FINITE GROUPS WITH A NILPOTENT MAXIMAL SUBGROUP

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(received 8 January 1964)

Let G be a finite group all of whose proper subgroups are nilpotent. Then by a theorem of Schmidt-Iwasawa the group G is soluble. But what can we say about a finite group G if only one maximal subgroup is nilpotent?

Let G be a finite group with a nilpotent maximal subgroup M. Then the following results are known:

THEOREM OF J. G. THOMPSON [4]. *If M has odd order, then G is soluble.*

THEOREM OF DESKINS [1]. *If M has class $\leqq 2$, then G is soluble.*

THEOREM OF JANKO [3]. *If the 2-Sylow subgroup M_2 of M is abelian, then G is soluble.*

Now we can give a very simple proof of the following

THEOREM. *Let G be a finite group with a nilpotent maximal subgroup M. If a 2-Sylow subgroup M_2 of M has class $\leqq 2$, then G is soluble.*

This result is the best possible because the simple group LF(2, 17) has a 2-Sylow subgroup of class 3 which is a maximal subgroup.

The theorem was announced without proof in [3]. The proof of the theorem will be independent of the theorem of Deskins. We shall give at first some definitions.

Definition 1. A finite group G is called p-nilpotent if it has a normal Sylow p-complement.

Definition 2. Let $Z(G_p)$ be the centre of a p-Sylow subgroup G_p of a finite group G. Then the group G is called p-normal if $Z(G_p)$ is the centre of every p-Sylow subgroup of G in which it is contained.

Definition 3. Let α be an automorphism of a group G. Then α is called fixed-point-free if and only if α fixes only the unit element of G.

In the proof of the theorem we shall use the theorem of J. G. Thompson and also the following two results:

THEOREM OF GRÜN-WIELANDT-P. HALL [2]. *Let G be a finite p-normal group and $Z(G_p)$ the centre of a p-Sylow subgroup G_p of G. Then G is p-nilpotent if and only if the normalizer $N(Z(G_p))$ of $Z(G_p)$ is p-nilpotent.*

THEOREM OF ZASSENHAUS [5]. *If a finite group G has a fixed-point-free automorphism of order 2, then G is an abelian group.*

PROOF OF THE THEOREM. Using induction on the order we can assume that there is no non-trivial normal subgroup of G contained in M. If M_p is a p-Sylow subgroup of M, then M_p is normal in M and M_p must be a p-Sylow subgroup of G. Hence M is a Hall subgroup of G.

Suppose now that M is not a Sylow subgroup of G. In this case we can prove that G is p-normal for every prime p which divides the order of M. Let $Z = Z(M_p)$ be the centre of the p-Sylow subgroup M_p of M. Suppose that

$$Z \leqq M_p^x = x^{-1}M_p x$$

for a certain $x \in G$. Then

$$C(Z) \geqq \{M, (M^x)_{p'}\} = M,$$

where $C(Z)$ denotes the centralizer of Z and $1 \neq (M^x)_{p'}$ is a p-Sylow complement of M^x. Consequently $M_{p'}^x = M_{p'}$ and $x \in N(M_{p'})$. On the other hand $N(M_{p'}) = M$ whence $x \in M$. So Z is the centre of $(M_p)^x = M_p$ and the group G is p-normal. By the theorem of Grün-Wielandt-P. Hall the group G is p-nilpotent for every prime p which divides the order $|M|$ of M. Let N_p denote a normal p-Sylow complement of G. Then we consider the intersection

$$N = \bigcap_{p||M|} N_p.$$

The group N is obviously a normal complement of M.

By the theorem of J. G. Thompson we can suppose that M has even order. Let τ be a central involution of M. Because $C(\tau) = M$ the involution τ acts fixed-point-free on N and by the theorem of Zassenhaus N is abelian and so G is soluble.

We suppose now that M is a Sylow subgroup of G. By the theorem of J. G. Thompson and by our assumption we can suppose that M is a 2-Sylow subgroup of class $\leqq 2$.

If G is 2-normal (and because $N(Z(M)) = M$), then G is 2-nilpotent by the theorem of Grün-Wielandt-P. Hall. Let N be a normal 2-complement of G. Then again by the theorem of Zassenhaus N is abelian and G is soluble.

If G is not 2-normal, then there is an $x \in G$ such that $Z(M) \leqq M \cap M^x = D$ and $M^x \neq M$. But then (because $M' \leqq Z(M)$) D is normal in M and $N(D) \cap M^x \neq D$ which gives $N(D) = G$. But this is impossible because we have assumed that M does not contain non-trivial normal subgroups of G. The proof is complete.

References

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https://doi.org/10.1017/S1446788700025271 Published online by Cambridge University Press