Glasgow Math. J. **45** (2003) 493–501.
DOI: 10.1017/S001708950300140X. Printed in the United Kingdom

ULTRAPOWERS AND SUBSPACES OF THE DUAL OF A BANACH SPACE*

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(Received 13 June, 2002; accepted 19 September, 2002)

Abstract. For a Banach space X we consider three ways in which a subspace of X^* can represent locally the whole dual space X^*. We obtain characterizations in terms of ultrapowers and we study the relationship between the subspaces of X^* and the subspaces of the dual of an ultrapower of X.

2000 *Mathematics Subject Classification.* Primary: 46B08, 46B10. Secondary: 46B04, 46B20.

1. Introduction. We consider the question of representing the dual space X^* of a Banach space X by means of a closed subspace Z of X^*. This is interesting because in many cases we do not have a good representation of X^* at hand.

For a subspace Z of X^*, the following three properties (see Definition 1) represent increasing stages in the accuracy of the representation of X^* by Z:

(a) Z is a norming subspace,
(b) X^* is finitely dual representable in Z,
(c) Z is a local dual of X.

The *principle of local reflexivity* is equivalent to saying that X, as a subspace of X^{**}, is a local dual of X^*. Moreover (b) implies (a) and (c) implies (b), but the converse implications fail [**4**, **5**]. Observe that these properties not only depend on the isometric properties of Z, but also on the position of Z inside X^*.

Here we study these three properties in terms of ultrapowers. This is interesting because the *principle of local reflexivity for ultrapowers* [**7**, Theorem 7.3] can be stated by saying that the ultrapower $(X^*)_{\mathfrak{U}}$ as a subspace of $(X_{\mathfrak{U}})^*$ is a local dual of $X_{\mathfrak{U}}$.

Properties (a), (b) and (c) are local properties, in the sense that they can be defined in terms of ε-isometries that satisfy some conditions on finite dimensional subspaces (Definition 1).

We show first that the conditions mentioned in the previous paragraph can be replaced by some "approximate" conditions (Proposition 2). We apply this result to obtain characterizations of (a), (b) and (c) in terms of ultrapowers (Theorem 3). As a consequence we show that the subspace Z of X^* has one of these properties if and only

*Supported in part by D.G.I. (Spain) Proyecto BFM2001-1147.

if the ultrapower $Z_{\mathfrak{U}}$ has the corresponding property as a subspace of the dual space $(X_{\mathfrak{U}})^*$ (Theorem 6).

We apply these results to show that the canonical copy of $L_1[0, 1]$ contained in the space of Radon measures $M[0, 1] \equiv C[0, 1]^*$ is a local dual of $C[0, 1]$, and that the canonical copy of $C[0, 1]$ in $L_\infty[0, 1]$ is a local dual of $L_1[0, 1]$. These results were proved in [**5**] assuming the Continuum Hypothesis. Here we remove this condition.

Similarly, we show that (the canonical copies) $L_1(\mathbb{T})$ is a local dual of $C(\mathbb{T})$, and that $C(\mathbb{T})$ is a local dual of $L_1(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle in the complex plane. Observe that $C(\mathbb{T})$ is not isometric to $C[0, 1]$.

In the paper X and Y are Banach spaces, B_X the closed unit ball of X, S_X the unit sphere of X, and X^* the dual of X. We identify X with a subspace of X^{**}. By a *subspace* we always mean a closed subspace.

We denote by $\mathcal{B}(X, Y)$ the space of all (bounded linear) operators from X into Y. Given $T \in \mathcal{B}(X, Y)$, T^* is the conjugate operator.

An operator $T \in \mathcal{B}(X, Y)$ is an *ε-isometry* $(0 < \varepsilon < 1)$ if it satisfies $(1+\varepsilon)^{-1} < \|Tx\| < 1 + \varepsilon$ for all $x \in S_X$. A Banach space X is said to be *finitely representable in* Y if for each $\varepsilon > 0$ and each finite dimensional subspace M of X there is an ε-isometry $T: M \longrightarrow Y$.

We denote by $\mathbb{N}$ the set of all positive integers. An ultrafilter $\mathfrak{U}$ on a set I is *countably incomplete* if there is a countable partition $\{I_n : n \in \mathbb{N}\}$ of I such that $I_n \notin \mathfrak{U}$ for every $n \in \mathbb{N}$. We always assume that the ultrafilters are countably incomplete. Observe that every infinite set admits a countably incomplete ultrafilter [**7**].

Given an ultrafilter $\mathfrak{U}$ on a set I and a family $(X_i)_{i \in I}$ of Banach spaces, $\ell_\infty(I, X_i)$ is the Banach space of all bounded families $(x_i)_{i \in I}$ such that $x_i \in X_i$ for every $i \in I$ endowed with the supremum norm $\|(x_i)\| := \sup\{\|x_i\| : i \in I\}$. We denote by $N_{\mathfrak{U}}(I, X_i)$ the closed subspace of all families $(x_i) \in \ell_\infty(I, X_i)$ which converge to 0 following $\mathfrak{U}$. The *ultraproduct of* $(X_i)_{i \in I}$ *following* $\mathfrak{U}$ is defined as the quotient:

$$(X_i)_{\mathfrak{U}} := \frac{\ell_\infty(I, X_i)}{N_{\mathfrak{U}}(I, X_i)}.$$

The element of $(X_i)_{\mathfrak{U}}$ including as a representative the family $(x_i) \in \ell_\infty(I, X_i)$ is denoted by $[x_i]$, and its norm in $X_{\mathfrak{U}}$ is given by $\|[x_i]\| = \lim_{\mathfrak{U}} \|x_i\|$. Given a uniformly bounded family of operators $(T_i) \in \ell_\infty(I, \mathcal{B}(X_i, Y_i))$, its ultraproduct is the operator $[T_i]: (X_i)_{\mathfrak{U}} \longrightarrow (Y_i)_{\mathfrak{U}}$ defined by $[T_i]([x_i]) = [Tx_i]$. If $X_i = X$ for all $i \in I$ then we speak of the ultrapower of X, denoted $X_{\mathfrak{U}}$. The ultrapower $X_{\mathfrak{U}}$ contains an isometric copy of X generated by the constant families of $\ell_\infty(I, X)$. We identify this copy with X. The ultrapower $(X^*)_{\mathfrak{U}}$ is contained in $(X_{\mathfrak{U}})^*$, but in general they do not coincide. Actually, $[f_i] \in (X^*)_{\mathfrak{U}}$ is identified with the functional $f \in (X_{\mathfrak{U}})^*$ defined by $f([x_i]) := \lim_{\mathfrak{U}} f_i(x_i)$; $(X^*)_{\mathfrak{U}} = (X_{\mathfrak{U}})^*$ if and only if X is superreflexive [**7**]. Note that for every $x \in X$ and every $f \in X^*$, the duality action as elements of $X_{\mathfrak{U}}$ and $(X_{\mathfrak{U}})^*$ is preserved, namely, $\langle f, x\rangle = \langle [f], [x]\rangle$. We refer to [**7**] for additional information about ultrapowers.

2. Main results. Let F and Z be subspaces of X^*, and let G be a subspace of X. For an operator $L: F \longrightarrow Z$ we shall consider the following *exact conditions*:

(I) $\langle Lf, x\rangle = \langle f, x\rangle$ for all $x \in G$ and all $f \in F$.

(II) $L(f) = f$ for all $f \in F \cap Z$.

https://doi.org/10.1017/S001708950300140X Published online by Cambridge University Press

We shall say that $L: F \longrightarrow Z$ *satisfies* (I) *or* (II) *with respect to* G.

Our aim is to study the concepts introduced below in terms of ultrapowers.

DEFINITION 1. Let Z be a subspace of X^*.

(a) We say that Z is *norming* if

$$\|x\| = \sup\{|\langle f, x\rangle| : f \in B_Z\}, \text{ for every } x \in X.$$

(b) We say that X^* is *finitely dual representable in* Z if for every couple of finite dimensional subspaces F of X^* and G of X, and for every $0 < \varepsilon < 1$, there is an ε-isometry $L: F \longrightarrow Z$ that satisfies (I) with respect to G.

(c) We say that Z is a *local dual of* X if for every couple of finite dimensional subspaces F of X^* and G of X, and every $\varepsilon > 0$, there is an ε-isometry $L: F \longrightarrow Z$ that satisfies (I) and (II) with respect to G.

The notions defined above were introduced in [**4**, Definition 1] and [**5**, Definition 2.1].

Clearly, if Z is a local dual of X, then X^* is finitely dual representable in Z, and therefore Z is norming. However, the converse implications do not hold [**4**, **5**].

Let F and Z be subspaces of X^*, let G be a subspace of X and let $0 < \varepsilon < 1$. For an ε-isometry $L: F \longrightarrow Z$ we shall consider the following *approximate conditions:*

(I′) $|\langle Lf, x\rangle - \langle f, x\rangle| \leq \varepsilon\|f\|\,\|x\|$ for all $x \in G$ and all $f \in F$.

(II′) $\|L(f) - f\| \leq \varepsilon\|f\|$ for all $f \in F \cap Z$.

We shall say that $L: F \longrightarrow Z$ *satisfies* (I’) *or* (II’) *with respect to* G.

PROPOSITION 2. *Let Z be a subspace of X^*.*

(a) *Z is norming if and only if for every finite dimensional subspace G of X and for every $\varepsilon > 0$, there exist a normalized basis $\{x_1, \ldots, x_n\}$ of G and functionals $\{g_1, \ldots, g_n\}$ in Z such that $\|g_i\| \leq 1 + \varepsilon$ and $\langle g_i, x_j\rangle = \delta_{ij}$ for $i, j = 1, \ldots, n$.*

(b) *X^* is finitely dual representable in Z if and only if for every couple of finite dimensional subspaces F of X^* and G of X, and every $0 < \varepsilon < 1$, there is an ε-isometry $L: F \longrightarrow Z$ that satisfies* (I’) *with respect to G.*

(c) *Z is a local dual of X if and only if for every couple of finite dimensional subspaces F of X^* and G of X, and every $0 < \varepsilon < 1$, there is an ε-isometry $L: F \longrightarrow Z$ that satisfies* (I’) *and* (II’) *with respect to G.*

Proof. (a) Let us assume that Z is norming. By Auerbach’s Lemma [**10**, Proposition 1.c.3], there are normalized vectors $\{x_1, \ldots, x_n\}$ in G and $\{f_1, \ldots, f_n\}$ in G^* such that $\langle f_i, x_j\rangle = \delta_{ij}$ for $i, j = 1, \ldots, n$.

Let $T: G \longrightarrow X$ denote the embedding map and select $\{h_1, \ldots, h_n\}$ in $(1 + \varepsilon/2)B_{X^*}$ such that $T^*h_i = f_i$ for $i = 1, \ldots, n$.

Since Z is norming, B_Z is weak*-dense in B_{X^*} [**3**]. Therefore, as $G^* \equiv X^*/G^\perp$ is finite dimensional, $T^*(B_Z)$ is norm-dense in B_{G^*}. Thus $(1 + \varepsilon/2)T^*(B_{X^*})$ is contained in $(1 + \varepsilon)T^*(B_Z)$, and it is enough to choose $\{g_1, \ldots, g_n\}$ in $(1 + \varepsilon)B_Z$ such that $T^*g_i = T^*h_i$ for $i = 1, \ldots, n$.

The converse implication is trivial.

(b) For the non-trivial implication, we fix finite dimensional subspaces F of X^* and G of X. Observe that the existence of L_ε for every $\varepsilon > 0$ implies that Z is norming.

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Thus, by Proposition 2 (a), we can choose a normalized basis $\{x_1, \dots, x_n\}$ of G and functionals $\{g_1, \dots, g_n\}$ in Z such that $\|g_i\| < 2$ and $\langle g_i, x_j\rangle = \delta_{ij}$ for $i, j = 1, \dots, n$.

Let $0 < \varepsilon < 1$ and let $\delta := \varepsilon/(8n)$. We define $L: F \longrightarrow Z$ by

$$Lf := L_\delta f + \sum_{i=1}^{n} \langle f - L_\delta f, x_i\rangle g_i \text{ for } f \in F.$$

Clearly $\langle Lf, x\rangle = \langle f, x\rangle$, for all $f \in F$ and all $x \in G$. Moreover, for every $f \in F$,

$$\left\| \sum_{i=1}^{n} \langle f - L_\delta f, x_i\rangle g_i \right\| \leq 2n\delta\|f\|.$$

Therefore, L is a ε-isometry that satisfies (I) with respect to G.

(c) is Theorem 2.5 in **[5]**. □

Part (c) in Proposition 2 was useful in **[4]** and **[5]** to find examples of local dual spaces of some Banach spaces.

Let Z be a subspace of X^* and let $\mathfrak{U}$ be an ultrafilter on a set Λ. Since B_{X^*} is weak* compact, it follows that every bounded family $(f_\alpha)_{\alpha\in\Lambda} \subset X^*$ is weak* converging following $\mathfrak{U}$. This fact allows us to introduce the natural operator $\mathcal{Q} : Z_{\mathfrak{U}} \to X^*$ defined by

$$\mathcal{Q}([z_\alpha]) := w^* - \lim_{\alpha\to\Lambda} z_i.$$

The operator $\mathcal{Q}$ plays a central role in the following theorem, which is our main result.

THEOREM 3. *Let Z be a subspace of X^*.*

(a) *Z is norming if and only if there is an ultrafilter $\mathfrak{U}$ such that $\mathcal{Q}$ maps the closed unit ball of $Z_{\mathfrak{U}}$ onto B_{X^*}.*

(b) *X^* is finitely dual representable in Z if and only if there is an ultrafilter $\mathfrak{U}$ and an isometry $T \in \mathcal{B}(X^*, Z_{\mathfrak{U}})$ such that $\mathcal{Q}T = I_{X^*}$.*

(c) *Z is a local dual of X if and only if there is an ultrafilter $\mathfrak{U}$ and an isometry $T \in \mathcal{B}(X^*, Z_{\mathfrak{U}})$ such that $\mathcal{Q}T = I_{X^*}$ and $T|_Z$ is the natural embedding.*

Proof. In all the direct implications, $\mathfrak{U}$ will be an ultrafilter on the set of all pairs $\alpha = (E_\alpha, F_\alpha)$ of finite dimensional subspaces of X and X^* refining the order filter.

(a) Assume Z is norming, namely, B_Z is w^*-dense in B_{X^*}. Let $f \in B_{X^*}$. For every index α, we take a basis $\{x_1, \dots, x_n\}$ of E_α and the w^*-neighborhood of f given by $\mathcal{V}_\alpha := \{g \in X^* : |\langle g - f, x_i\rangle| < n^{-1},\ i = 1, \dots, n\}$. Take $g_\alpha \in \mathcal{V}_\alpha \cap B_Z$. Note that for every w^*-neighborhood $\mathcal{V}$ of f, there is an index α verifying $\mathcal{V}_\alpha \subset \mathcal{V}$, so $\mathcal{Q}([g_\alpha]) = f$. For the converse, take a norm one element $x \in X$ and choose a norm one functional $f \in X^*$ such that $1 = \langle f, x\rangle$. Following the hypothesis, there is a family $(g_i)_{i\in I} \subset B_Z$ such that $\mathcal{Q}([g_i]) = f$. Hence $1 = \lim_{i\to\mathfrak{U}}\langle g_i, x\rangle$, so Z is norming.

(b) Suppose that X^* is finitely dual representable in Z. For every $\alpha = (E_\alpha, F_\alpha)$, we write $n_\alpha = \dim E_\alpha + \dim F_\alpha$. Then there exists a n_α^{-1}-isometry $T_\alpha : F_\alpha \longrightarrow Z$ that satisfies (I) with respect to E_α.

We define $T \in \mathcal{B}(X^*, Z_{\mathfrak{U}})$ by $Tf := [(Tf)_\alpha]$, where

$$(Tf)_\alpha = \begin{cases} T_\alpha f, & \text{if } f \in F_\alpha; \\ 0, & \text{otherwise.} \end{cases}$$

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Clearly T is an isometry. Moreover, for every $x \in X$ and $f \in X^*$, we have

$$\langle QTf, x\rangle = \lim_{\alpha \longrightarrow \mathfrak{U}} \langle (Tf)_\alpha, x\rangle = \langle f, x\rangle.$$

Hence $QT = I_{X^*}$.

Conversely, assume that there exist an ultrafilter $\mathfrak{U}$ on a set Λ and an isometry $T: X^* \longrightarrow Z_{\mathfrak{U}}$ verifying $QT = I_{X^*}$. By part (a), Z is norming.

Consider a pair of finite dimensional subspaces F of X^* and G of X. Let $0 < \varepsilon < 1$. Since $T|_F$ has finite rank, by [**7**, Lemma 7.3] there is a uniformly bounded family of operators $(L_\alpha)_{\alpha \in \Lambda}$ in $\mathcal{B}(T(F), Z)$ such that $T = [L_\alpha T]$, and moreover, there exists $\Lambda_1 \in \mathfrak{U}$ so that $T_\alpha := L_\alpha T$ is an ε-isometry for all $\alpha \in \Lambda_1$.

By Proposition 2 (a), there exist a normalized basis $\{x_1, \dots, x_m\}$ of G and functionals $\{g_1, \dots, g_m\}$ in Z such that $\|g_i\| < \sqrt{2}$ and $\langle g_i, x_j\rangle = \delta_{ij}$ for $i, j = 1, \dots, m$. Moreover, there exist a normalized basis $\{h_1, \dots, h_n\}$ of F and vectors $\{y_1, \dots, y_n\}$ in X such that $\|y_i\| < \sqrt{2}$ and $\langle h_i, y_j\rangle = \delta_{ij}$ for $i, j = 1, \dots, n$.

Since $QT = I_{X^*}$, we can select $\alpha \in \Lambda_1$ so that

$$|\langle T_\alpha h_i - h_i, x_j\rangle| < \frac{\varepsilon}{2mn}, \quad \text{for all } i \text{ and } j.$$

Thus, for every $h \in F$ and every $x \in G$, as $h = \sum_{i=1}^n \langle h, y_i\rangle h_i$ and $x = \sum_{j=1}^m \langle g_j, x\rangle x_j$, we get $|\langle T_\alpha h - h, x\rangle| \le \varepsilon \|h\| \|x\|$, and applying Proposition 2 (b), the proof is done.

(c) For the direct implication, for every index α we choose a n_α^{-1}-isometry T_α verifying (I) and (II) with respect to E_α. Proceeding as in (b), we get $QT = I_{X^*}$. Moreover, since $T_\alpha g = g$ for all $g \in Z \cap F_\alpha$, we get $Tg = [g]$.

For the converse implication, we proceed as in part (b), but instead of choosing $\alpha \in \Lambda_1$, we choose $\alpha \in \Lambda_1 \cap \Lambda_2$, where $\Lambda_2 := \{\alpha \in \Lambda : \|T_\alpha h_i - h_i\| < \sqrt{2}^{-1} n^{-1}\varepsilon\}$; notice that since $T|_Z$ is the natural embedding of Z into $Z_{\mathfrak{U}}$, then $\Lambda_2 \in \mathfrak{U}$. Therefore, the ε-isometry T_α verifies (I′) and (II′) with respect to G. Proposition 2 (c) shows that Z is a local dual of X. □

REMARK 4. In Theorem 3, TQ is a norm-one projection on $Z_{\mathfrak{U}}$ with range isometric to X^*.

REMARK 5. It has been already noted in [**7**] and [**1**, 8.17 Theorem] that the classical principle of local reflexivity implies the existence of an ultrafilter $\mathfrak{U}$ and an isometry $J: X^{**} \longrightarrow X_{\mathfrak{U}}$ such that JQ is a norm-one projection whose range is isometric to X^{**} and $JQ|_X$ is the canonical embedding of X into $X_{\mathfrak{U}}$. Theorem 3 and Remark 4 shed light upon the projection JQ: its existence is equivalent to the classical principle of local reflexivity.

The previous characterizations are the key to prove the converse implications in the following result.

THEOREM 6. *Let Z be a subspace of X^* and let $\mathfrak{U}$ be an ultrafilter on a set I.*

(a) *Z is norming if and only if $Z_{\mathfrak{U}}$ is norming.*

(b) *X^* is finitely dual representable in Z if and only if $(X_{\mathfrak{U}})^*$ is finitely dual representable in $Z_{\mathfrak{U}}$.*

(c) *Z is a local dual of X if and only if $Z_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$.*

Proof. (a) Assume that Z is a norming subspace of X^*. Let $[x_i] \in X_{\mathfrak{U}}$ and $\varepsilon > 0$. For each $i \in I$ there is a norm one element $f_i \in Z$ such that $\langle f_i, x_i\rangle \ge \|x_i\| - \varepsilon$, so

$\langle [f_i], [x_i] \rangle \geq \|[x_i]\| - \varepsilon$, which proves that $Z_{\mathfrak{U}}$ is norming. For the converse, assume that $Z_{\mathfrak{U}}$ is a norming subspace of $(X_{\mathfrak{U}})^*$. Given $x \in X$, there is $[f_i] \in Z_{\mathfrak{U}}$ such that $\|f_i\| = 1$ for all i and $\|x\| = \langle [f_i], [x] \rangle$. Thus $\|x\| = \lim_{i \to \mathfrak{U}} \langle f_i, x \rangle$, hence Z is a norming subspace of X^*.

(b) The proof is essentially contained in the proof of (c).

(c) Assume that Z is a local dual of X and fix a couple of finite dimensional subspaces F of $X^*_{\mathfrak{U}}$ and G of $X_{\mathfrak{U}}$ and $0 < \varepsilon < 1$. Since $X^*_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$, there is an $\varepsilon/4$-isometry $L_1 \colon F \longrightarrow (X^*)_{\mathfrak{U}}$ satisfying (I) and (II) with respect to G.

We fix a basis $\{[f_i^1], \ldots, [f_i^m]\}$ of $L_1(F)$ and a basis $\{[x_i^1], \ldots, [x_i^n]\}$ of G. For every $i \in I$, we write $F^i := \operatorname{span}\{f_i^1 \ldots f_i^n\} \subset X^*$ and $G^i := \operatorname{span}\{x_i^1 \ldots x_i^n\} \subset X$.

Now, for every $i \in I$ we select an $\varepsilon/4$-isometry $L^i \colon F^i \longrightarrow Z$ satisfying (I) and (II) with respect to G^i, and define an $\varepsilon/4$-isometry $L_2 \colon L_1(F) \longrightarrow Z_{\mathfrak{U}}$ by $L_2[f_i] := [L^i f_i]$.

Clearly $L := L_2 L_1 \colon F \longrightarrow Z_{\mathfrak{U}}$ is an ε-isometry satisfying (I) and (II) with respect to G. Thus $Z_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$.

Conversely, assume that $Z_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$. Let $F \subset X^*$ and $G \subset X$ be finite dimensional subspaces and $0 < \varepsilon < 1/2$. Take $0 < \varepsilon' < 2^{-3}\varepsilon$ and choose an ε'-net $\{f_j\}_{j=1}^n$ in B_F and an ε'-net $\{x_j\}_{j=1}^n$ in B_G. Note that $\{f_j\}_{j=1}^n$ includes a basis of F. Since we can consider F and G as subspaces of $(X^*)_{\mathfrak{U}}$ and $X_{\mathfrak{U}}$, there is an ε'-isometry $L \colon F \longrightarrow Z_{\mathfrak{U}}$ verifying (I) and (II) with respect to G.

Since L has finite rank, by [**7**, Lemma 7.3] there exist a set $\Lambda_1 \in \mathfrak{U}$ and a bounded family of uniformly bounded operators $(L_i)_{i \in I}$ in $\mathcal{B}(F, Z)$ such that $L = [L_i]$ and such that, for each $i \in \Lambda_1$, L_i is an ε-isometry. Moreover, there exists $\Lambda_2 \in \mathfrak{U}$ such that, for every $i \in \Lambda_2$,

$$|\langle L_i f_j, x_k \rangle - \langle f_j, x_k \rangle| \leq \varepsilon' \|f_j\| \, \|x_k\| \text{ for all } j, k = 1, \ldots, n, \text{ and}$$
$$\|L_i(f_j) - f_j\| \leq \varepsilon' \|f_j\| \text{ for all } j = 1, \ldots, n.$$

We fix $i \in \Lambda_1 \cap \Lambda_2$ and denote $T = L_i$. Thus T is a ε-isometry. Let us show that T satisfies (I$'$) and (II$'$) for G.

First, we take $f \in S_F$ and $x \in S_G$. We pick f_k and x_l so that $\|f - f_k\| < \varepsilon'$ and $\|x - x_l\| < \varepsilon'$. Thus, as

$$\langle (I - T)f,\ x \rangle = \langle (I - T)f,\ x - x_l \rangle + \langle (I - T)(f - f_k),\ x_l \rangle + \langle (I - T)f_k,\ x_l \rangle$$

we obtain $|\langle (I - T)f, x \rangle| \leq 2(2 + \varepsilon')\varepsilon' + \varepsilon' < \varepsilon$. Second, we take $f \in S_F \cap Z$, and pick f_k so that $\|f - f_k\| < \varepsilon'$. Therefore

$$\|Tf - f\| \leq \|Tf - Tf_k\| + \|Tf_k - f_k\| + \|f_k - f\| < 2\varepsilon' + \varepsilon' + \varepsilon' < \varepsilon.$$

Hence T is a ε-isometry that satisfies (I$'$) and (II$'$) with respect to G, and the proof is complete. $\square$

REMARK 7. Note that part (c) in Theorem 6 implies that $(X^*)_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$, which constitutes the principle of local reflexivity for ultrapowers [**7**].

In the following we are going to prove that there is local duality between $C[0, 1]$ and $L_1[0, 1]$. The same result was already given by the authors in [**5**, Proposition 2.8], but the proof offered here needs not the Continuum Hypothesis, which is an important improvement with respect to the proof in [**5**]. In order to show that, for every $n \in \mathbb{N}$, we consider the intervals

$$I_i^n = \begin{cases} \left[\dfrac{i-1}{2^n}, \dfrac{i}{2^n}\right) & \text{if } i = 1, \dots 2^n - 1, \\[2ex] \left[\dfrac{2^n-1}{2^n}, 1\right] & \text{if } i = 2^n. \end{cases}$$

Let us denote by χ_i^n the characteristic functions of the intervals I_i^n. Recall that $C[0, 1]^*$ can be identified with the space $\mathcal{M}[0, 1]$ of all Borel measures on $[0, 1]$. We identify $L_1[0, 1]$ with a subspace of $\mathcal{M}[0, 1]$ in the usual way.

For every $n \in \mathbb{N}$, we define operators $G_n \colon \mathcal{M}[0, 1] \longrightarrow L_1[0, 1]$ by

$$G_n\lambda := \sum_{i=1}^{2^n} 2^n \lambda\left(I_i^n\right)\chi_i^n.$$

LEMMA 8. (a) *the sequence* $(G_n f)$ *converges in norm to* f, *for every* $f \in L_1[0, 1]$.

(b) $(G_n\lambda)$ *converges to* λ *in the weak***-topology, for every* $\lambda \in \mathcal{M}[0, 1]$.

Proof. (a) is well known [**2**].

(b) Let $\lambda \in \mathcal{M}[0, 1]$ be a positive measure and $f \in C[0, 1]$. We denote $M_i^n := \sup\{f(t) : t \in I_i^n\}$, $m_i^n := \inf\{f(t) : t \in I_i^n\}$ and

$$\rho_n(f) := \max\left\{M_i^n - m_i^n : i = 1, \dots, 2^n\right\}.$$

It is enough to prove that $|\langle f, \lambda - G_n\lambda\rangle| \le \rho_n(f)\|\lambda\|$.

We consider the Borel-measurable functions M^n and m^n on $[0, 1]$, defined by

$$M^n(t) := \sum_{i=1}^{2^n} M_i^n \chi_i^n(t) \text{ and } m^n(t) := \sum_{i=1}^{2^n} m_i^n \chi_i^n(t).$$

Clearly,

$$\langle m^n, \lambda\rangle \le \langle f, \lambda\rangle \le \langle M^n, \lambda\rangle$$

and

$$\langle m^n, G_n\lambda\rangle \le \langle f, G_n\lambda\rangle \le \langle M^n, G_n\lambda\rangle.$$

Since $\langle m^n, \lambda\rangle = \langle m^n, G_n\lambda\rangle$ and $\langle M^n, \lambda\rangle = \langle M^n, G_n\lambda\rangle$, we get

$$|\langle f, \lambda - G_n\lambda\rangle| \le \langle M^n - m^n, \lambda\rangle = \sum_{i=1}^{2^n}\left(M_i^n - m_i^n\right)\lambda\left(I_i^n\right) \le \rho_n(f)\|\lambda\|.$$

□

Let Z be a local dual of X, and let $J \colon X \longrightarrow X^{**}$ the canonical embedding of X into X^{**}. We denote by $\Upsilon \colon X \longrightarrow Z^*$ the isometry that maps x onto $J(x)|_Z$. It has been proved in [**5**, Proposition 2.10] that $\Upsilon(X)$ is a local dual of Z.

THEOREM 9. (a) $L_1[0, 1]$ *is a local dual of* $C[0, 1]$.

(b) $C[0, 1]$ *is a local dual of* $L_1[0, 1]$.

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Proof. (a) Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{N}$ and define $T: \mathcal{M}[0,1] \longrightarrow L_1[0,1]_{\mathfrak{U}}$ by $T\lambda := [G_n\lambda]$. Lemma 8 fulfills the hypotheses of part (c) in Theorem 3, which proves that $L_1[0,1]$ is a local dual of $C[0,1]$.

(b) Note that $\Upsilon(C[0,1])$ is the canonical copy of $C[0,1]$ contained in $L_1[0,1]^*$, so statement (a) and [**5**, Proposition 2.10] proves (b). $\square$

We denote the torus $\{e^{it} : t \in [0, 2\pi]\}$ by $\mathbb{T}$. Note that the following Theorem cannot be obtained from Theorem 9 because $C[0,1]$ is not isometrically isomorphic to $C(\mathbb{T})$.

THEOREM 10. (a) *$L_1(\mathbb{T})$ is a local dual of $C(\mathbb{T})$;*
(b) *$C(\mathbb{T})$ is a local dual of $L_1(\mathbb{T})$.*

Proof. (b). Let $\{\psi_n\}_{n=1}^{\infty}$ a positive summability kernel on $\mathbb{T}$ [**9**, Definition 2.2] such that $\psi_n(e^{it}) = \psi_n(e^{-it})$ for all $t \in [0, 2\pi]$ and all $n \in \mathbb{N}$. The following facts are well known:

(i) $\|\psi_n * f - f\|_1 \xrightarrow{n} 0$ for all $f \in L_1(\mathbb{T})$,
(ii) $\|\psi_n * g - g\|_\infty \xrightarrow{n} 0$ for all $g \in C(\mathbb{T})$,
(iii) $\psi_n * g \in C(\mathbb{T})$ for all $g \in L_\infty(\mathbb{T})$;

moreover, statement (i), the symmetry of each ψ_n and Fubini's theorem yield:

(iv) $\langle \psi_n * g, f\rangle = \langle g, \psi_n * f\rangle \xrightarrow{n} \langle g, f\rangle$ for all $g \in L_\infty(\mathbb{T})$ and all $f \in L_1(\mathbb{T})$.

Thus, for every $n \in \mathbb{N}$, statement (iv) allows us to define the operator $T_n: L_\infty(\mathbb{T}) \longrightarrow C(\mathbb{T})$ by $f_n(g) = \psi_n * g$. Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{N}$ and define the operators $T: L_\infty(\mathbb{T}) \longrightarrow C(\mathbb{T})_{\mathfrak{U}}$ by $T(g) = [T_n(g)]$, and $Q: C(\mathbb{T})_{\mathfrak{U}} \longrightarrow L_\infty(\mathbb{T})$ by $Q([g_n]) = w^*\text{-}\lim_{n\to\mathfrak{U}} g_n$. On the one hand, statement (ii) yields

$$\|T(g) - g\| = \lim_{n\to\mathfrak{U}} \|T_n(g) - g\|_\infty = 0 \text{ for all } g \in C(\mathbb{T}).$$

So the restriction of T to $C(\mathbb{T})$ is the natural embedding of $C(\mathbb{T})$ into $C(\mathbb{T})_{\mathfrak{U}}$. On the other hand, (iv) leads to

$$\langle T(g), [f]\rangle = \lim_{n\to\mathfrak{U}} \langle T_n(g), f\rangle = \langle g, f\rangle, \quad \text{for all } g \in L_\infty(\mathbb{T}) \text{ and all } f \in L_1(\mathbb{T}),$$

which shows that QT is the identity on $L_\infty(\mathbb{T})$. Hence, the hypotheses of Theorem 3 (c) are fulfilled, so $C(\mathbb{T})$ is a local dual of $L_1(\mathbb{T})$.

(a) The proof is analogous to that of part (b) in Theorem 9, by using [**5**, Proposition 2.10]. $\square$

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