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LOCAL C^r-RIGHT EQUIVALENCE OF C^{r+1} FUNCTIONS

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Abstract. Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be C^{r+1} functions, $r \in \mathbb{N}$. We will show that if $\nabla f(0) = 0$ and there exist a neighbourhood U of $0 \in \mathbb{R}^n$ and a constant $C > 0$ such that

$$\left|\partial^m (g - f)(x)\right| \leq C\,|\nabla f(x)|^{r+2-|m|} \qquad \text{for } x \in U,$$

and for any $m \in \mathbb{N}_0^n$ such that $|m| \leq r$, then there exists a C^r diffeomorphism $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \varphi$ in a neighbourhood of $0 \in \mathbb{R}^n$.

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1. Introduction. Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. We say that f and g are *C^r-right equivalent* if there exists a C^r diffeomorphism $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \varphi$ in a neighbourhood of $0 \in \mathbb{R}^n$. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A norm in $\mathbb{R}^n$ we denote by $|\cdot|$ and by $\operatorname{dist}(x, V)$ – the distance of a point $x \in \mathbb{R}^n$ to a set $V \subset \mathbb{R}^n$. By $C^k(n)$, where $k, n \in \mathbb{N}$, we denote the set of C^k functions $(\mathbb{R}^n, 0) \to \mathbb{R}$. Let $\mathcal{J}_f C^k(n)$ be the ideal in $C^k(n)$ generated by $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$. The ideal $\mathcal{J}_f C^k(n)$ is called *the Jacobi ideal in $C^k(n)$* (we will call it in short *the Jacobi ideal*).

In this paper, we address the question under what conditions C^r-right equivalence of C^{r+1} functions holds. There exists result which deals with C^r-right equivalence of C^{r+2}, namely J. Bochnak has used Tougeron's Implicit Theorem to prove the following theorem [**1**, Theorem 1]
Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be C^{r+2} functions such that $\nabla f(0) = 0$, $r \in \mathbb{N}$. If $(g - f) \in \mathfrak{m}(\mathcal{J}_f C^{r+1}(n))^2$ then f and g are C^r-right equivalent. By $\mathfrak{m}$ we mean maximal ideal in the set of C^{r+1} functions $(\mathbb{R}^n, 0) \to \mathbb{R}$.

Results presented in this paper are proven in the classical spirit of Kuiper–Kuo Theorem which deals with C^0-right equivalence of C^r functions with isolated singularity at $0 \in \mathbb{R}^n$ ([**2**, **3**] see also [**7**]). Moreover, in compare to Bochnak Theorem, we assume geometric condition for $(g - f)$ instead algebraic condition. More precisely, we will prove the following

THEOREM 1. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be C^{r+1} functions, $r \in \mathbb{N}$. If $\nabla f(0) = 0$ and there exist a neighbourhood U of $\in \mathbb{R}^n$ and a constant $C > 0$ such that*

$$\left|\partial^m (g - f)(x)\right| \leq C\,|\nabla f(x)|^{r+2-|m|} \qquad \textit{for } x \in U, \tag{1}$$

and for any $m \in \mathbb{N}_0^n$ such that $|m| \leq r$, then f and g are C^r-right equivalent.

After slight modification of the above theorem, we will obtain some sufficient condition for C^0-right equivalence (Theorem 2). Moreover, we will see that Theorem 1 implies Theorem 3, where we assume that $(g-f)$ belongs to Jacobi ideal of f to some power depends on r. It is worth mention about author's result **[6]**, where it has proved that if two analytic functions f, g are such that $(g-f) \in (f)^{r+2}$ then f and g are C^r-right equivalent, (f) denote ideal generated by f. In this paper, we will also prove that if two real analytic functions are C^1-right equivalent then they have the same exponent in the Łojasiewicz gradient inequality (Proposition 1).

2. Auxiliary results. Let us start this section from some obvious lemma. Let $M, m, k, r \in \mathbb{N}$, $k \geq r$ and $M > r$. Moreover, let $p, q_1, \ldots, q_m \in C^k(n)$ and let $\mathcal{Q}C^k(n)$ denote the ideal in $C^k(n)$ generated by $q_1, \ldots, q_m$.

LEMMA 1. *If $p \in (\mathcal{Q}C^k(n))^M$ then*

(*i*) $\frac{\partial^r p}{\partial x_{i_1} \ldots \partial x_{i_r}} \in (\mathcal{Q}C^{k-r}(n))^{M-r}$ *for* $i_1, \ldots, i_r \in \{1, \ldots, n\}$,

(*ii*) $|p(x)| \leq C|(q_1(x), \ldots, q_n(x))|^M$ *in a neighbourhood of* $0 \in \mathbb{R}^n$ *and for some positive constant* C.

From the above, we obtain at once

COROLLARY 1. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be C^{r+1} functions, $r \in \mathbb{N}$. If $(g-f) \in (\mathcal{J}_f C^r(n))^{r+2}$, then there exist a neighbourhood U of $0 \in \mathbb{R}^n$ and a constant $C > 0$ such that*

$$\left|\partial^m (g-f)(x)\right| \leq C\,|\nabla f(x)|^{r+2-|m|} \quad \text{for } x \in U,$$

and for any $m \in \mathbb{N}_0^n$ such that $|m| \leq r$.

The next two lemmas come from **[6]** (respectively Lemmas 2 and 3).

LEMMA 2. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a locally lipschitzian. Then, there exist a neighbourhood U of $0 \in \mathbb{R}^n$ and a constant $C > 0$ such that for any $x \in U$, $|f(x)| \leq C\,\mathrm{dist}(x, V_f)$ (V_f denote zero set of f).*

LEMMA 3. *Let $\xi, \eta : U \to \mathbb{R}$ be $C^{|k|}$ functions, where $U \subset \mathbb{R}^n$ is a neighbourhood of $0 \in \mathbb{R}^n$ and $k \in \mathbb{N}_0^n$. Assume that there exist constants $A_1, A_2, A_3 > 0$ such that*

$$A_1|\eta(x)|^2 \leq |\xi(x)| \leq A_2|\eta(x)|^2, \quad |\partial\xi(x)| \leq A_3|\eta(x)| \quad \text{for } x \in U.$$

Then, there exist a neighbourhood $U_1 \subseteq U$ and a constant $B > 0$ such that

$$\left|\partial^k \left(\frac{1}{\xi(x)}\right)\right| \leq B|\eta(x)|^{-|k|-2} \quad \text{for } x \in U_1.$$

The last lemma in this section is slight modification of [**7**, Lemma 1].

LEMMA 4. *Let $G \subset \mathbb{R} \times \mathbb{R}^n$ be an open set, $W : G \to \mathbb{R}^n$ be a continuous mapping and let $V \subset \mathbb{R}^n$ be a closed set. If a system*

$$\frac{dy}{dt} = W(t, y), \tag{2}$$

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has a global uniqueness of solutions property in $G \backslash (\mathbb{R} \times V)$ and if

$$|W(t,x)| \le C \operatorname{dist}(x, V) \quad \textit{for } (t,x) \in U,$$

for some constant $C > 0$ and some neighbourhood $U \subset G$ of set $(\mathbb{R} \times V) \cap G$, then (2) *has a global uniqueness of solutions property in G.*

3. Proof of Theorem 1. Let Z be the zero set of ∇f and let $U \subset \mathbb{R}^n$ be a neighbourhood of $0 \in \mathbb{R}^n$ such that f and g are defined. By Lemma 2 there exists a positive constant A such that

$$|\nabla f(x)| \le A \operatorname{dist}(x, Z) \quad \text{for } x \in U. \tag{3}$$

Define the function $F : \mathbb{R} \times U \to \mathbb{R}$ by the formula

$$F(\xi, x) = f(x) + \xi(g - f)(x),$$

obviously

$$\nabla F(\xi, x) = ((g - f)(x), \nabla f(x) + \xi \nabla (g - f)(x)) .$$

Let $G = \{(\xi, x) \in \mathbb{R} \times U : |\xi| < \delta\}$ where $\delta \in \mathbb{R}$, $\delta > 2$. From the above, diminishing U if necessary, we have that there exists a constant $C_1 > 0$ such that

$$|\nabla f(x)| \le C_1 |\nabla F(\xi, x)| \quad \text{for } (\xi, x) \in G. \tag{4}$$

Indeed,

$$|\nabla F(\xi, x)| \ge |\nabla f(x) + \xi \nabla (g - f)(x)| \ge |\nabla f(x)| - |\xi| |\nabla (g - f)(x)|.$$

Since $r \ge 1$ then from (1) we get that there exists a constant $C_2 > 0$ such that

$$|\nabla (g - f)(x)| \le C_2 |\nabla f(x)|^{r+1} \le C_2 |\nabla f(x)|^2 \quad \text{for } x \in U.$$

Hence, diminishing U if necessary,

$$|\nabla F(\xi, x)| \ge |\nabla f(x)| - |\xi| C_2 |\nabla f(x)|^2 \ge \frac{1}{C_1} |\nabla f(x)| \quad \text{for } (\xi, x) \in G.$$

Moreover, from definition of ∇F we get at once, that there exists a positive constant C_3 such that

$$|\nabla F(\xi, x)| \le C_3 |\nabla f(x)| \quad \text{for } (\xi, x) \in G. \tag{5}$$

Now we will show that the mapping $X : G \to \mathbb{R}^n \times \mathbb{R}$ defined by

$$X(\xi, x) = (X_1, \ldots, X_{n+1}) = \begin{cases} \frac{(g-f)(x)}{|\nabla F(\xi,x)|^2} \nabla F(\xi, x) & \text{for } x \notin Z \\ 0 & \text{for } x \in Z \end{cases}$$

is a C^r mapping. The proof of this fact will be divided into several steps.

STEP 1. The mapping X is continuous in G.

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Indeed, let $h_i(\xi, x) = (g - f)(x)\frac{\partial F}{\partial x_i}(\xi, x)$. Then, from (1) and (5) we have

$$|h_i(\xi, x)| = \left|\frac{\partial F}{\partial x_i}(\xi, x)\right| |(g - f)(x)| \leq CC_3|\nabla f(x)|^{r+3},$$

for $(\xi, x) \in G\backslash(\mathbb{R} \times Z)$. Moreover, from definition of X, (3) and (4) we obtain that there exists a constant $A' > 0$ such that for any $(\xi, x) \in G\backslash(\mathbb{R} \times Z)$

$$|X_i(\xi, x)| \leq CC_1^2C_3|\nabla f(x)|^{r+1} \leq A' \operatorname{dist}(x, Z)^{r+1}. \tag{6}$$

The above inequality also holds for $(\xi, x) \in G \cap (\mathbb{R} \times Z)$, therefore X is continuous in G.

Step 2. Let $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}_0^{n+1}$ be such that $|\alpha| \leq r$, then there exists a constant $A'' > 0$ such that

$$|\partial^\alpha X_i(\xi, x)| \leq A'' \operatorname{dist}(x, Z)^{r+1-|\alpha|} \quad \text{for } (\xi, x) \in G\backslash(\mathbb{R} \times Z).$$

Indeed, let us take $(\xi, x) \in G\backslash(\mathbb{R} \times Z)$, from Leibniz rule we have

$$\partial^\alpha X_i(\xi, x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}(h_i(\xi, x))\partial^\beta \left(\frac{1}{|\nabla F(\xi, x)|^2}\right). \tag{7}$$

Diminishing G if necessary, from Lemma 3 we obtain

$$\left|\partial^\beta \left(\frac{1}{|\nabla F(\xi, x)|^2}\right)\right| \leq \frac{A''_\beta}{|\nabla F(\xi, x)|^{|\beta|+2}},$$

for some constants $A''_\beta > 0$. Therefore, from (7) we have

$$|\partial^\alpha X_i(\xi, x)| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial^{\alpha-\beta}(h_i(\xi, x))| \frac{A''_\beta}{|\nabla F(\xi, x)|^{2|\beta|+2}}. \tag{8}$$

From (1) and (5) we have

$$|\partial^{\alpha-\beta}(h_i(\xi, x))| \leq B_{\alpha-\beta}|\nabla f(x)|^{r+3-|\alpha|+|\beta|}, \tag{9}$$

for some positive constants $B_{\alpha-\beta}$. Finally, from (8), (9), (4) and (3) we obtain

$$\begin{aligned}
|\partial^\alpha X_i(\xi, x)| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} B_{\alpha-\beta}|\nabla f(x)|^{r+3-|\alpha|+|\beta|} \frac{A''_\beta}{|\nabla F(\xi, x)|^{|\beta|+2}} \\
&\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} A''_\beta B_{\alpha-\beta}|\nabla f(x)|^{r+3-|\alpha|+|\beta|-|\beta|-2} \\
&\leq \frac{A''}{A}|\nabla f(x)|^{r+1-|\alpha|} \leq A'' \operatorname{dist}(x, Z)^{r+1-|\alpha|},
\end{aligned}$$

for some constant $A'' > 0$ and for $(\xi, x) \in G\backslash(\mathbb{R} \times Z)$.

Step 3. Partial derivatives $\partial^\alpha X_i$ vanish for $(\xi, x) \in G \cap (\mathbb{R} \times Z)$ and $|\alpha| \leq r$.

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Indeed, we will carry out induction with respect to $|\alpha|$. Let $t \in \mathbb{R}$, $x \in Z$ and let $x_m^t = (x_1, \ldots, x_m + t, \ldots, x_n)$. For $|\alpha| = 0$ hypothesis is obvious. Assume that hypothesis is true for $|\alpha| \leq r - 1$. Then, from Step 2 we have

$$\frac{|\partial^\alpha X_i(\xi, x_m^t) - \partial^\alpha X_i(\xi, x)|}{|t|} = \frac{|\partial^\alpha X_i(\xi, x_m^t)|}{|t|} \leq \frac{A'' \operatorname{dist}(x_m^t, Z)^{r+1-|\alpha|}}{|t|}$$
$$\leq \frac{A''|t|^{r+1-|\alpha|}}{|t|} = A''|t|^{r-|\alpha|}.$$

Since $r - |\alpha| \geq r - r + 1 = 1$, we obtain $\partial^\gamma X_i(\xi, x) = 0$ for $x \in Z$ and $|\gamma| = |\alpha| + 1$. This completes Step 3.

In summary from Step 1, 2 and 3 we obtain that X_i are C^r functions in G. Therefore, X is a C^r mapping in G.

Define a vector field $W : G \to \mathbb{R}^n$ by the formula

$$W(\xi, x) = \frac{1}{X_1(\xi, x) - 1}(X_2(\xi, x), \ldots, X_{n+1}(\xi, x)). \tag{10}$$

Diminishing U if necessary, we may assume that $A' \operatorname{dist}(x, Z) < \frac{1}{2}$. From (6) we obtain

$$|X_1(\xi, x) - 1| \geq 1 - |X(\xi, x)| \geq 1 - A' \operatorname{dist}(x, Z) > \frac{1}{2}, \tag{11}$$

for $(\xi, x) \in G$. Hence, the field W is well defined and it is a C^r mapping.

Consider the following system of ordinary differential equations

$$\frac{dy}{dt} = W(t, y). \tag{12}$$

Since $r \geq 1$, then W is at least of class C^1 on G, so it has a uniqueness of solutions property in G. By solving (12) we obtain that a general solution is of C^r-class. Moreover, by definition of G any solution is defined on interval $[0, 1]$. Hence, there exists a C^r diffeomorphism $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ given by formula $\varphi(x) = y_x(1)$, where $y_x : [0, 1] \to \mathbb{R}^n$ is solution of system (12) with initial condition $y_x(0) = x$.

Note that for any $x \in U$,

$$F(t, y_x(t)) = const. \quad \text{in } [0, 1]. \tag{13}$$

Indeed, from definition of W we derive the formula

$$[1, W(\xi, x)] = \frac{1}{X_1(\xi, x) - 1}(X(\xi, x) - e_1) \quad \text{for } (\xi, x) \in G,$$

where $e_1 = [1, 0, \ldots, 0] \in \mathbb{R}^{n+1}$ and $[1, W] : G \to \mathbb{R} \times \mathbb{R}^n$. Thus, if we denote by $\langle a, b\rangle$ the scalar product of two vectors a, b, then for $t \in [0, 1]$, we have

$$\begin{aligned}\frac{dF(t, y_x(t))}{dt} &= \langle(\nabla F)(t, y_x(t)), [1, W(t, y_x(t))]\rangle \\ &= \frac{1}{X_1(t, y_x(t)) - 1}\left(\langle(\nabla F)(t, y_x(t)), X(t, y_x(t))\rangle - \frac{\partial F}{\partial \xi}(t, y_x(t))\right) \\ &= \frac{1}{X_1(t, y_x(t)) - 1}(g(y_x(t)) - f(y_x(t)) - g(y_x(t)) + f(y_x(t))) = 0.\end{aligned}$$

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This gives (13). Finally, (13) yields

$$f(x) = F(0, x) = F(0, y_x(0)) = F(1, y_x(1)) = F(1, \varphi(x)) = g(\varphi(x)),$$

for $x \in U$. This ends the proof. □

4. Additional results. Under assumptions of Theorem 1, note that in the situation when $r = 0$, we have that ∇f is a continuous mapping and we can't use Lemma 2, so we should assume that ∇f is a locally lipschitzian mapping. Moreover, condition (1) has the form

$$|(g - f)(x)| \leq C\,|\nabla f(x)|^2, \quad x \in U,$$

so inequalities (4) and (5) are false. But when we will assume additionally that

$$|\nabla(g - f)| \leq C'\,|\nabla f(x)|^2, \quad x \in U,$$

for some constant $C' > 0$, then those inequalities will be already true. Moreover, from those inequalities we obtain that vector field (10) is continuous in G and locally lipschitzian in $G\backslash(\mathbb{R} \times Z)$. Therefore system (12) has a global uniqueness of solutions property only in $G\backslash(\mathbb{R} \times Z)$. But from (11) and Lemma 4 we have that (12) has a global uniqueness of solutions property in G. Therefore, due to the above, we obtain the following sufficient condition for C^0-right equivalence. Additionally to obtain that mapping ∇F is locally lipschitzian we should assume that ∇g is locally lipschitzian.

THEOREM 2. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be C^1 functions such that $\nabla f, \nabla g$ are locally lipschitzian mappings. If $\nabla f(0) = 0$ and there exist a neigbourhood U of $0 \in \mathbb{R}^n$ and constants $C, C' > 0$ such that*

$$|(g - f)(x)| \leq C\,|\nabla f(x)|^2, \quad |\nabla(g - f)(x)| \leq C'\,|\nabla f(x)|^2 \quad \textit{for } x \in U,$$

then f and g are C^0-right equivalent.

From Theorem 1 and Corollary 1 we obtain immediately

THEOREM 3. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be C^{r+1} functions, $r \in \mathbb{N}$. If $\nabla f(0) = 0$ and $(g - f) \in (\mathcal{J}_f C^r(n))^{r+2}$ then f and g are C^r-right equivalent.*

It seems that Bochnak Theorem [**1**, Theorem 1] is stronger than Theorem 3, because in the first theorem we assume that power of Jacobi ideal is constant, whereas in the last theorem this power is depending on r. But on the other hand in Theorem 3 assumption about class of f, g is weaker than in Bochnak Theorem. So, it is difficult to say which theorem is better.

5. Łojasiewicz exponent in the gradient inequality. Under the additional assumption of analyticity of functions, we will show that if two functions are C^1-right equivalent then their Łojasiewicz exponents in the gradient inequality are the same.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function. It is known that there exist a neighbourhood of $0 \in \mathbb{R}^n$ and constants $C, \eta > 0$ such that the following *Łojasiewicz*

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gradient inequality holds

$$|\nabla f(x)| \geq C|f(x)|^{\eta}, \qquad \text{for } x \in U.$$

The smallest exponent η in the above inequality is called *Łojasiewicz exponent in the gradient inequality* and is denoted by $\varrho_0(f)$ (cf. [**4**, **5**]).

PROPOSITION 1. *Let* $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *be analytic functions. If* f *and* g *are* C^1-*right equivalent, then* $\varrho_0(f) = \varrho_0(g)$.

Proof. By the assumption there exists a C^1 diffeomorphism $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $g = f \circ \varphi$ and $f = g \circ \varphi^{-1}$. Moreover, there exist a neighbourhood U of $0 \in \mathbb{R}^n$ and a constant $C > 0$ such that

$$|\nabla g(x)| \geq C|g(x)|^{\varrho_0(g)} \qquad \text{for } x \in U.$$

By $J(\varphi)$ we denote the Jacobian matrix of mapping φ and by $\|J(\varphi)\|$ the norm of this matrix. Note that, diminishing U if necessary, there exists a constant $A > 0$ such that

$$\|J(\varphi(x))\| \leq A \qquad \text{for } x \in U.$$

Moreover, $\nabla g = \nabla(f \circ \varphi) = \nabla f(\varphi) \cdot J(\varphi)$ and from the above,

$$|\nabla g(x)| \leq |\nabla f(\varphi(x))| \, \|J(\varphi(x))\| \leq A|\nabla f(\varphi(x))| \qquad \text{for } x \in U.$$

Hence, diminishing U if necessary

$$|\nabla f(\varphi(x))| \geq \frac{1}{A}|\nabla g(x)| \geq \frac{C}{A}|g(x)|^{\varrho_0(g)} = \frac{C}{A}|f(\varphi(x))|^{\varrho_0(g)} \quad \text{for } x \in U.$$

Therefore, $\varrho_0(f) \leq \varrho_0(g)$. Analogously, we get $\varrho_0(f) \geq \varrho_0(g)$. Hence, we have $\varrho_0(f) = \varrho_0(g)$. □

From Proposition 1 and Theorem 3 we obtain

COROLLARY 2. *Let* $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *be analytic functions. If* $\nabla f(0) = 0$ *and* $(g - f) \in \mathcal{J}_f^3$ *then* $\varrho_0(f) = \varrho_0(g)$*, where* J_f *denotes the Jacobi ideal of* f *in the ring of germs of analytic functions* $(\mathbb{R}^n, 0) \to \mathbb{R}$.

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