This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University, Montreal, P. Q.

AN ISOPERIMETRIC INEQUALITY FOR TETRAHEDRA

Z. A. Melzak

1. Let T be a tetrahedron and let $V(T)$ and $L(T)$ denote its volume and the sum of its edge-lengths. In this note we prove

THEOREM 1.

$$V(T)/L^3(T) \leq 6^{-4}\, 2^{-1/2}$$

with equality if and only if the tetrahedron T is regular.

An equivalent statement is: of all tetrahedra, the sum of whose edge-lengths is kept fixed, the regular one has the greatest volume. The proof is completely elementary and it seems to provide a good esercise in vector algebra.

2. Our object is to find the tetrahedra T which maximize the isoperimetric ratio $V(T)/L^3(T)$. The existence of such maximal tetrahedra is easily proved by the standard continuity-and-compactness argument. Let the vertices of T be v_1, v_2, v_3, v_4 and define the vertex vectors v_{ij} $(i, j = 1, 2, 3, 4;\ i \neq j)$ to be $v_i v_j / |v_i v_j|$. In this way one obtains twelve unit vectors lying along the edges of T and originating at the vertices.

LEMMA 1. The sum of vertex vectors originating at any vertex is orthogonal to the plane containing the face opposite that vertex.

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This is proved by a perturbation argument: a vertex is displaced so that the volume stays fixed and it is examined under what conditions the sum of the lengths of the three edges meeting at that vertex is minimized. Consider the vertex v_1, say, and let P be the plane through v_1, parallel to the face $v_2v_3v_4$. Let x be a vector in P, originating at v_1; as v_1 gets displaced through the vector x the volume of the tetrahedron remains unchanged while the sum of the lengths of the three edges meeting in P changes by

$$(v_{12} + v_{13} + v_{14}) \cdot x + 0(|x|^2) .$$

Since the direction of x in P is arbitrary, it follows from the maximality of T that the vector $v_{12} + v_{13} + v_{14}$, which is clearly $\neq 0$, must be orthogonal to P. The same argument applies at the other three vertices and so the lemma is proved.

3. Let the four outward-bound unit normal vectors to the faces of T be n_1, n_2, n_3 and n_4. The twelve vertex vectors are then the cross-products $n_i \times n_j$ (i, j = 1, 2, 3, 4; i $\neq$ j). Applying Lemma 1 at each vertex, and paying some attention to the signs, we obtain four vector equations

$$(1) \quad \begin{aligned} n_1 \times n_2 + n_2 \times n_3 + n_3 \times n_1 &= a_4 n_4 \\ n_1 \times n_4 + n_4 \times n_2 + n_2 \times n_1 &= a_3 n_3 \\ n_4 \times n_1 + n_1 \times n_3 + n_3 \times n_4 &= a_2 n_2 \\ n_2 \times n_4 + n_4 \times n_3 + n_3 \times n_2 &= a_1 n_1 \end{aligned}$$

where the a_i are some four negative scalars. Denote the vectors appearing in the left-hand sides of (1) by N_4, N_3, N_2, N_1. Making use of the standard properties of scalar, vector, and triple products, we find that

$$N_4 \cdot n_3 - N_3 \cdot n_4 = N_4 \cdot n_4 - N_3 \cdot n_3$$

so that

$$(n_3 \cdot n_4)(a_4 - a_3) = a_4 - a_3 .$$

Since $n_3 \cdot n_4 \neq 1$ it follows that $a_3 = a_4$. Symmetric arguments show then that

(2) $$a_1 = a_2 = a_3 = a_4 = a ,$$

say. We next verify that

$$N_1 + N_2 + N_3 + N_4 = 0$$

which together with (2) implies

(3) $$n_1 + n_2 + n_3 + n_4 = 0 .$$

Now, we have

$$N_1 \cdot n_2 - N_3 \cdot n_1 = a(n_1 \cdot n_2 - n_1 \cdot n_3)$$

and the left-hand side is $[n_1 + n_3, n_2, n_4]$ which vanishes by (3). Therefore $n_1 \cdot n_2 = n_1 \cdot n_3$. Similar arguments prove that all the scalar products $n_i \cdot n_j$ $(i \neq j)$ are equal. This, together with (3), leads immediately to $n_i \cdot n_j = -1/3$ $(i \neq j)$. Therefore all the dihedral angles of T are those of a regular tetrahedron, and so T itself is regular. This completes the proof of Theorem 1.

The University of British Columbia