname{Hom}_A(P, -)\colon A\text{-mod} \to \operatorname{End}_A(P)^{op}$-mod is fully faithful on (A, R)-inj $\cap R$-proj (see [Cru22b, Theorem 4.1] with Lemma 5.1.2).

Remark 5.1.4. Let A be a finite-dimensional algebra so that $\operatorname{domdim} A \geq 2$. Let P be a faithful projective-injective A-module. The Schur functor $\operatorname{Hom}_A(P, -)\colon A\text{-mod} \to \operatorname{End}_A(P)^{op}$-mod is fully faithful on A-proj and on A-inj if and only if A is a Morita algebra (see [Cru22a]).

Following the work developed in [MS08, 2.2], we will see in the following lemma that every split quasi-hereditary algebra has a (partial) tilting module with a double centralizer property. In the worst case scenario, this (partial) tilting module coincides with the characteristic tilting module.

Lemma 5.1.5. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra and let T be a characteristic tilting module of A. Then, there is an exact sequence $0 \to A \to M \to X \to 0$, where $M \in \mathcal{F}(\tilde{\Delta}_A) \cap \mathcal{F}(\tilde{\nabla}_A)$ and $X \in \mathcal{F}(\tilde{\Delta}_A)$. Moreover, there exists a (partial) tilting module $Q \in A$-mod$\cap R$-proj such that $DM \in \operatorname{add} Q$ and Q-$\operatorname{domdim}(A, R) \geq 2$. In particular, there exists a double centralizer property on Q.*

Proof. Denote by $R(A)$ the Ringel dual $\operatorname{End}_A(T)^{op}$. Let $P \twoheadrightarrow T$ be a right projective presentation of T over $R(A)$. Then, $P \in \mathcal{F}(\tilde{\Delta}_{R(A)^{op}})$. Note that $T \simeq \operatorname{Hom}_A(A, T) \simeq \operatorname{Hom}_{A^{op}}(DT, DA) \in \mathcal{F}(\tilde{\Delta}_{R(A)^{op}})$. Since $\mathcal{F}(\tilde{\Delta}_{R(A)^{op}})$ is resolving, so the kernel of $P \to T$ belongs to $\mathcal{F}(\tilde{\Delta}_{R(A)^{op}})$. Since $\operatorname{Hom}_{A^{op}}(DT, -)$ gives an exact equivalence between $\mathcal{F}(\tilde{\nabla}_{A^{op}})$ and $\mathcal{F}(\tilde{\Delta}_{R(A)^{op}})$, there exists an exact sequence of right A-modules $0 \to K \to M' \to DA \to 0$, where M' is a (partial) tilting module and $K \in \mathcal{F}(\tilde{\nabla}_{A^{op}})$. Applying D, we obtain the desired exact sequence. By [Rou08, Proposition 4.26], since $DK \in \mathcal{F}(\tilde{\Delta}_A)$, there exists an exact sequence $0 \to DK \to M'' \to K'' \to 0$, where M' is a (partial) tilting module and $K' \in \mathcal{F}(\tilde{\Delta}_A)$. Put $Q = DM' \oplus M''$. Hence, Q is (partial) tilting module, and the (A, R)-exact sequence $0 \to A \to DM' \to M''$ remains exact under $\operatorname{Hom}_A(Q, -)$. This means that Q-$\operatorname{domdim}(A, R) \geq 2$. By Corollary 3.1.6 and Lemma 5.1.2, the second assertion follows. □

5.2. Q-$\operatorname{codomdim}_{(A,R)} M$ controlling the behavior of $\operatorname{Hom}_A(Q, -)$

To address how the relative dominant dimension with respect to a module Q can be used as a tool to deduce what extension groups are preserved by the functor $\operatorname{Hom}_A(Q, -)$, the following result is crucial.

Lemma 5.2.1. *Let $M \in A$-mod. Suppose that $\operatorname{Tor}_i^B(Q, F_Q M) = \mathrm{L}_i\,\mathbb{I}(F_Q M) = 0$ for $1 \leq i \leq q$. For any $X \in \{Y \in A\text{-mod} \mid \operatorname{Ext}_A^{i>0}(Q, Y) = 0\}$, there are isomorphisms $\operatorname{Ext}_A^i(\mathbb{I}F_Q M, X) \simeq \operatorname{Ext}_B^i(F_Q M, F_Q X)$, $0 \leq i \leq q$, and an exact sequence*

$$\begin{aligned} 0 \to \operatorname{Ext}_A^{q+1}(\mathbb{I}F_Q M, X) \to \operatorname{Ext}_B^{q+1}(F_Q M, F_Q X) \to \operatorname{Hom}_A(\operatorname{Tor}_{q+1}^B(Q, F_Q M), X) \\ \to \operatorname{Ext}_A^{q+2}(\mathbb{I}F_Q M, X) \to \operatorname{Ext}_B^{q+2}(F_Q M, F_Q X). \quad (9) \end{aligned}$$

Proof. Let $X \in A$-mod such that $\operatorname{Ext}_A^{i>0}(Q, X) = 0$. The desired isomorphism for $i = 0$ follows immediately by Tensor-Hom adjunction. To obtain the result for higher values, we will use Theorem 10.49 of [Rot09]. Fix $f = \operatorname{Hom}_A(-, X)$ and $g = Q \otimes_B -$. So, f is a contravariant left exact and g is covariant. We note that gP is f-acyclic for any $P \in B$-proj. In fact, $\mathrm{R}^{j>0} f(gP) = \operatorname{Ext}_A^{j>0}(gP, X) = 0$, since $gP = Q \otimes_B P \in \operatorname{add}_A Q$. So, for each $a \in B$-mod, there is a spectral sequence $E_2^{i,j} = (\mathrm{R}^i f)(L_j g)(a)$ which converges to $\mathrm{R}^{i+j}(f \circ g)(a)$. By Tensor-Hom adjunction, $f \circ g(N) = \operatorname{Hom}_A(Q \otimes_B N, X) \simeq \operatorname{Hom}_B(N, \operatorname{Hom}_A(Q, X)) = \operatorname{Hom}_B(-, F_Q X)(N)$ for every $N \in B$-mod. Hence, the spectral sequence $E_2^{i,j} = \operatorname{Ext}_A^i(\operatorname{Tor}_j^B(Q, a), X)$ converges to $\operatorname{Ext}_B^{i+j}(a, F_Q X)$. For each $M \in A$-mod, fix $a = F_Q M$. By assumption, $\operatorname{Tor}_i^B(Q, F_Q M) = 0$ for $1 \leq i \leq q$. Hence, $E_2^{i,j} = 0$, $1 \leq i \leq q$. By the dual of Lemma A.4 of [Cru22b], the result follows. ◻

5.3. *Ringel duality in cover theory*

As we saw, fully faithfulness of $\operatorname{Hom}_A(Q, -)$ on relative injectives is related to the existence of a double centralizer property on Q. In this subsection, we explore the meaning of fully faithfulness of a functor F_Q on $\mathcal{F}(\tilde{\nabla})$ for relative projective objects Q in $\mathcal{F}(\tilde{\nabla})$. This leads us to one of our main results, connecting Ringel duality with Rouquier's cover theory.

Theorem 5.3.1. *Let R be a commutative Noetherian ring. Let $(A, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ be a split quasi-hereditary R-algebra with a characteristic tilting module T. Denote by $R(A)$ the Ringel dual $\operatorname{End}_A(T)^{op}$ (of A). Assume that $Q \in \operatorname{add} T$ is a (partial) tilting module of A, and denote by B the endomorphism algebra of Q. Then, the following assertions hold.*

(a) $Q\text{-codomdim}_{(A,R)} \mathcal{F}(\tilde{\nabla}) = Q\text{-codomdim}_{(A,R)} T = Q\text{-codomdim}_{(A,R)} \bigoplus_{\lambda \in \Lambda} \nabla(\lambda)$.

(b) *If $Q\text{-codomdim}_{(A,R)} T \geq n \geq 2$, then the functor F_Q induces isomorphisms*

$$\operatorname{Ext}_A^j(M, N) \to \operatorname{Ext}_B^j(F_Q M, F_Q N), \quad \forall M, N \in \mathcal{F}(\tilde{\nabla}),\ 0 \leq j \leq n-2.$$

(c) *If $Q\text{-codomdim}_{(A,R)} T \geq 3$, then the functor F_Q induces an exact equivalence $\mathcal{F}(\tilde{\nabla}) \to \mathcal{F}(F\tilde{\nabla})$.*

(d) *If $Q\text{-codomdim}_{(A,R)} T \geq n \geq 2$, then $(R(A), \operatorname{Hom}_A(T, Q))$ is an $(n-2)$-$\mathcal{F}(\tilde{\Delta}_{R(A)})$ split quasi-hereditary cover of $\operatorname{End}_A(Q)^{op}$. The converse holds if R is a field.*

Proof. The proof of (a) is analogous to [Cru24c, Theorem 6.2.1]. Thanks to F_Q and D being exact on short exact sequences of modules belonging to $\mathcal{F}(\tilde{\nabla})$, we obtain that we can apply Lemma 3.1.8 to the filtrations by costandard modules. Further, for every $X \in R$-proj so that $R^t \simeq X \oplus Y$ for some $t \in \mathbb{N}$,

$$\begin{aligned} Q\text{-codomdim}_{(A,R)} \nabla(\lambda) &= Q\text{-codomdim}_{(A,R)} \nabla(\lambda)^t \\ &= \inf\{Q\text{-codomdim}_{(A,R)} \nabla(\lambda) \otimes_R X, Q\text{-codomdim}_{(A,R)} \nabla(\lambda) \otimes_R Y\}. \end{aligned}$$

Therefore, $Q\text{-codomdim}_{(A,R)} \bigoplus_{\lambda \in \Lambda} \nabla(\lambda) = Q\text{-codomdim}_{(A,R)} \mathcal{F}(\tilde{\nabla})$. Now using the exact sequences (1) together with Lemma 3.1.8 and the reasoning of Theorem [Cru24c, Theorem 6.2.1], assertion (a) follows.

By Proposition 2.1.2, $DQ \otimes_A Q \in R$-proj. Any (A, R)-injective module being projective over R belongs to $Q^\perp$ (see, for example, [Cru22b, Proposition 2.10]). By Lemma 5.1.1 and Theorem 3.1.1, F_Q is fully faithful on $\mathcal{F}(\tilde{\Delta})$. By Theorem 3.1.4, $\operatorname{Tor}_i^B(Q, F_Q M) = 0$, $1 \leq i \leq n-2$ for every $M \in \mathcal{F}(\tilde{\nabla})$. By Lemma 5.2.1, $\operatorname{Ext}_B^i(F_Q M, F_Q X) \simeq \operatorname{Ext}_A^i(\mathbb{I} F_Q M, X)$ for $0 \leq i \leq n-2$, where $M, X \in \mathcal{F}(\tilde{\nabla})$. Since χ_M is an isomorphism for every $M \in \mathcal{F}(\tilde{\nabla})$, (b) follows.

By the exactness of F_Q on $\mathcal{F}(\tilde{\nabla})$ and according to Lemma 2.0.1, $F_Q(\nabla(\lambda) \otimes_R X) \simeq F_Q \nabla(\lambda) \otimes_R X$ for every $\lambda \in \Lambda$ and $X \in R$-proj, and the restriction of the functor F_Q on $\mathcal{F}(\tilde{\nabla})$ has image in $\mathcal{F}(F_Q \tilde{\nabla})$. By (b), it is enough to prove that for each module M in $\mathcal{F}(F_Q \tilde{\nabla})$, there exists $N \in \mathcal{F}(\tilde{\nabla})$ so that $F_Q N \simeq M$. By (b), the functor $\mathbb{I} = Q \otimes_B -$ is exact on short exact sequences of modules belonging to $\mathcal{F}(F_Q \tilde{\nabla})$.

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Thanks to $Q \otimes_B F_Q \nabla(\lambda) \otimes_R X \simeq \nabla(\lambda) \otimes_R X$ for every $X \in R$-proj and $\lambda \in \Lambda$, we obtain that $\mathbb{I}$ sends $\mathcal{F}(F_Q \tilde{\nabla})$ to $\mathcal{F}(\tilde{\nabla})$. So, (c) follows.

Assume now that Q-codomdim$_{(A,R)}\, T \geq n \geq 2$. Fix $B = \operatorname{End}_A(Q)^{op}$. By Ringel duality (see, for instance, [Cru24b, Lemma 7.1]), for each $M \in \mathcal{F}(\tilde{\nabla})$, we have

$$\operatorname{Hom}_{R(A)}(\operatorname{Hom}_A(T,Q), \operatorname{Hom}_A(T,M)) \simeq \operatorname{Hom}_A(Q,M) \text{ as } (B, R(A))\text{-bimodules.} \tag{10}$$

In particular, $\operatorname{Hom}_{R(A)}(\operatorname{Hom}_A(T,Q), R(A)) \simeq \operatorname{Hom}_A(Q,T)$ as $(B, R(A))$-bimodules. By (c), F_Q is fully faithful on $\mathcal{F}(\tilde{\nabla})$. Hence, $\operatorname{End}_B(\operatorname{Hom}_A(Q,T))^{op} \simeq \operatorname{End}_A(T)^{op}$ and $\operatorname{End}_{R(A)}(\operatorname{Hom}_A(T,Q))^{op} \simeq B$. So, $(R(A), \operatorname{Hom}_A(T,Q))$ is a split quasi-hereditary cover of B. By (b) and [Cru24b, Lemma 7.1],

$$\begin{aligned}\operatorname{Ext}^i_B(\operatorname{Hom}_A(Q,T), \operatorname{Hom}_{R(A)}(\operatorname{Hom}_A(T,Q), \operatorname{Hom}_A(T,M))) &\simeq \operatorname{Ext}^i_B(\operatorname{Hom}_A(Q,T), \operatorname{Hom}_A(Q,M)) \\ &\simeq \operatorname{Ext}^i_B(F_Q T, F_Q M), \quad 0 \leq i \leq n-2, \forall M \in \mathcal{F}(\tilde{\nabla}),\end{aligned} \tag{11}$$

and these Ext groups vanish if $1 \leq i \leq n-2$. By [Cru24b, Lemma 7.1] and [Cru24c, Proposition 3.0.3, 3.0.4], the first part of (d) follows. Since the maps $\operatorname{Hom}_A(T, DA) \to \operatorname{Hom}_B(F_Q T, F_Q DA)$ and $DT \to \operatorname{Hom}_B(F_Q T, DQ)$ are equivalent, the second assertion follows by equations (10) and (11), together with Theorem 3.1.4 using Ext when R is a field (see [ASS06, A.4]). ◻

Remark 5.3.2. T-codomdim$_{(A,R)}\, \mathcal{F}(\tilde{\nabla}) = T$-codomdim$_{(A,R)}\, T = +\infty$ for a characteristic tilting module T. Of course, the Ringel dual is an infinite cover of itself.

Remark 5.3.3. The cover constructed in Theorem 5.3.1 makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{F}(\tilde{\nabla}_A) & \xrightarrow{\operatorname{Hom}_A(T,-)} & \mathcal{F}(\tilde{\Delta}_{R(A)}) \\ & {\scriptstyle \operatorname{Hom}_A(Q,-)} \searrow \qquad \swarrow {\scriptstyle \operatorname{Hom}_{R(A)}(\operatorname{Hom}_A(T,Q),-)} & \\ & \mathcal{F}(F\tilde{\nabla}_A) & \end{array} . \tag{12}$$

Since every projective module is the image of a (partial) tilting under the Ringel dual functor, every quasi-hereditary cover can be recovered/discovered using this approach. More precisely, every split quasi-hereditary algebra A is Morita equivalent to the Ringel dual of its Ringel dual $R(R(A))$, and every projective over $R(R(A))$ can be written as $\operatorname{Hom}_{R(A)}(T_{R(A)}, Q)$ for some $Q \in \operatorname{add} T_{R(A)}$, where $T_{R(A)}$ is a characteristic tilting module of $R(A)$. Hence, every split quasi-hereditary cover can be written in the form $(R(A), \operatorname{Hom}_A(T,Q))$ for some split quasi-hereditary algebra A, T a characteristic tilting module and $Q \in \operatorname{add} T$. Further, the second part of Theorem 5.3.1(d) indicates that the quality of faithful split quasi-hereditary covers of finite-dimensional algebras is controlled by the relative codominant dimension of characteristic tilting modules with respect to (partial) tilting modules.

5.3.1. An analogue of Lemma 5.1.2 for Ringel duality

Lemma 5.3.4. *Let R be a commutative Noetherian ring. Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra with a characteristic tilting module T. Denote by $R(A)$ the Ringel dual $\operatorname{End}_A(T)^{op}$ (of A). Assume that $Q \in \operatorname{add} T$ is a (partial) tilting module of A, and fix $B = \operatorname{End}_A(Q)^{op}$. Then, the following assertions hold.*

(a) *If $D\chi_T : DT \to \operatorname{Hom}_B(\operatorname{Hom}_A(Q,T), DQ)$ is an isomorphism, then $(R(A), \operatorname{Hom}_A(T,Q))$ is a split quasi-hereditary cover of B.*

(b) *If $D\chi^r_{DT} : DDT \to \operatorname{Hom}_B(\operatorname{Hom}_A(DQ, DT), DDQ)$ is an isomorphism, then $\operatorname{Hom}_A(T,Q)$ satisfies a double centralizer property between $R(A)$ and B.*

Proof. By projectivization, $\operatorname{Hom}_A(T,Q) \in R(A)$-proj and $\operatorname{End}_{R(A)}(\operatorname{Hom}_A(T,Q))^{op} \simeq \operatorname{End}_A(Q)^{op} = B$. By (a) and Proposition 2.1.2, we have as $(R(A), R(A))$-bimodules

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$$R(A) = \operatorname{Hom}_A(T,T) \simeq \operatorname{Hom}_{A^{op}}(DT,DT) \simeq \operatorname{Hom}_{A^{op}}(DT, \operatorname{Hom}_B(\operatorname{Hom}_A(Q,T),DQ)) \tag{13}$$

$$\simeq \operatorname{Hom}_{A^{op}}(DT, \operatorname{Hom}_{B^{op}}(Q, D\operatorname{Hom}_A(Q,T))) \simeq \operatorname{Hom}_{B^{op}}(DT\otimes_A Q, D\operatorname{Hom}_A(Q,T)) \tag{14}$$

$$\simeq \operatorname{Hom}_B(\operatorname{Hom}_A(Q,T),\operatorname{Hom}_A(Q,T)). \tag{15}$$

Since $F_Q R(A) = \operatorname{Hom}_{R(A)}(\operatorname{Hom}_A(T,Q),\operatorname{Hom}_A(T,T)) \simeq \operatorname{Hom}_A(Q,T)$, assertion (a) follows.

Now using the isomorphism χ^r_{DT} and Proposition 2.1.2, we obtain

$$R(A) = \operatorname{Hom}_A(T,T) \simeq \operatorname{Hom}_A(T, \operatorname{Hom}_{B^{op}}(\operatorname{Hom}_A(T,Q),Q)) \simeq \operatorname{Hom}_A(T, \operatorname{Hom}_B(DQ, D\operatorname{Hom}_A(T,Q)))$$
$$\simeq \operatorname{Hom}_B(DQ\otimes_A T, D\operatorname{Hom}_A(T,Q)) \simeq \operatorname{Hom}_B(\operatorname{Hom}_A(T,Q),\operatorname{Hom}_A(T,Q)).$$

□

We have not yet addressed the case of $Q\text{-codomdim}_{(A,R)} T = 1$. For this case, we can recover the Ringel dual being a cover using deformation theory.

Corollary 5.3.5. *Let R be a commutative regular Noetherian domain with quotient field K. Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra with a characteristic tilting module T. Let $R(A)$ be the Ringel dual of A, $\operatorname{End}_A(T)^{op}$. Assume that $Q \in \operatorname{add} T$ is a (partial) tilting module of A so that $Q\text{-codomdim}_{(A,R)} T \geq 1$ and $K\otimes_R Q\text{-codomdim}_{(K\otimes_R A)} K\otimes_R T \geq 2$. Then, $(R(A), \operatorname{Hom}_A(T,Q))$ is a split quasi-hereditary cover of $\operatorname{End}_A(Q)^{op}$. Moreover, $(R(A), \operatorname{Hom}_A(T,Q))$ is a 0-$\mathcal{F}(\tilde{\Delta})$ cover of $\operatorname{End}_A(Q)^{op}$.*

Proof. If $Q\text{-codomdim}_{(A,R)} T \geq 2$, then this is nothing more than Theorem 5.3.1. Assume that $Q\text{-domdim}_{(A,R)} T = 1$. By Theorem 3.1.1, χ_T is surjective. In view of Lemma 5.3.4, it is enough to prove that $D\chi_T$ is an isomorphism. Since $T \in R\text{-proj}$, χ_T is an (A,R)-epimorphism, and therefore, $D\chi_T$ is an (A,R)-monomorphism. By assumption, $K\otimes_R Q\text{-codomdim}_{K\otimes_R A} K\otimes_R T \geq 2$. Hence, thanks to the flatness of K, $K\otimes_R D\chi_T$ is an isomorphism.

Denote by X the cokernel of $D\chi_T$. Since $D\chi_T$ is split over R, $X \in \operatorname{add}_R \operatorname{Hom}_B(\operatorname{Hom}_A(Q,T),DQ)$. As we saw, $K\otimes_R X = 0$. In particular, X is a torsion R-module. We cannot deduce right away that $\operatorname{Hom}_B(\operatorname{Hom}_A(Q,T),DQ)$ is projective over R, but we can embed $\operatorname{Hom}_B(\operatorname{Hom}_A(Q,T),DQ)$ into $\operatorname{Hom}_R(\operatorname{Hom}_A(Q,T),DQ)$ which is projective over R due to both $\operatorname{Hom}_A(Q,T)$ and DQ being projective over R. So, $\operatorname{Hom}_B(\operatorname{Hom}_A(Q,T),DQ)$ is a torsion-free R-module. However, the localization $\operatorname{Hom}_B(\operatorname{Hom}_A(Q,T),DQ)_{\mathfrak{p}}$ is projective over $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of height one, and so $K\otimes_R X = 0$ yields that $X_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p}$ of height one. Applying Proposition 3.4 of [AB59] to $D\chi_T$, we obtain that $X_{\mathfrak{p}}$ must be zero, and consequently, $D\chi_T$ is an isomorphism. Denote by $F_{\operatorname{Hom}_A(T,Q)}$ the Schur functor and $G_{\operatorname{Hom}_A(T,Q)}$ its right adjoint of this cover. Observe that $\operatorname{Hom}_A(T,DA)$ is a characteristic tilting module of $R(A)$. Since $D\chi_T$ is a monomorphism and

$$\operatorname{Hom}_B(\operatorname{Hom}_A(Q,T),DQ) \simeq \operatorname{Hom}_B(\operatorname{Hom}_A(Q,T),\operatorname{Hom}_A(Q,DA)) \tag{16}$$
$$\simeq \operatorname{Hom}_B(\operatorname{Hom}_{R(A)}(F_T Q, F_T T), \operatorname{Hom}_{R(A)}(F_T Q, F_T DA))$$
$$\simeq G_{\operatorname{Hom}_A(T,Q)} F_{\operatorname{Hom}_A(T,Q)} \operatorname{Hom}_A(T,DA), \tag{17}$$

the claim follows (see also [Cru24c, Proposition 3.1.1, Proposition 3.1.2]). □

5.4. Ringel self-duality as an instance of uniqueness of covers

We will now see how Ringel self-duality can be related to the uniqueness of covers. For the definitions of relative Morita R-algebras and relative gendo-symmetric R-algebras, we refer to [Cru22b].

Corollary 5.4.1. *Let (A,P,DP) be a relative Morita R-algebra. Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra, Assume that $\operatorname{domdim}_{(A,R)} T, \operatorname{codomdim}_{(A,R)} T \geq 3$ for a characteristic tilting module T. Then, there exists an exact equivalence $\mathcal{F}(F_P\tilde{\Delta}_A) \to \mathcal{F}(F_P\tilde{\nabla}_A)$ if and only if A is Ringel self-dual.*

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Proof. By Theorems 5.3.1 and Proposition 2.3.1, (A, P) is a 1-faithful split quasi-hereditary cover of $\operatorname{End}_A(P)^{op}$ and $(R(A), \operatorname{Hom}_A(T, P))$ is a 1-faithful split quasi-hereditary cover of $\operatorname{End}_A(P)^{op}$. As illustrated in Remark 5.3.3, F restricts to exact equivalences

$$\mathcal{F}(\tilde{\nabla}_A) \to \mathcal{F}(F_P\tilde{\nabla}_A) \quad \text{and} \quad \mathcal{F}(\tilde{\Delta}_A) \to \mathcal{F}(F_P\tilde{\Delta}_A).$$

Therefore, there exists an exact equivalence between $\mathcal{F}(F_P\tilde{\Delta}_A)$ and $\mathcal{F}(F_P\tilde{\nabla}_A)$ if and only if there exists an exact equivalence between $\mathcal{F}(\tilde{\Delta}_A)$ and $\mathcal{F}(\tilde{\nabla}_A)$. Hence, A is Ringel self-dual. □

This is an indication that the phenomenon of Ringel self-duality behaves better the larger the dominant dimension of the characteristic tilting module. As before, for deformations, we can weaken the conditions on the dominant and codominant dimension of the characteristic tilting module.

Corollary 5.4.2. *Let R be an integral regular domain with quotient field K. Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra. Let (A, P, V) be a relative Morita R-algebra. Fix $B = \operatorname{End}_A(P)^{op}$. Assume the following conditions hold.*

(i) *$(K \otimes_R A, K \otimes_R P)$ is a 1-faithful split quasi-hereditary cover of B;*
(ii) *$K \otimes_R P$-codomdim $K \otimes_R T \geq 3$ for a characteristic tilting module T;*
(iii) *$\operatorname{domdim}_{(A,R)} T$, $\operatorname{codomdim}_{(A,R)} T \geq 2$ for a characteristic tilting module T;*
(iv) *There exists an exact equivalence $\mathcal{F}(F_P\tilde{\Delta}_A) \to \mathcal{F}(F_P\tilde{\nabla}_A)$.*

Then, A is Ringel self-dual.

Proof. We need to show that $\mathcal{F}(\tilde{\Delta}_A)$ and $\mathcal{F}(\tilde{\nabla}_A)$ are exact equivalent. Note that $\operatorname{domdim}_{A(\mathfrak{m})} T(\mathfrak{m}) \geq 2$ and $\operatorname{domdim}_{A^{op}(\mathfrak{m})} DT(\mathfrak{m}) = \operatorname{codomdim}_{A(\mathfrak{m})} T(\mathfrak{m}) \geq 2$ for every $\mathfrak{m} \in \operatorname{MaxSpec} R$. By Theorem 5.3.1,

$$(R(K \otimes_R A), \operatorname{Hom}_{K\otimes_R A}(K \otimes_R T, K \otimes_R P)) = (K \otimes_R R(A), K \otimes_R \operatorname{Hom}_A(T, P))$$

is a 1-faithful split quasi-hereditary cover of $K \otimes_R B$, and $(R(A(\mathfrak{m})), \operatorname{Hom}_A(T, P)(\mathfrak{m}))$ is a 0-faithful split quasi-hereditary cover of $B(\mathfrak{m})$ for every $\mathfrak{m} \in \operatorname{MaxSpec} R$. By Proposition 2.3.1(b), $(A(\mathfrak{m}), P(\mathfrak{m}))$ is a 0-faithful split quasi-hereditary cover of $B(\mathfrak{m})$ for every $\mathfrak{m} \in \operatorname{MaxSpec} R$. By [Cru24c, Proposition 5.0.5, Theorem 5.0.9], $(R(A), \operatorname{Hom}_A(T, P))$ and (A, P) are 1-faithful split quasi-hereditary covers of B. By Remark 5.3.3 and [Cru24c, Proposition 3.1.4], there exists an exact equivalence,

$$\mathcal{F}(\tilde{\Delta}_A) \to \mathcal{F}(F_P\tilde{\Delta}_A) \xrightarrow{(iv)} \mathcal{F}(F_P\tilde{\nabla}_A) \to \mathcal{F}(\tilde{\nabla}_A).$$ □

6. Wakamatsu tilting conjecture for quasi-hereditary algebras

Theorem 3.1.4 is the main advantage of Definition 3.0.1 compared to [KSX01, Definition 2.1] giving a meaning to what this relative dominant dimension measures. Another point of view to be referred to is the Wakamatsu tilting conjecture (see [Wak88]). In this context, the Wakamatsu tilting conjecture says that if Q has finite projective A-dimension and it admits no self-extensions in any degree, then Q-domdim (A, R) measures how far Q is from being a tilting module. In particular, for split quasi-hereditary algebras, this amounts to saying that for a module Q in the additive closure of a characteristic tilting module, Q-domdim (A, R) measures how far Q is from being a characteristic tilting module of A.

Theorem 6.0.1. *Let R be a Noetherian commutative ring and $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra. Assume that T is a characteristic tilting module and $Q \in \operatorname{add}_A T$ is a partial tilting module.*

If Q-domdim $(A, R) = +\infty$, then Q is a characteristic tilting module of A.

Proof. Consider first that R is a field. By assumption, DQ-$\operatorname{codomdim}_{(A^{op},R)} DA = +\infty$. $\operatorname{Hom}_{A^{op}}(DQ, -)$ is exact on $\mathcal{F}(\tilde{\nabla}_{A^{op}})$, and so, DQ-$\operatorname{codomdim}_{(A^{op},R)} \mathcal{F}(\tilde{\nabla}_{A^{op}}) = +\infty$ by Lemma 3.1.8. By Theorem 5.3.1, $(\operatorname{End}_{A^{op}}(DT)^{op}, \operatorname{Hom}_{A^{op}}(DT, DQ))$ is an $+\infty$ faithful split quasi-hereditary cover of $\operatorname{End}_{A^{op}}(DQ)^{op}$. By [Cru24c, Corollary 4.2.2], $\operatorname{End}_{A^{op}}(DT)^{op}$ is Morita equivalent to $\operatorname{End}_{A^{op}}(DQ)^{op}$. In particular, by projectization, DT and DQ have the same number of indecomposable modules. Therefore, $\operatorname{add}_{A^{op}} DQ = \operatorname{add}_{A^{op}} DT$, and so Q is a characteristic tilting module. Assume now that R is an arbitrary Noetherian commutative ring. By Theorem 3.2.5 and the first part, $Q(\mathfrak{m})$ is a characteristic tilting module for every $\mathfrak{m} \in \operatorname{MaxSpec} R$; that is, $\operatorname{add} Q(\mathfrak{m}) = \mathcal{F}(\Delta(\mathfrak{m})) \cap \mathcal{F}(\nabla(\mathfrak{m}))$. By Proposition 2.1.3, the result follows. □

6.1. *Measuring* Q-$\operatorname{domdim}_{(A,R)} T$ *using projective resolutions*

We can also try to compute Q-$\operatorname{domdim}_{(A,R)} T$ using projective resolutions over a Ringel dual of A. Moreover, this perspective leads us to reformulate that Q-$\operatorname{domdim}_{(A,R)} T$ measures how much the projective module $\operatorname{Hom}_A(T, Q)$ controls projective resolutions of characteristic tilting modules over the Ringel dual.

Proposition 6.1.1. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary R-algebra with a characteristic tilting module T. Denote by $R(A)$ the Ringel dual $\operatorname{End}_A(T)^{op}$ (of A). Suppose that $Q \in \operatorname{add}_A T$ is a partial tilting module. Then, $\operatorname{Hom}_A(T, Q)$-$\operatorname{codomdim}_{(R(A),R)} DT = Q$-$\operatorname{domdim}_{(A,R)} T$.*

Proof. Observe that $\operatorname{End}_{R(A)}(\operatorname{Hom}_A(T, Q))^{op} \simeq \operatorname{End}_A(Q)^{op}$ since $Q \in \operatorname{add} T$. Denote by B this endomorphism algebra and by H the Ringel dual functor $\operatorname{Hom}_A(T, -) \colon A\text{-mod} \to R(A)\text{-mod}$. Note that the maps χ^r_{DT} and $\chi_{DT}\colon \operatorname{Hom}_A(T, Q) \otimes_B \operatorname{Hom}_{R(A)}(HQ, DT)$ are equivalent. Indeed by Ringel duality, Tensor-Hom adjunction and [Cru22b, Proposition 2.2], the following R-modules are isomorphic:

$$\begin{aligned}\operatorname{Hom}_A(DQ, DT) \otimes_B DQ &\simeq \operatorname{Hom}_A(T, Q) \otimes_B DQ \simeq \operatorname{Hom}_A(T, Q) \otimes_B \operatorname{Hom}_A(Q, DA)\\ &\simeq \operatorname{Hom}_A(T, Q) \otimes_B \operatorname{Hom}_{R(A)}(HQ, H(DA)) \simeq \operatorname{Hom}_A(T, Q) \otimes_B \operatorname{Hom}_{R(A)}(HQ, DT),\\ \text{and } \operatorname{Tor}_i^B(\operatorname{Hom}_A(T, Q), &\operatorname{Hom}_{R(A)}(\operatorname{Hom}_A(T, Q), DT)) \simeq \operatorname{Tor}_i^B(\operatorname{Hom}_A(T, Q), DQ),\ \forall i \in \mathbb{N}. \quad (18)\end{aligned}$$

The result now follows from Theorems 3.1.3, 3.1.1 and 3.1.4. □

7. Going from bigger covers to smaller covers

In this section, we explore whether the quality of a split quasi-hereditary cover is preserved by truncation by split heredity ideals and by Schur functors between split highest weight categories.

7.1. *Truncation of split quasi-hereditary covers*

Theorem 7.1.1. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary Noetherian R-algebra. Assume that (A, P) is an i-$\mathcal{F}(\tilde{\Delta})$ cover of $\operatorname{End}_A(P)^{op}$ for some integer $i \geq 0$. Let J be a split heredity ideal of A. Then, $(A/J, P/JP)$ is an i-$\mathcal{F}(\tilde{\Delta}_{A/J})$ cover of $\operatorname{End}_{A/J}(P/JP)^{op}$, where $\mathcal{F}(\tilde{\Delta}_{A/J}) = \mathcal{F}(\tilde{\Delta}) \cap A/J$-mod.*

Proof. Fix $B := \operatorname{End}_A(P)^{op}$. The map $A \twoheadrightarrow A/J$ induces the fully faithful functor $A/J\text{-mod} \to A\text{-mod}$. Hence, $\operatorname{End}_{A/J}(P/JP)^{op} \simeq \operatorname{End}_A(P/JP)^{op}$. We wish to express $B_J := \operatorname{End}_{A/J}(P/JP)^{op}$ as a quotient of B. To see this, consider the exact sequence of (A, B)-bimodules $\delta\colon 0 \to JP \to P \to P/JP \to 0$. Applying $\operatorname{Hom}_A(P, -)$ to δ, we get the exact sequence $\gamma\colon 0 \to \operatorname{Hom}_A(P, JP) \to B \to \operatorname{Hom}_A(P, P/JP) \to 0$, while by applying $\operatorname{Hom}_A(-, P/JP)$ to δ we get the exact sequence

$$0 \to \operatorname{End}_A(P/JP) \to \operatorname{Hom}_A(P, P/JP) \to \operatorname{Hom}_A(JP, P/JP). \quad (19)$$

Thanks to $J = J^2$, we have $\operatorname{Hom}_A(JP, X) = 0$ for every $X \in A/J$-mod. Combining (19) with γ, we obtain the exact sequence $0 \to \operatorname{Hom}_A(P, JP) \to B \to B_J \to 0$. Since δ is exact as (A, B)-bimodules, the latter is exact as (B, B)-bimodules. Thus, the functor B_J-mod $\to B$-mod is fully faithful. Denote by $G_{P/JP}$ the functor $\operatorname{Hom}_{B_J}(\operatorname{Hom}_{A/J}(P/JP, A/J), -) = \operatorname{Hom}_B(\operatorname{Hom}_{A/J}(P/JP, A/J), -)\colon B_J\text{-mod} \to A/J\text{-mod}$ and $F_{P/JP} = \operatorname{Hom}_{A/J}(P/JP, -) = \operatorname{Hom}_A(P/JP, -)\colon A/J\text{-mod} \to B_J\text{-mod}$.

To assert that the truncated cover is a 0-$\mathcal{F}(\tilde{\Delta})$ cover, it is enough to compare the restrictions of the functors F_P and $G_P \circ F_P$ to $\mathcal{F}(\tilde{\Delta}) \cap A/J$-mod with the restriction of the functors $F_{P/JP}$ and $G_{P/JP} \circ F_{P/JP}$ to $\mathcal{F}(\tilde{\Delta}_{A/J})$, respectively. For each $X \in A/J$-mod, applying $\operatorname{Hom}_A(-, X)$ to δ instead

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of $\mathrm{Hom}_A(-, P/JP)$ yields that $F_{P/JP}X \simeq F_P X$. By applying $\mathrm{Hom}_B(-, F_P X)$ to $0 \to F_P J \to F_P A \to F_P(A/J) \to 0$ we obtain the exact sequence $0 \to G_{P/JP}F_{P/JP}X \to G_P F_P X \to \mathrm{Hom}_B(F_P J, F_P X)$. Fixing $X \in \mathcal{F}(\tilde{\Delta}_{A/J})$ we obtain that $\mathrm{Hom}_B(F_P J, F_P X) \simeq \mathrm{Hom}_A(J, X) = 0$ since (A, P) is a 0-$\mathcal{F}(\tilde{\Delta})$ cover of B. These isomorphisms are functorial, so if we denote by η^J the unit of the adjunction $F_{P/JP} \dashv G_{P/JP}$, then η^J_X is an isomorphism for every $X \in \mathcal{F}(\tilde{\Delta}_{A/J})$. This shows that $(A/J, P/JP)$ is a 0-$\mathcal{F}(\tilde{\Delta}_{A/J})$ cover of B_J.

Our aim now is to compute $\mathrm{R}^j G_{P/JP}(F_{P/JP}X)$ for $j \leq i$ and every $X \in \mathcal{F}(\tilde{\Delta}_{A/J})$. Hence, fix an arbitrary $X \in \mathcal{F}(\tilde{\Delta}_{A/J})$. Applying $\mathrm{Hom}_B(-, F_P X)$ to γ, we obtain $\mathrm{Ext}^1_B(B_J, F_P X) = 0$ and $\mathrm{Ext}^l_B(B_J, F_P X) \simeq \mathrm{Ext}^{l-1}_B(F_P JP, F_P X)$ for every $l > 1$. Observe that $JP \simeq J \otimes_A P$ as left A-modules since $P \in A$-proj. Moreover, $JP \in \mathrm{add}_A J$, and thus, it is projective as left A-module. Thus, $\mathrm{Ext}^l_B(B_J, F_P X) \simeq \mathrm{Ext}^{l-1}_A(JP, X) = 0$ for every $0 < l-1 \leq i$. Hence, $\mathrm{Ext}^l_B(B_J, F_P X) = 0$ for every $1 \leq l \leq i+1$. Let $\cdots \to P_i \to \cdots \to P_1 \to P_0 \to F_{P/JP}A/J \to 0$ be a projective B_J-resolution of $F_{P/JP}A/J$. Denote by Ω^{j+1} the kernel of $P_j \to P_{j-1}$, with $P_{-1} := \Omega^0 = F_{P/JP}A/J$. Note that $\mathrm{Ext}^l_B(P_j, F_P X) = 0$ for $1 \leq l \leq i+1$. Taking into account that B_J-mod $\to B$-mod is a fully faithful functor, applying $\mathrm{Hom}_B(-, F_P X)$ and $\mathrm{Hom}_{B_J}(-, F_P X)$ to the projective B_J-resolution of $F_P A/J$ yields

$$\mathrm{Ext}^l_B(\Omega^j, F_P X) \simeq \mathrm{Ext}^{l+1}_B(\Omega^{j-1}, F_P X), \quad \mathrm{Ext}^s_{B_J}(\Omega^j, F_P X) \simeq \mathrm{Ext}^{s+1}_{B_J}(\Omega^{j-1}, F_P X), \tag{20}$$

for $1 \leq l \leq i$, s, $j \geq 1$ and the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}_B(P_j, F_{P/JP}X) & \longrightarrow & \mathrm{Hom}_B(\Omega^{j+1}, F_{P/JP}X) & \longrightarrow & \mathrm{Ext}^1_B(\Omega^j, F_{P/JP}X) & \longrightarrow & 0 \\ \| & & \| & & & & \\ \mathrm{Hom}_{B_J}(P_j, F_{P/JP}X) & \longrightarrow & \mathrm{Hom}_{B_J}(\Omega^{j+1}, F_{P/JP}X) & \longrightarrow & \mathrm{Ext}^1_{B_J}(\Omega^j, F_{P/JP}X) & \longrightarrow & 0 \end{array}. \tag{21}$$

By the commutative diagram, $\mathrm{Ext}^1_B(\Omega^j, F_P X)$ is zero if and only if $\mathrm{Ext}^1_{B_J}(\Omega^j, F_P X)$ is zero. By assumption and the previous discussion, for each $1 \leq l \leq i$,

$$\begin{aligned} 0 = \mathrm{Ext}^l_B(F_P A/J, F_P X) = \mathrm{Ext}^l_B(\Omega^0, F_P X) &\simeq \mathrm{Ext}^1_B(\Omega^{l-1}, F_P X) \\ &= \mathrm{Ext}^1_{B_J}(\Omega^{l-1}, F_{P/JP}X) \simeq \mathrm{Ext}^l_{B_J}(F_{P/JP}A/J, F_{P/JP}X). \end{aligned} \tag{22}$$

By [Cru24c, Proposition 3.1.5], this concludes the proof. ☐

Remark 7.1.2. The module P/JP might not be injective even if P is.

Remark 7.1.3. It follows from the proof of Theorem 7.1.1 that if (A, P) is a cover of B such that $(A/J, P/JP)$ is a 0-$\mathcal{F}(\tilde{\Delta}_{A/J})$ cover of B_J, then (A, P) is a (-1)-$\mathcal{F}(\tilde{\Delta})$ cover of B.

This gives another reason to be interested in zero faithful split quasi-hereditary covers. These are exactly the covers for which double centralizer properties occur in every step of the split heredity chain. In particular, this gives another perspective on why zero faithful split quasi-hereditary covers possess so much nicer properties compared with (-1)-faithful split quasi-hereditary covers.

7.2. Relative codominant dimension with respect to a module in the image of a Schur functor preserving the highest weight structure

In general, the Schur functor induced from a cosaturated idempotent (see Theorem 2.1.4) does not preserve dominant or codominant dimension. In this subsection, we relate the codominant dimension of a characteristic tilting module T of A with eP-codomdim$_{eAe}\, eT$, when P is a faithful projective-injective A-module and e is a cosaturated idempotent.

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Theorem 7.2.1. *Let k be a field and $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over k. Assume that A admits a cosaturated idempotent e and suppose that P is a projective-injective faithful module. Let $M \in \mathcal{F}(\nabla)$. If* $\operatorname{codomdim}_A M \geq i$, *then* $eP\text{-}\operatorname{codomdim}_{eAe} eM \geq i$ *for* $i \in \{1, 2\}$.

Proof. Denote by $B = \operatorname{End}_A(P)^{op}$ and by $C = \operatorname{End}_{eAe}(eP)^{op}$. Since P is a (partial) tilting module, the map $B \to C$, given by multiplication by e, is surjective according to Theorem 2.1.4. Thus, C is a quotient of B. In particular, C-mod is a full subcategory of B-mod. Again by Theorem 2.1.4, the map $\operatorname{Hom}_A(P, M) \to \operatorname{Hom}_{eAe}(eP, eM)$ is a surjective left B-homomorphism. Denote such a map by φ_M. We can consider the following commutative diagram

$$\begin{array}{ccccccc} e\cdot(P\otimes_B \operatorname{Hom}_A(P,M)) & = & (eP)\otimes_B \operatorname{Hom}_A(P,M) & \xrightarrow[(e\cdot P)\otimes_B\varphi_M]{} & (eP)\otimes_B \operatorname{Hom}_{eAe}(eP,eM) & = & (eP)\otimes_C \operatorname{Hom}_{eAe}(eP,eM) \\ \downarrow e\delta_{DM} & & & & & & \downarrow \chi_{eM} \\ eM & & & = & & & eM \end{array}$$

with the composition of the upper rows being surjective (see also Remark 3.1.2). In fact, thanks to the C-mod being a full subcategory of B-mod, we have the isomorphisms

$$\begin{aligned} D((eP)\otimes_C \operatorname{Hom}_{eAe}(eP,eM)) &\simeq \operatorname{Hom}_C(\operatorname{Hom}_{eAe}(eP,eM), D(eP)) = \operatorname{Hom}_B(\operatorname{Hom}_{eAe}(eP,eM), D(eP)) \\ &\simeq D((eP)\otimes_B \operatorname{Hom}_{eAe}(eP,eM)). \end{aligned} \tag{23}$$

Since $\operatorname{domdim}_{A^{op}} DM = \operatorname{codomdim}_A M \geq 1$(resp. 2) if and only if δ_{DM} is surjective (resp. bijective), we obtain that $e\delta_{DM}$ is surjective if $i = 1$ and bijective if $i = 2$. So, if $i = 1$, it follows that χ_{eM} is surjective, by the commutative diagram. Assume that $i = 2$. Then, $(e\cdot P)\otimes_B \varphi_M$ must be injective, and so it is an isomorphism. This implies that χ_{eM} is also an isomorphism. □

For larger values of relative dominant dimension, the most natural approach to consider is to see when the exact sequence giving the value of dominant dimension under the Schur functor $eA\otimes_A -$ gives information about the relative dominant dimension of eAe with respect to eP.

Proposition 7.2.2. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over a field k, e a cosaturated idempotent of A and P a projective-injective A-module. Suppose that there exists an exact sequence*

$$\delta\colon 0 \to A \to P_0 \to \cdots \to P_{n-1}, \quad \textit{with } P_i \in \operatorname{add}_A P. \tag{24}$$

Then, the chain complex $\operatorname{Hom}_{eAe}(e\delta, eP)$ *is exact if and only if* $P \in \operatorname{add} D(eA)$. *In particular, if the chain complex* $\operatorname{Hom}_{eAe}(e\delta, eP)$ *is exact, then eP is a projective-injective eAe-module.*

Proof. Assume that $P \in \operatorname{add}_A D(eA)$. Then, $eP \in \operatorname{add}_{eAe} D(eAe)$; that is, eP is injective over eAe. It is clear that the functor $\operatorname{Hom}_{eAe}(-, eP)$ is exact.

Conversely, suppose that $e\delta$ remains exact under $\operatorname{Hom}_{eAe}(-, eP)$. Let X_0 be the cokernel of $A \to P_0$. Consider the commutative diagram

$$\begin{array}{ccccc} \operatorname{Hom}_A(P_1,P) & \longrightarrow & \operatorname{Hom}_A(P_0,P) & \twoheadrightarrow & \operatorname{Hom}_A(A,P) \\ \downarrow & \searrow\!\!\!\!\twoheadrightarrow \quad \operatorname{Hom}_A(X_0,P) \quad \hookrightarrow\!\!\nearrow & \downarrow & & \downarrow \\ & \qquad\qquad \downarrow & & & \\ & \nearrow \quad \operatorname{Hom}_{eAe}(eX_0,eP) \quad \hookrightarrow\!\!\searrow & & & \\ \operatorname{Hom}_{eAe}(eP_1,eP) & \longrightarrow & \operatorname{Hom}_{eAe}(eP_0,eP) & \twoheadrightarrow & \operatorname{Hom}_{eAe}(eA,eP). \end{array} \tag{25}$$

The vertical maps are surjective maps due to Theorem 2.1.4. By assumption, the bottom row of (25) is exact. Hence, the lower triangle is an epi-mono factorization. Therefore, $\operatorname{Hom}_A(X_0, P) \to \operatorname{Hom}_{eAe}(eX_0, eP)$ is surjective. By Snake Lemma, we obtain that the map

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$\mathrm{Hom}_A(A,P) \to \mathrm{Hom}_{eAe}(eA,eP)$ is, in addition to being surjective, an injective map. Since eA has a filtration by standard modules over eAe, $\mathrm{Ext}^{i>0}_{eAe}(eA,eP)$. By Lemma 2.10 of [GK15], for every $M \in A$-mod,

$$\mathrm{Hom}_A(M,P) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_{eAe}(eA,eP)) \simeq \mathrm{Hom}_{eAe}(eM,eP). \tag{26}$$

Then, by Theorem 3.10 of [Psa14], there exists an exact sequence $0 \to P \to \mathrm{Hom}_{eAe}(eA, D(eAe)) \simeq D(eA)$. Since P is injective, this exact sequence splits, and we obtain that $P \in \mathrm{add}_A D(eA)$. □

The above condition is very restrictive, since in most cases of our interest, a module $Q \in \mathcal{F}(\Delta)\cap\mathcal{F}(\nabla)$ is projective if and only if it is injective. Still, the injective dimension gives information on a lower bound:

Corollary 7.2.3. *Let k be a field and A a finite-dimensional k-algebra. Let $Q \in A$-mod with $Q \in {}^\perp Q$. Suppose that $M \in {}^\perp Q$ and assume that there exists an A-exact sequence $0 \to M \to Q_1 \to \cdots \to Q_n$, with $Q_i \in \mathrm{add}\, Q$. Then, Q-$\mathrm{domdim}_A M \geq n - \mathrm{idim}_A Q$.*

Proof. Assume that $n > \mathrm{idim}_A Q$; otherwise, there is nothing to prove. Denote by X_i the cokernel of $Q_{i-1} \to Q_i$ where by convention we consider $Q_0 := M$. By dimension shifting,

$$\mathrm{Ext}^{i>0}_A(X_{n-\mathrm{idim}_A Q}, Q) \simeq \mathrm{Ext}^{i+1>0}_A(X_{n-\mathrm{idim}_A Q+1}, Q) \simeq \mathrm{Ext}^{i+\mathrm{idim}_A Q>0}_A(X_n, Q) = 0. \tag{27}$$

So, the exact sequence $M \hookrightarrow Q_1 \to \cdots \to Q_{n-\mathrm{idim}_A Q}$ satisfies the assumptions of Proposition 3.1.12. □

Another option is to consider the homology over $\mathrm{End}_{eAe}(eP)$ by viewing it as a quotient of $\mathrm{End}_A(P)$.

Remark 7.2.4. The surjective map $\psi\colon kS_d \twoheadrightarrow \mathrm{End}_{S_k(n,d)}(V^{\otimes d})$ (see Subsection 8.1) may not be a homological epimorphism if $n < d$. Indeed, by [dlPX06, Proposition 2.2(a)], ψ is a homological epimorphism if and only if $\ker\psi$ is an idempotent ideal and $\mathrm{Tor}^{kS_d}_{i>0}(\ker\psi, kS_d/\ker\psi) = 0$. Fix $n = 2$, $d = 3$ and k a field of characteristic three. Then, $\ker\psi$ is the ideal generated by $a := e + (132) + (123) - (12) - (13) - (23)$. As $a^2 = 0$, $\ker\psi$ is not an idempotent ideal. So, such a method does not work for Schur algebras.

This means that we should use another approach to obtain lower bounds of codominant dimension with respect to eP using the codominant dimension with respect to P. We can use, instead, truncation of covers. This is fruitful for values of Hemmer-Nakano dimension greater than or equal to zero, but this poses no problem in our situation since the smaller cases can be treated using Theorem 7.2.1.

Theorem 7.2.5. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over a field k. Let e be a cosaturated idempotent of A. If A has positive dominant dimension with faithful projective-injective Af, then*

$$eAf\text{-}\mathrm{codomdim}_{eAe}\, eT \geq \mathrm{codomdim}_A T,$$

where T is the characteristic tilting module of A.

Proof. We can assume without loss of generality that A is a basic algebra and Af is also basic. If $\mathrm{codomdim}_A T = 1$, then the result follows from Theorem 7.2.1. Assume that $\mathrm{codomdim}_A T \geq 2$.

By Theorem 2.1.4, eT is the characteristic tilting module of eAe. Hence, the endomorphism algebra $\mathrm{End}_{eAe}(eT)^{op}$ is the Ringel dual of eAe which we denote by $R(eAe)$. Also, by Theorem 2.1.4, there exists an exact sequence $0 \to X \to R(A) \to R(eAe) \to 0$, where X is an ideal of the Ringel dual of A. More precisely, X is the set of all endomorphisms $g \in \mathrm{End}_A(T)$ satisfying $eg = 0$. Fix $P = Af$. We claim that $X\,\mathrm{Hom}_A(T,P)$ is the kernel of the surjective map $\mathrm{Hom}_A(T,P) \to \mathrm{Hom}_{eAe}(eT,eP)$. Denote this surjection by ψ. Let $g \in X$ and $l \in \mathrm{Hom}_A(T,P)$; then $e(lg) = (el)(eg) = 0$. So, it is clear that $X\,\mathrm{Hom}_A(T,P) \subset \ker\psi$. Now, let $l \in \mathrm{Hom}_A(T,P)$ such that $el = 0$; that is, $l \in \ker\psi$. By assumption, we can write $i \circ \pi = \mathrm{id}_P$, where $\pi \in \mathrm{Hom}_A(T,P)$. So, $e(i \circ l) = ei \circ el = 0$. This means that $i \circ l \in X$.

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Now $l = \pi \circ i \circ l = (i \circ l) \cdot \pi \in X\operatorname{Hom}_A(T, P)$. Now, a k-basis of $\operatorname{End}_A(T)$ can be constructed using its filtration by modules $\operatorname{Hom}_A(\Delta(\nu), \nabla(\nu))$, $\nu \in \Lambda$ and the liftings of $\Delta(\lambda) \hookrightarrow T(\lambda) \twoheadrightarrow \nabla(\lambda)$ along these filtrations (see [Cru24b, Proposition 5.5] and [KSX01]). In particular, these maps factor through $T(\lambda)$, $\lambda \in \Lambda$. Analogously, $\operatorname{End}_{eAe}(eT)$ has a k-basis of the maps factoring through $eT(\lambda) \neq 0$, $\lambda \in \Lambda'$ (see Theorem 2.1.4(ii)(II) and its notation). So, X has a basis whose maps $T \to T$ factor through $T(\lambda)$, $\lambda \in \Lambda \setminus \Lambda'$. Let g_λ denote the idempotent $T \twoheadrightarrow T(\lambda) \hookrightarrow T$ and $g_e = \sum_{\lambda \in \Lambda \setminus \Lambda'} g_\lambda$. Then, we showed that $X = R(A)g_eR(A)$. In particular, X has a filtration by split heredity ideals of quotients of $R(A)$.

As $\operatorname{codomdim}_A T \geq 2$, Theorem 5.3.1 implies that $(R(A), \operatorname{Hom}_A(T, P))$ is a $(\operatorname{codomdim}_A T - 2)$-$\mathcal{F}(\Delta_{R(A)})$ cover of $\operatorname{End}_A(P)^{op}$. By induction on the filtration of X by split heredity ideals and using Theorem 7.1.1, we obtain that $(\operatorname{End}_{eAe}(eT), \operatorname{Hom}_{eAe}(eT, eP)) \simeq (R(A)/X, \operatorname{Hom}_A(T, P)/X\operatorname{Hom}_A(T, P))$ is a $(\operatorname{codomdim}_A T - 2)$-$\mathcal{F}(\Delta_{R(eAe)})$ cover of $\operatorname{End}_{R(A)/X}(\operatorname{Hom}_A(T, P)/X\operatorname{Hom}_A(T, P))^{op}$ which is isomorphic to $\operatorname{End}_{R(eAe)}(\operatorname{Hom}_{eAe}(eT, eP))^{op} \simeq \operatorname{End}_{eAe}(eP)^{op}$. By Theorem 5.3.1, the result follows. □

8. Applications

In this section, we will use the technology on relative dominant dimension with respect to a partial tilting module including [Cru22b, Cru24c] to construct a split quasi-hereditary cover of certain quotients of Iwahori-Hecke algebras. This gives a new point of view to the Schur functors from module categories of q-Schur algebras studied in detail in [Cru24c]. This technology together with the results from [Cru24c] will allow us to give a new proof for the Ringel self-duality of the blocks of the BGG category $\mathcal{O}$.

8.1. Generalized Schur algebras in the sense of Donkin

Throughout, assume that n, d are natural numbers, R is a commutative Noetherian ring with an invertible element u and $q = u^{-2}$. In this subsection, we continue [Cru24c, Cru22b] studying q-Schur algebras and Iwahori-Hecke algebras over commutative Noetherian rings using relative dominant dimensions. In particular, we use the notation of [Cru22b, Cru24a, Cru24c].

The study of Iwahori-Hecke algebras (denoted by $H_{R,q}(d)$) can be traced back to [Iwa64], and there are several equivalent ways to define them (here we follow [PW91]). Properties on q-Schur algebras (denoted by $S_{R,q}(n, d)$) can be found in [DJ91, DJ89, Don98] and in [Gre81] for $q = 1$. For instance, the module $V^{\otimes d} := (R^n)^{\otimes d}$ admits a $(S_{R,q}(n, d), H_{R,q}(d))$-bimodule structure so that $S_{R,q}(n, d) = \operatorname{End}_{H_{R,q}(d)}(V^{\otimes d})$.

Quantum Schur–Weyl duality says that $\psi\colon H_{R,q}(d) \to \operatorname{End}_{S_R(n,d)}(V^{\otimes d})^{op}$ is a surjective map (see, for example, [DPS98, Theorem 6.2]). If $n \geq d$, $V^{\otimes d}$ is a faithful module over $H_{R,q}(d)$, and in such a case, ψ is actually an isomorphism. So this means that there are double centralizer properties:

$$S_{R,q}(n, d) = \operatorname{End}_{H_{R,q}(d)}(V^{\otimes d}), \qquad H_{R,q}(d) \simeq \operatorname{End}_{S_{R,q}(n,d)}(V^{\otimes d})^{op}, \qquad \text{if } n \geq d; \quad (28)$$

$$S_{R,q}(n, d) = \operatorname{End}_{H_{R,q}(d)/\ker\psi}(V^{\otimes d}), \quad H_{R,q}(d)/\ker\psi \simeq \operatorname{End}_{S_{R,q}(n,d)}(V^{\otimes d})^{op}, \qquad \text{if } n < d. \quad (29)$$

The double centralizer property in (28) follows from $(S_{R,q}(n, d), V^{\otimes d})$ being a cover of $H_{R,q}(d)$ (see, for example, [Cru22b, Theorem 7.20], [Cru24c, Proposition 2.4.4]). But if $n < d$, then $V^{\otimes d}$ is not necessarily projective, and so the former pair cannot be a cover anymore in general. Our aim now is to unify these double centralizer properties and their cohomological higher versions using relative dominant dimension and cover theory generalizing the approach taken in [KSX01]. In our treatment, we will take care of the quantum case and the classical case ($q = 1$) simultaneously having always the integral setup in mind.

8.1.0.1. Background on q-Schur algebras.

The ring R is an $\mathbb{Z}[X, X^{-1}]$-algebra by defining the map of R-algebras $\mathbb{Z}[X, X^{-1}] \to R$ which sends $z \in \mathbb{Z}$ to $z1_R$ and X to $u \in R$. Hence, all q-Schur algebras and Iwahori-Hecke algebras can be constructed from the ones over Laurent polynomials (see, for example, [Cru22b, Lemma 7.17]):

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$$H_{R,q}(d) \simeq R \otimes_{\mathbb{Z}[X,X^{-1}]} H_{\mathbb{Z}[X,X^{-1}],X^{-2}}(d), \quad S_{R,q}(n,d) \simeq R \otimes_{\mathbb{Z}[X,X^{-1}]} S_{\mathbb{Z}[X,X^{-1}],X^{-2}}(n,d). \tag{30}$$

The q-Schur algebra admits a split quasi-hereditary structure whose standard modules are called the q-Weyl modules, and are indexed by the partitions of d in at most n parts ordered by the dominance order (see, for example, [PW91, Theorem 11.5.2] and [Cru24c, Theorem 7.2.2]). We denote the standard modules of $S_{R,q}(n,d)$ by $\Delta_{S_{R,q}(n,d)}(\lambda)$, $\lambda \in \Lambda^+(n,d)$, or just by $\Delta(\lambda)$ when there are no ambiguities. By $\Lambda^+(n,d)$ we mean the set of all partitions of d in at most n parts. It admits a cellular structure with the standard modules being the cell modules (ordered with the opposite of the dominance order). Furthermore, for each $\mathfrak{m} \in \operatorname{MaxSpec} R$, the $q_\mathfrak{m}$-Schur algebra $S_{R(\mathfrak{m}),q_\mathfrak{m}}(n,d)$ (with $n \geq d$) belongs to the class $\mathcal{A}$ defined in [FK11], where $q_\mathfrak{m}$ denotes the image of q in $R/\mathfrak{m}$. By [Cru22b, Theorem 7.20] and [Cru24c, Corollary 7.2.4], if $n \geq d$, the pair $(S_{R,q}(n,d), V^{\otimes d})$ is a relative gendo-symmetric R-algebra and $\operatorname{domdim}(S_{R,q}(n,d), R) = 2 \operatorname{domdim}_{(S_{R,q}(n,d),R)} T = 2 \inf\{s \in \mathbb{N} \mid 1 + q + \cdots + q^s \notin R^\times,\ s < d\}$, where T is the characteristic tilting module of $S_{R,q}(n,d)$.

Remark 8.1.1. For any $n, d \in \mathbb{N}$ and any $M \in \operatorname{add}_{S_{R,q}(n,d)} T$, we have $M\text{-}\operatorname{domdim}_{(S_{R,q}(n,d),R)} T = M\text{-}\operatorname{codomdim}_{(S_{R,q}(n,d),R)} T$. Indeed, this follows from applying Theorem 3.2.5 (and its dual) with Proposition 3.1.7 to the $q_\mathfrak{m}$-Schur algebras $S_{R(\mathfrak{m}),q_\mathfrak{m}}(n,d)$, $\mathfrak{m} \in \operatorname{MaxSpec} R$.

8.1.0.2. Covers of Iwahori-Hecke algebras.

When $n \geq d$, the pair $(S_{R,q}(n,d), V^{\otimes d})$ is (at least) a (-1)-$\mathcal{F}(\tilde{\Delta})$ split quasi-hereditary cover of $H_{R,q}(d)$ for every commutative ring R. The quality of the Schur functor $F_{V^{\otimes d}} = \operatorname{Hom}_{S_{R,q}(n,d)}(V^{\otimes d}, -) \colon S_{R,q}(n,d)\text{-mod} \to H_{R,q}(d)\text{-mod}$ was completely determined for all local commutative rings R in [Cru24c, Subsection 7.2.1]. The field case is due to [FK11, FM19]. In the general case, the quality depends on whether the ground ring is partial divisible or not. For more details, we refer to [Cru24c, Subsections 7.1 and 7.2].

8.1.0.3. Covers of quotients of Iwahori-Hecke algebras.

We now use Theorem 7.2.5 to transfer information from $n \geq d$ to the case $n < d$. Let k be a field. Put $q = u^{-2}$ for some $u \in k$ and assume that $n < d$. By [Don98, 2.2(1), 4.7 and A3.11(i)], $S_{k,q}(d,d)$ admits a cosaturated idempotent f so that $f(k^d)^{\otimes d} \simeq (K^n)^{\otimes d}$ as $S_{k,q}(n,d)$-modules and $S_{k,q}(n,d) \simeq fS_{k,q}(d,d)f$ as k-algebras. By Theorem 2.1.4, $\operatorname{add} fT_d = \operatorname{add} T_n$ and $(k^n)^{\otimes d} \in \operatorname{add} T_n$ since $(k^d)^{\otimes d}$ is a projective-injective of $S_{k,q}(d,d)$, where T_n (resp. T_d) is the characteristic tilting module of $S_{k,q}(n,d)$ (resp. $S_{k,q}(d,d)$). When $q = 1$, see also [Erd94, 3.9, 4.2]. By Proposition 2.1.3, $(R^n)^{\otimes d} \in \mathcal{F}(\tilde{\Delta}) \cap \mathcal{F}(\tilde{\nabla})$ for every commutative Noetherian R.

Theorem 8.1.2. *Let R be a commutative Noetherian ring with an invertible element $u \in R$ and n, d be natural numbers. Put $q = u^{-2}$. Let T_n be a characteristic tilting module of $S_{R,q}(n,d)$. Then,*

$$V^{\otimes d}\text{-}\operatorname{domdim}_{(S_{R,q}(n,d),R)} T_n \geq \inf\{s \in \mathbb{N} \mid 1 + q + \cdots + q^s \notin R^\times,\ s < d\} \geq 1.$$

Proof. By [Cru24c, Corollary 7.2.4], we may assume that $n < d$. By Propositions 2.1.3 and 2.1.2 and Theorem 3.2.5,

$$V^{\otimes d}\text{-}\operatorname{domdim}_{(S_{R,q}(n,d),R)} T_n = \inf\{(R(\mathfrak{m})^n)^{\otimes d}\text{-}\operatorname{domdim}_{(S_{R(\mathfrak{m}),q_\mathfrak{m}}(n,d)} T_n(\mathfrak{m}) | \mathfrak{m} \in \operatorname{MaxSpec} R\},$$

where $q_\mathfrak{m}$ is the image of q in $R(\mathfrak{m})$. By Theorem 7.2.5, Remark 8.1.1 and the above discussion,

$$V^{\otimes d}\text{-}\operatorname{domdim}_{(S_{R,q}(n,d),R)} T_n \geq \inf\{(R(\mathfrak{m})^d)^{\otimes d}\text{-}\operatorname{domdim}_{(S_{R(\mathfrak{m}),q_\mathfrak{m}}(d,d)} T_{d,\mathfrak{m}} | \mathfrak{m} \in \operatorname{MaxSpec} R\} \tag{31}$$

$$= \inf\{\operatorname{domdim}_{(S_{R(\mathfrak{m}),q_\mathfrak{m}}(d,d)} T_{d,\mathfrak{m}} | \mathfrak{m} \in \operatorname{MaxSpec} R\} \tag{32}$$

$$= \inf\{\operatorname{domdim}_{S_{R(\mathfrak{m}),q_\mathfrak{m}}(d,d)} T_d(\mathfrak{m}) | \mathfrak{m} \in \operatorname{MaxSpec} R\}, \tag{33}$$

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where $T_{d,\mathfrak{m}}$ is the characteristic tilting module of $S_{R(\mathfrak{m}),q_{\mathfrak{m}}}(d,d)$ and $\operatorname{add} T_d(\mathfrak{m}) = \operatorname{add} T_{d,\mathfrak{m}}$, $\mathfrak{m} \in \operatorname{MaxSpec} R$. By Theorem 6.13 of [Cru22b] and Corollary 7.2.4 of [Cru24c], it follows that

$$V^{\otimes d}\text{-domdim}_{(S_{R,q}(n,d),R)}\, T_n \geq \inf\{s \in \mathbb{N} \mid 1 + q + \cdots + q^s \notin R^\times,\ s < d\}. \qquad \square$$

This inequality is sharp in general since this becomes an equality in case $n \geq d$ (see [Cru24c, Corollary 7.2.4]). We are now ready to prove one of our main results.

Theorem 8.1.3. *Let R be a commutative Noetherian regular ring with an invertible element $u \in R$ and n, d be natural numbers. Put $q = u^{-2}$. Let T be a characteristic tilting module of $S_{R,q}(n,d)$ and $R(S_{R,q}(n,d)) := \operatorname{End}_{S_{R,q}(n,d)}(T)^{op}$ the Ringel dual of the q-Schur algebra $S_{R,q}(n,d)$.*

Then, $(R(S_{R,q}(n,d)), \operatorname{Hom}_{S_{R,q}(n,d)}(T, V^{\otimes d}))$ is a $(V^{\otimes d}\text{-domdim}_{(S_{R,q}(n,d),R)}\, T - 2)$-$\mathcal{F}(\tilde{\Delta}_{R(S_{R,q}(n,d))})$ split quasi-hereditary cover of $\operatorname{End}_{S_{R,q}(n,d)}(V^{\otimes d})^{op}$.

Proof. If $V^{\otimes d}\text{-domdim}_{(S_{R,q}(n,d),R)}\, T \geq 2$, the result follows from Theorems 8.1.2, 5.3.1 and Proposition 3.1.7. Assume now that $V^{\otimes d}\text{-domdim}_{(S_{R,q}(n,d),R)}\, T = 1$. Theorem 8.1.2 implies that $1+q$ is not invertible and $d > 1$. So we need to verify, in particular, the case $1 + q = 0$ and $d > 1$.

Let L be a field with an element $u \in L$ satisfying $u^{-2} = -1$. Consider the Laurent polynomial ring $R := L[X, X^{-1}]$. The invertible elements of $L[X, X^{-1}]$ are of the form lX^z, $z \in \mathbb{Z}$ and $l \in L$ and $1 + X^{-2} \neq 0$. So, $\inf\{s \in \mathbb{N} \mid 1 + q + \cdots + q^s \notin L[X, X^{-1}]^\times,\ s < d\} = 1$ with $q = X^{-2}$. Let Q be the quotient field of $L[X, X^{-1}]$. Then,

$$\inf\{s \in \mathbb{N} \mid 1 + q + \cdots + q^s \neq 0 \in Q,\ s < d\} \geq 2.$$

Therefore, $(Q^n)^{\otimes d}\text{-domdim}_{S_{Q,q}(n,d),Q}\, Q \otimes_{L[X,X^{-1}]} T \geq 2$ and $V^{\otimes d}\text{-domdim}_{(S_{R,q}(n,d),R)}\, T \geq 1$ for a characteristic tilting module T of $S_{R,q}(n,d)$. By Corollary 5.3.5, we obtain that $(R(S_{R,q}(n,d)), \operatorname{Hom}_{S_{R,q}(n,d)}(T, V^{\otimes d}))$ is a 0-$\mathcal{F}(\tilde{\Delta}_{R(S_{R,q}(n,d))})$ split quasi-hereditary cover of $\operatorname{End}_{S_{R,q}(n,d)}(V^{\otimes d})^{op}$. Observe that L is the residue field $L[X, X^{-1}]/(X - u^{-1})$ and the image of X in this residue field is u^{-1}. It follows by applying $L[X, X^{-1}]/(X - u^{-1}) \otimes_{L[X,X^{-1}]} -$ to the previous cover together with Theorem 5.0.7 of [Cru24c] that $(R(S_{L,-1}(n,d)), \operatorname{Hom}_{S_{L,-1}(n,d)}(T, V^{\otimes d}))$ is a (-1)-$\mathcal{F}(\tilde{\Delta}_{R(S_{L,-1}(n,d))})$ split quasi-hereditary cover of $\operatorname{End}_{S_{L,-1}(n,d)}(V^{\otimes d})^{op}$. By [Cru24c, Proposition 5.0.5], we conclude that the result holds in all cases. $\square$

The special case $q = 1$ puts the Schur–Weyl duality between $S_R(n,d)$ and RS_d without restrictions on the parameters n and d into our context. Moreover, this generalizes the results of Hemmer and Nakano in [HN04] and [FK11, Theorem 3.9] on the Schur algebra ($S_R(n,d)$ with parameters $n \geq d$). In general, this formulation helps us to put the quantum Schur–Weyl duality between q-Schur algebras and Iwahori-Hecke algebras on $V^{\otimes d}$ completely in the setup of cover theory generalising the results of [FM19, PS05]. In fact, if $n \geq d$ and k is a field with $u \in k$, then by ([Don93, Proposition 3.7] for $q = 1$ and [Don98, Proposition 4.1.4, Proposition 4.1.5]), the Ringel dual of $S_{k,q}(n,d)$ is the opposite algebra of $S_{k,q}(n,d)$, and we can identify the projective relative injective module $\operatorname{Hom}_{S_{R,q}(n,d)}(T, V^{\otimes d})$ with $DV^{\otimes d}$. In this case, quantum Schur–Weyl duality gives $\operatorname{End}_{S_{R,q}(n,d)}(V^{\otimes d}) \simeq H_{R,q}(d)^{op}$. So, for $n \geq d$, Theorem 8.1.3 is translated to $(S_{R,q}(n,d)^{op}, DV^{\otimes d})$ being a split quasi-hereditary cover of $H_{R,q}(d)^{op}$. Therefore, for $n \geq d$, this statement is nothing new, and since $S_R(n,d)$ is relative gendo-symmetric, the quality of this cover coincides with the cover $(S_{R,q}(n,d), V^{\otimes d})$. The novelty lies in the case $n < d$.

However, this formulation gives us better insights into older results present in the literature, and it explains, in some sense, why different conventions about the Schur functor can be considered. For simplicity, let us assume that $q = 1$. Let $H \colon S_R(n,d)\text{-mod} \to R(S_R(n,d))$ be the equivalence of categories given by Ringel self-duality in [Don93, Proposition 3.7]. Denote by F' the Schur functor associated with the cover $(R(S_R(n,d)), \operatorname{Hom}_{S_R(n,d)}(T, V^{\otimes d}))$ and recall that $F = F_{V^{\otimes d}}$ is the multiplication by an idempotent e (see, for instance, [Cru22b, 7.5, 7.6]). Then, $F\Delta(\lambda) = e\Delta(\lambda)$ and by [CPS96, Lemma 1.6.12],

$$F'H\Delta(\lambda) \simeq F' \operatorname{Hom}_{S_R(n,d)}(T, \nabla(\lambda')) \simeq F\nabla(\lambda') = eD\Delta(\lambda')^\iota \simeq D(e\Delta(\lambda'))^{e\iota e} \simeq \operatorname{sgn} \otimes_R e\Delta(\lambda). \tag{34}$$

Here, λ' is the conjugate partition of λ, sgn is the free module R with the S_d-action $\sigma \cdot 1_R = \operatorname{sgn}(\sigma)1_R$ and M^ι is the right module M with right action $m \cdot a = \iota(a)m$, $m \in M$ and $a \in S_R(n,d)$ for the involution ι (see [Cru24a, Section 5]). The same notation is used for modules over RS_d. By (34), $F'H = \operatorname{sgn} \otimes_R - \circ F$ on $S_R(n,d)$-proj. Since all functors F, H, F' and $\operatorname{sgn} \otimes_R -\colon RS_d\text{-mod} \to RS_d\text{-mod}$ are exact, we obtain $F'H = \operatorname{sgn} \otimes_R - \circ F$ on $S_R(n,d)$-mod. This means that the cover $(R(S_R(n,d)), \operatorname{Hom}_{S_R(n,d)}(T, V^{\otimes d}))$ is equivalent to $(S_R(n,d), V^{\otimes d})$ (in the sense of [Cru24c, Definition 4.3.1]) if $n \geq d$.

When $n < d$, the Ringel dual of $S_R(n,d)$ is no longer, in general, a Schur algebra; it is instead a generalized Schur algebra in the sense of Donkin. The construction of the Ringel dual of $S_R(n,d)$ is as follows: let $U_\mathbb{Z}$ be the Konstant $\mathbb{Z}$-form of the enveloping algebra of the semi-simple complex Lie algebra $\mathfrak{sl}_d(\mathbb{C})$. Then, the Ringel dual of $S_\mathbb{Z}(n,d)$ is the free Noetherian $\mathbb{Z}$-algebra $U_\mathbb{Z}/I_\mathbb{Z}$, where $I_\mathbb{Z}$ is the largest ideal of $U_\mathbb{Z}$ so that the simple modules of $\mathbb{Q} \otimes_\mathbb{Z} U_\mathbb{Z}/I_\mathbb{Z}$ are isomorphic to the Weyl modules indexed by the weights belonging to $\Lambda^+(n,d)$ (see [Don86, 3.1] and [Don93, Proposition 3.11]). For an arbitrary commutative ring R, $R \otimes_\mathbb{Z} U_\mathbb{Z}/I_\mathbb{Z}$ is the Ringel dual of $S_R(n,d)$ known as **generalized Schur algebra** associated with $\mathfrak{sl}_d$ and the set $\Lambda^+(n,d)$. Since the Ringel dual of the Schur algebra is a quotient of $R \otimes_\mathbb{Z} U_\mathbb{Z}$, Theorem 8.1.3 suggests why Schur–Weyl duality between $S_R(n,d)$ and RS_d can be deduced by studying the action of the Konstant $\mathbb{Z}$-form on $V^{\otimes d}$.

Remark 8.1.4. From Theorem 8.1.3, it follows that the basic algebra of $\operatorname{End}_{S_{R,q}(n,d)}(V^{\otimes d})^{op}(\mathfrak{m})$, $A_\mathfrak{m}$, has a cellular structure with cell modules of $H \operatorname{Hom}_{S_{R,q}(n,d)}(V^{\otimes d}, \nabla(\lambda))(\mathfrak{m})$, $\lambda \in \Lambda$, for every $\mathfrak{m} \in \operatorname{MaxSpec} R$, (see [Cru24b, Theorem 7.5]) where $H\colon \operatorname{End}_{S_{R,q}(n,d)}(V^{\otimes d})^{op}(\mathfrak{m})\text{-mod} \to A_\mathfrak{m}\text{-mod}$ is an equivalence of categories. If $q = 1$, using the fact that all Schur algebras can be recovered using change of rings from $S_\mathbb{Z}(n,d)$ [Cru24b, Proposition 7.6] implies that $\operatorname{End}_{S_R(n,d)}(V^{\otimes d})^{op}_\mathfrak{m}$ is Morita equivalent to a Noetherian algebra which has a cellular structure with cell modules labeled by $\lambda \in \Lambda$ for all $\mathfrak{m} \in \operatorname{MaxSpec} R$.

The existence of the quasi-hereditary cover described in Theorem 8.1.3 allows us to study the multiplicities of simple $\operatorname{End}_{S_k(n,d)}(V^{\otimes d})$-modules through the multiplicities of simple $R(S_k(n,d))$-modules for a field k. In particular, this explains the background for the techniques used in [Erd94] to determine decomposition numbers in the symmetric group. For example, [Erd94, 4.5] can be deduced using the Schur functor constructed in Theorem 8.1.3, the Ringel duality functor and BGG reciprocity.

If R is a field, the value of the cover in Theorem 8.1.3 is optimal. But, as we saw even for the case $n \geq d$, the situation can be improved in some cases. We leave as an open question to determine the relative dominant dimension $V^{\otimes d}\text{-domdim}_{(S_{R,q}(n,d),R)} T$, when $n < d$ and, in particular, the optimal value of the cover in Theorem 8.1.3.

8.2. *Relative dominant dimension as a tool for Ringel self-duality*

We are now going to use the results of Section 5.3 together with integral representation theory to reprove Ringel self-duality for two classical cases. The method is particularly effective when applied to the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. Such situations highlight the strength of working with split quasi-hereditary covers whose exact categories of modules admit a filtration by summands of direct sums of standard modules with large Hemmer-Nakano dimension. For the undefined notation and terminology in this subsection, we refer to [Cru24c].

8.2.1. Bernstein-Gelfand-Gelfand category $\mathcal{O}$

The **BGG category** $\mathcal{O}$ (see [BGG76, Hum08, Jan79] and the references therein) of a semi-simple Lie algebra $\mathfrak{g}_K = \mathfrak{n}_K^+ \oplus \mathfrak{h}_K \oplus \mathfrak{n}_K^-$ over a splitting field of characteristic zero K is the full subcategory of $U(\mathfrak{g}_K)$-Mod whose modules M are finitely generated over $U(\mathfrak{g}_K)$, locally $\mathfrak{n}_K^+$-finite and semi-simple over $\mathfrak{h}_K$; that is, $M = \bigoplus_{\lambda \in \mathfrak{h}_K^*} M_\lambda$ for weight spaces M_λ. Here, $\mathfrak{h}_K^*$ denotes the dual vector space of $\mathfrak{h}_K$.

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Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra. When K is an R-algebra, using the Chevalley basis, any semi-simple Lie algebra can be deformed to a Lie algebra $\mathfrak{g}_R$ so that $K \otimes_R \mathfrak{g}_R \simeq \mathfrak{g}_K$, and the same applies to its Cartan decomposition. The BGG category $\mathcal{O}$ admits an integral version $\mathcal{O}_{[\lambda],R}$, introduced in [GJ81, 1.4] as follows. Let $\lambda \in \mathfrak{h}_R^*$. We denote by $[\lambda]$ the set of elements of $\mathfrak{h}_R^*$, μ that satisfy $\mu - \lambda \in \Lambda_R$, where Λ_R is the integral weight lattice associated to the root system of $\mathfrak{g}_R$ relative to $\mathfrak{h}_R$. We define $\mathcal{O}_{[\lambda],R}$ to be the full subcategory of $U(\mathfrak{g}_R)$-Mod whose modules M are finitely generated over the enveloping algebra $U(\mathfrak{g}_R)$, $U(\mathfrak{n}_R^+)$-locally finite and satisfy $M = \sum_{\mu\in[\lambda]} M_\mu$.

The category $\mathcal{O}_{[\lambda],R}$ has Verma modules $\Delta(\mu)$, $\mu \in [\lambda]$, and each $\Delta(\mu)(\mathfrak{m})$ is a Verma module of $\mathcal{O}$ (see [Cru24c, Subsection 7.3.8]). Since R is a local ring, $\mathcal{O}_{[\lambda],R}$ decomposes into blocks following decompositions of $[\lambda]$ into blocks. The blocks of $[\lambda]$ were completely determined in [GJ81, 1.8.2], and all of them can be written in the form $\mathcal{D} = W_{\overline{\mu}} \cdot \mu + \nu$ with $\mu \in \mathfrak{h}_R^*$, $\nu \in \mathfrak{m}\mathfrak{h}_R^*$, where $W_{\overline{\mu}}$ is the Weyl group of the subroot system making $\overline{\mu}$ an integral weight. For each block $\mathcal{D}$ of $[\lambda]$, we define in [Cru24c, Definition 7.3.37] a projective Noetherian R-algebra $A_\mathcal{D}$ so that the module category $A_\mathcal{D}(\mathfrak{m})$-mod is equivalent to a block of the BGG category $\mathcal{O}$ whose composition factors are top $\Delta(\overline{\mu})$ with $\mu \in \mathcal{D}$ and $\overline{\mu}$ is the image of μ in $\mathfrak{h}_R^*/\mathfrak{m}\mathfrak{h}_R^*$. In particular, every block of the BGG category $\mathcal{O}$ can be obtained in this way.

The algebra $A_\mathcal{D}$ is a split quasi-hereditary cellular R-algebra with standard modules $\Delta_A(\mu)$, $\mu \in \mathcal{D}$ (see [Cru24c, Theorems 7.3.38 and 7.3.42]). Moreover, the pair $(A_\mathcal{D}, P_A(\omega))$ is a split quasi-hereditary cover of the commutative R-algebra $C := \operatorname{End}_{A_\mathcal{D}}(P_A(\omega))^{op}$ (see [Cru24c, Theorem 7.3.44]), where the projective module $P_A(\omega)$ is indexed by an antidominant weight (that is, $\overline{\omega}$ is an antidominant weight in $\mathfrak{h}_{R(\mathfrak{m})}^*$). Further, the pair $(A_\mathcal{D}, P_A(\omega))$ is a relative gendo-symmetric R-algebra with 2 $\operatorname{domdim}_{(A_\mathcal{D},R)} T = \operatorname{domdim}(A_\mathcal{D}, R) = 2$ if $|\mathcal{D}| > 1$; otherwise, it is infinite (see [Cru24c, Theorem 7.3.43, Theorem 7.3.44]), where T is a characteristic tilting module of $A_\mathcal{D}$. Here, C is a deformation of a subalgebra of a coinvariant algebra, in the sense that $C(\mathfrak{m})$ is a subalgebra of a coinvariant algebra. We can consider an integral version of the Soergel's combinatorial functor $\mathbb{V}_\mathcal{D} = \operatorname{Hom}_{A_\mathcal{D}}(P_A(\omega), -)\colon A_\mathcal{D}\text{-mod} \to C\text{-mod}$. This integral version has better properties than the classical one, and in particular, if $\mathcal{D}$ is nice enough, $\mathbb{V}_\mathcal{D}$ induces a full embedding from $\mathcal{F}(\Delta)$ to C-mod preserving also Ext groups up to a certain degree smaller than or equal to the rank of the Cartan subalgebra (see [Cru24c, Theorem 7.3.45, Remark 7.3.47]).

8.2.1.1. Ringel self-duality of BGG category $\mathcal{O}$.
The blocks of the classical BGG category $\mathcal{O}$ are Ringel self-dual ([Soe98, Corollary 2.3] and [FKM00, Proposition 4]). We present now a different proof.

Theorem 8.2.1. *Let R be a local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra. Let $\mathcal{D}$ be a block of $[\lambda]$ for some $\lambda \in \mathfrak{h}_R^*$. The split quasi-hereditary R-algebra $A_\mathcal{D}$ is Ringel self-dual.*

Proof. Let $\mu \in \mathcal{D}$ and $\mathfrak{m} \in \operatorname{MaxSpec} R$. Both $\Delta(\mu)$ and $\nabla(\mu)$ belong to $A_\mathcal{D}/J$-mod, where J is an ideal admitting a filtration by split heredity ideals and such that $\Delta(\mu)$ is a projective $A_\mathcal{D}/J$-module. Further, since $\nabla(\mu)(\mathfrak{m})$ is the dual of $\Delta(\mu)(\mathfrak{m})$, its socle coincides with the top of $\Delta(\mu)(\mathfrak{m})$. Denote by f the nonzero $A_\mathcal{D}/J(\mathfrak{m})$-homomorphism $\Delta(\mu)(\mathfrak{m}) \twoheadrightarrow \operatorname{top}\Delta(\mu)(\mathfrak{m}) \hookrightarrow \nabla(\mu)(\mathfrak{m})$. As $\Delta(\mu)$ is a projective object in $A_\mathcal{D}/J$-mod, there exists an $A_\mathcal{D}$-homomorphism $\overline{f}$ making the following diagrams commutative:

$$\begin{array}{ccc} \Delta(\mu) & \twoheadrightarrow & \Delta(\mu)(\mathfrak{m}) \\ \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f} \\ \nabla(\mu) & \twoheadrightarrow & \nabla(\mu)(\mathfrak{m}) \end{array} \qquad \begin{array}{ccc} F\Delta(\mu) & \twoheadrightarrow & F\Delta(\mu)(\mathfrak{m}) \\ \downarrow{\scriptstyle F\overline{f}} & & \downarrow{\scriptstyle Ff} \\ F\nabla(\mu) & \twoheadrightarrow & F\nabla(\mu)(\mathfrak{m}) \end{array}, \qquad (35)$$

where F is the Schur functor $\mathbb{V}_\mathcal{D} = \operatorname{Hom}_{A_\mathcal{D}}(P_A(\omega), -)$ and ω is the antidominant weight. Observe that for any $X \in A_\mathcal{D}$-mod, $F(X(\mathfrak{m})) = \operatorname{Hom}_{A_\mathcal{D}}(P_A(\omega), X(\mathfrak{m})) \simeq \operatorname{Hom}_{A_\mathcal{D}(\mathfrak{m})}(P_{A_\mathcal{D}(\mathfrak{m})}(\overline{\omega}), X(\mathfrak{m}))$ (see,

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for example, [Cru24c, Lemma 2.1.1]) and Ff is isomorphic to the map $\mathrm{Hom}_{A_{\mathcal{D}}(\mathfrak{m})}(P_{A(\mathfrak{m})}(\overline{\omega}), f)$ which is nonzero since top $P_{A(\mathfrak{m})}(\overline{\omega})$ is the image of f. Moreover, $F\Delta(\mu)(\mathfrak{m}) \simeq R(\mathfrak{m})$ and $F\nabla(\mu)(\mathfrak{m}) \simeq R(\mathfrak{m})$. Hence, Ff is an isomorphism. Applying $R(\mathfrak{m}) \otimes_R -$ to the diagram (35), we obtain that $F\underline{f}(\mathfrak{m})$ is an isomorphism. Since both $F\Delta(\mu), F\nabla(\mu) \in R$-proj, Nakayama's Lemma yields that $F\underline{f}$ is an isomorphism. This shows that $F\Delta(\mu) \simeq F\nabla(\mu) = \mathrm{Hom}_{A_{\mathcal{D}}}(P(\omega), \nabla(\mu)) \simeq \mathrm{Hom}_{R(A_{\mathcal{D}})}(\mathrm{Hom}_{A_{\mathcal{D}}}(T, P(\omega)), \mathrm{Hom}_{A_{\mathcal{D}}}(T, \nabla(\mu)))$, for every $\mu \in \mathcal{D}$. Fix, for a moment, $R = K[X_1, X_2]_{(X_1, X_2)}$ and $\mathcal{D}$ to be the block $W_{\overline{\mu}}\mu + \frac{X_1}{1}\alpha_1 + \frac{X_2}{1}\alpha_2$, where α_1, α_2 are distinct simple roots (so we are excluding the case $\mathfrak{g} = \mathfrak{sl}_2$), where $\mu \in \mathfrak{h}_R^*$ is a preimage of an antidominant weight in $\mathfrak{h}_{R(\mathfrak{m})}^*$ which is not dominant without coefficients in $\mathfrak{m}$ in its unique linear combination of simple roots. Hence, we are excluding the simple blocks which are trivially Ringel self-dual. By [Cru24c, Theorem 7.3.45, Remark 7.3.47], $(A_{\mathcal{D}}, P_A(\omega))$ is a 1-$\mathcal{F}(\tilde{\Delta})$ cover of C.

Let T be a characteristic tilting module of $A_{\mathcal{D}}$. We claim that $(R(A_{\mathcal{D}}), \mathrm{Hom}_{A_{\mathcal{D}}}(T, P(\omega))$ is a 1-$\mathcal{F}(\tilde{\Delta}_{R(A_{\mathcal{D}})})$ cover of C, where $R(A_{\mathcal{D}})$ denotes the Ringel dual of $A_{\mathcal{D}}$. In the proof of [Cru24c, Theorem 7.3.45] (replacing $\mathbb{C}$ by K), we observe that, for any prime ideal $\mathfrak{p}$ of R with height at most one, $Q(R/\mathfrak{p}) \otimes_R A_{\mathcal{D}}$ is semi-simple because the weights in $Q(R/\mathfrak{p}) \otimes_R \mathcal{D}$ are both dominant and antidominant, where $Q(R/\mathfrak{p})$ is the quotient field of $R/\mathfrak{p}$. Therefore, the Ringel dual of $Q(R/\mathfrak{p}) \otimes_R A_{\mathcal{D}}$, $Q(R/\mathfrak{p}) \otimes_R R(A_{\mathcal{D}})$, according to Propositions 2.1.2 and 2.1.3, is semi-simple for any prime ideal $\mathfrak{p}$ of R with height at most one. Therefore, $Q(R/\mathfrak{p}) \otimes_R P(\omega)$-codomdim$_{Q(R/\mathfrak{p})\otimes_R A_{\mathcal{D}}}$ $Q(R/\mathfrak{p}) \otimes_R T = +\infty$ and $(Q(R/\mathfrak{p}) \otimes_R R(A_{\mathcal{D}}), Q(R/\mathfrak{p}) \otimes_R \mathrm{Hom}_{A_{\mathcal{D}}}(T, P(\omega))$ is a $+\infty-$faithful split quasi-hereditary cover of $Q(R/\mathfrak{p}) \otimes_R C$ for every $\mathfrak{p} \in \mathrm{Spec}\, R$ with height at most one. By [Cru24c, Theorems 7.3.43 and 7.3.44(f)] and Proposition 3.1.7, $P(\omega)$-codomdim$_{(A_{\mathcal{D}}, R)}$ $T = 1$. By Corollary 5.3.5, $(R(A_{\mathcal{D}}), \mathrm{Hom}_{A_{\mathcal{D}}}(T, P(\omega))$ is a 0-$\mathcal{F}(\tilde{\Delta}_{R(A_{\mathcal{D}})})$ cover of C. So, by [Cru24c, Theorem 5.1.1], the claim follows.

By [Cru24c, Corollary 4.3.6], $A_{\mathcal{D}}$ is Ringel self-dual. That is, there exists an equivalence of categories $H\colon A_{\mathcal{D}}$-mod $\to R(A_{\mathcal{D}})$-mod preserving the highest weight structure. Applying $R(\mathfrak{m}) \otimes_R -$ to H, we obtain that $A_{\mathcal{D}}(\mathfrak{m})$ is Ringel self-dual. That is, the blocks of category $\mathcal{O}$ over a field of characteristic zero are Ringel self-dual. We excluded the case $\mathfrak{g} = \mathfrak{sl}_2$. But the non-simple blocks of the category $\mathcal{O}$ associated with $\mathfrak{sl}_2$ are Morita equivalent to the Auslander algebra of $K[x]/(x^2)$ which is Ringel self-dual.

Return to the general case of R being an arbitrary local regular commutative Noetherian ring which is a $\mathbb{Q}$-algebra and $\mathcal{D}$ an arbitrary block. Since $R(\mathfrak{m})$ is a field of characteristic zero, $A_{\mathcal{D}}(\mathfrak{m})$ is Ringel self-dual. By [Cru24b, Lemma 7.4], $A_{\mathcal{D}}$ is Ringel self-dual. □

8.2.2. Schur algebras

By [Don93, Proposition 3.7], the Schur algebras $S_{\mathbb{Z}}(d, d)$ are Ringel self-dual, and therefore, the Schur algebra $S_R(d, d)$ is also Ringel self-dual for an arbitrary commutative Noetherian ring R.

The approach presented in Theorem 8.2.1 can also be applied to Schur algebras. However, we have to exclude the case of characteristic two for similar reasons why we dealt with the case $\mathfrak{sl}_2$ separately.

Theorem 8.2.2. *Assume that $n \geq d$. The Schur algebra $S_{\mathbb{Z}[\frac{1}{2}]}(n, d)$ is Ringel self-dual.*

Proof. The quotient field of $R := \mathbb{Z}[\frac{1}{2}]$ is $\mathbb{Q}$. So, for $S_{\mathbb{Z}[\frac{1}{2}]}(n, d)$, Conditions (i) and (ii) of Corollary 5.4.2 hold. By Remark 8.1.1, Condition (iii) holds since domdim$_{S_{\mathbb{Z}[\frac{1}{2}]}(n,d), R}$ $T = 2$. For each $\lambda \in \Lambda^+(n, d)$, we have $F\nabla(\lambda) \simeq e\nabla(\lambda) \simeq eD\Delta(\lambda)^\iota \simeq D(e\Delta(\lambda))^\iota \simeq \mathrm{sgn} \otimes_R e\Delta(\lambda')$ using the notation of Subsection 8.1. The last isomorphism is [CPS96, Lemma 1.6.12]. Moreover, $\mathrm{sgn} \otimes_R M \simeq M$ for any $M \in RS_d$-mod and $\mathrm{sgn} \otimes_R -\colon RS_d$-mod $\to RS_d$-mod is an isomorphism of categories. Therefore, for all $\lambda \in \Lambda^+(n, d)$, $F\nabla(\lambda) \simeq \mathrm{sgn} \otimes_R F\Delta(\lambda')$, and $\mathcal{F}(F\nabla) \simeq \mathcal{F}(F\Delta)$. So the result follows by Corollary 5.4.2. □

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