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Nonlinear Dynamos

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Abstract. This paper discusses nonlinear dynamos where the nonlinearity arises directly via the Lorentz force in the Navier-Stokes equation, and leads to a situation where the Lorentz force and the velocity and the magnetic field are in direct competition over substantial regions of the flow domain. Filamentary and non-filamentary dynamos are contrasted, and the concept of Alfvénic dynamos with almost equal magnetic and kinetic energies is reviewed via examples. So far these remain in the category of toy models; the paper concludes with a discussion of whether similar dynamos are likely to exist in astrophysical objects, and whether they can model the solar cycle.

Keywords. magnetic fields, MHD, Sun: magnetic fields, stars: magnetic fields

1. Introduction

Nonlinear dynamo calculations can be divided into three main classes, each with a well-defined purpose and a different set of advantages and shortcomings. The most astrophysically applicable theories typically use some form of mean-field electrodynamics; these include the so-called α–ω models (Krause & Rädler 1981), Parker's dynamo wave model (Parker 1955), the solar Babcock-Leighton dynamo and its flux-transport variants (see eg Dikpati & Charbonneau 1999), and Braginskii's nearly axisymmetric model for the Earth (Braginskii 1964). These theories all started life as kinematic models that took a prescribed velocity field and tested whether it gave growing or decaying solutions for the magnetic field; the resulting mathematical problem was linear and various parametrised terms arose which were in essence turbulent closures. The dynamical feedback of the the ensuing Lorentz force was neglected, so the basic theoretical outputs were an eigenvalue and its associated eigenfunction. The sign of the real part of the eigenvalue says whether there is a dynamo or not, the complex part if present determines the period, and the eigenfunction gives the configuration of the generated magnetic field. There is no information about the final field strength of such a dynamo, as the field grows exponentially without bound. Dynamics and nonlinearity are typically added via ad hoc parametrisations, a procedure which has been the subject of vigorous debate (see eg Vainshtein & Cattaneo 1992). The advantages of this approach are versatility and direct applicability in a wide variety of astrophysical objects. The disadvantage is that the underlying physics is questionable and relies on assumptions which are not satisfied in practice. The situation is reminiscent of the use of mixing-length theory for stellar convection; one knows that the theory is at best only qualitatively right, but somehow this does not really matter and stellar evolution theory proceeds very satisfactorily notwithstanding.

The second class of calculation is direct numerical simulation. The patron saint of these Proceedings has been a major contributor in this area, encouraging successive waves of students and postdocs by leading them in the solution of a sequence of important and realistic problems, first in convection, then in magnetoconvection, and most recently in dynamo theory.

The third approach consists of the solution of model problems which though not meant to apply literally to a specific astrophysical object, aspire nonetheless to isolate and elucidate the fundamental processes which may be at work. The important thing is that these models include the crucial physics, yet are simple enough to understand and interpret. This is the classic applied mathematician's approach, as epitomised in Lord Rayleigh's *Theory of Sound*. Nowadays the tongue-in-cheek terminology is that these are "toy models". The hope is that the processes studied can, if they look promising, be incorporated into direct numerical simulation of a specific object. This happens all too rarely, but it is a real treat when it works.

Magnetoconvection calculations constitute a side-class: they are not fundamentally different from the dynamo case, except that typically a mean field is imposed across the computational domain. The latter is normally guaranteed to be a conserved quantity, and the convection acts to amplify the magnetic energy by a large factor. Such a process is not regarded as a dynamo, because if the mean field were turned off, all field would eventually drain away by ohmic dissipation. To be a real dynamo, the field must be self-excited, and over the whole domain its average must be zero, because $\nabla \cdot \mathbf{B} = 0$. There can be patches of "mean field" in the sense of mean field theory, but they must average to zero over the whole domain. This point is often not properly understood. Any magnetoconvective calculation aimed at exhibiting genuine dynamo action must use initial conditions with no overall mean component.

This isolation of models into what is almost a philosophical classification was beautifully summarised by Spiegel (1977), who described it in terms of a political analogy; the three categories above are left-wing, centrist, and right-wing respectively. In this paper I will be concerned mainly with the right-wing end of the spectrum, but this is my own preference and is not meant to imply the other approaches are less worthwhile—with difficult problems attack on all fronts is vital!

In the next section, I will discuss filamentary dynamos, citing the example of *ABC* dynamos and showing the problems that arise when considering the generation of astrophysically significant fields. In subsequent sections I will then ask the question whether all dynamo fields have to be filamentary, and give a counterexample, the Archontis dynamo, where this is not the case. At the end I will try to be more astrophysically responsible and speculate about the forms nonlinear dynamos might take in real objects.

2. Filamentary dynamos

We start by writing down the induction and momentum equations for an electrically conducting fluid with both viscous and magnetic diffusivities, assuming a simple ohmic diffusion. For simplicity we take the incompressible case and use the standard notation for the various quantities appearing. The equations to be solved are then the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

and the momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla \left(p/\rho + \frac{1}{2} u^2 \right) + \mathbf{j} \times \mathbf{B} + \mathbf{F} + \nu \nabla^2 \mathbf{u} \,,$$

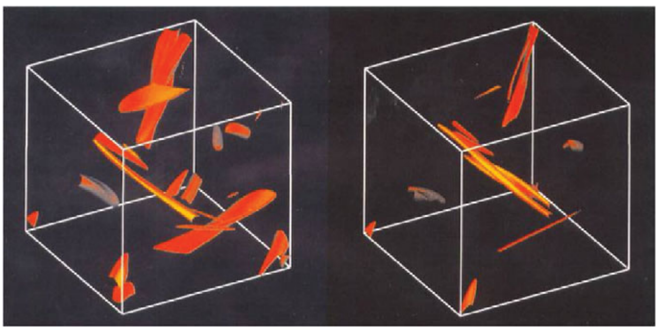

Figure 1. The *ABC* dynamo with (1,1,1) forcing, for $\nu = 1/5, \eta = 1/400$. Shown are isosurfaces where the magnetic energy attains 20% of its maximum value at that instant, for two separate times approximately 40 turnover units apart.

with $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0$, and $\mu_o \mathbf{j} = \nabla \times \mathbf{B}$. The driving force $\mathbf{F}$ is necessary to sustain $\mathbf{u}$ and $\mathbf{B}$ against viscous and ohmic dissipation. In astrophysics it would arise from some natural physical mechanism such as buoyancy, but in model calculations it is often externally imposed with a specific form. The assumption of incompressibility makes for considerable mathematical and computational simplification, and those anelastic or fully compressible computations which have been done show remarkably few differences compared to the incompressible case, at least as far as dynamo theory is concerned.

The *ABC* dynamo is a prototype example where the resulting magnetic field is typically filamentary. Here everything is 2π-periodic and a force

$$\mathbf{F} = \nu(A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x)$$

is supplied. In the absence of magnetic fields the *ABC* flow (with $\mathbf{u}$ equal to the above formula without the factor ν) is a solution to the Navier-Stokes equation with this forcing field. In the case $A : B : C = 1 : 1 : 1$, this solution is hydrodynamically unstable when the kinetic Reynolds number R_e exceeds 13.09 (Galloway & Frisch, 1987).

Figures 1 and 2 show the nature of the magnetic field generated for the case $\nu = 1/5, \eta = 1/400$ (this calculation was computed some time ago by Olga Podvigina; solutions were earlier given by Galanti, Sulem & Pouquet 1992). The viscosity is high enough that the non-magnetic case has the (1,1,1) *ABC* flow as a stable solution; the magnetic diffusivity is low enough to yield a dynamo when a divergence-free small seed field with zero mean is added. The solution evolves quite rapidly to a time-dependent but statistically steady state with a total magnetic energy around twice the total kinetic energy. The isosurfaces of magnetic energy show the field is highly filamentary.

Astrophysical dynamos operate in the regime where both the magnetic and kinetic Reynolds numbers are very large, so it is natural to ask if these solutions where the total magnetic and kinetic energies are comparable can persist to high kinetic Reynolds number. The observation that the fields are filamentary suggests we attempt to establish scaling laws: two possibilities are that the filaments have thickness $R_m^{-1/2}L$ and length L, where L is the characteristic size of the domain, or that the filaments form magnetic plugs with order L thickness and length but order $R_m^{-1/2}L$ edges in which the dissipation takes place. This gives formulae

$$\frac{\text{Total Magnetic Energy}}{\text{Total Kinetic Energy}} \simeq \frac{1}{R_e} \text{ (filaments of thickness } R_m^{-1/2}\text{)}$$

$$\frac{\text{Total Magnetic Energy}}{\text{Total Kinetic Energy}} \simeq \frac{1}{R_e}\left(\frac{\nu}{\eta}\right)^{1/2} \text{ (thickness order } L \text{ with } R_m^{-1/2} \text{ edges)} .$$

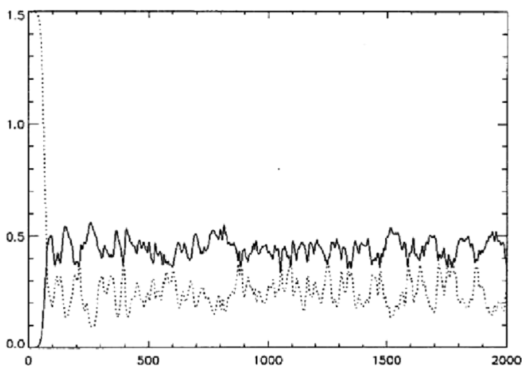

Figure 2. The same case as in Figure 1, but now showing the evolution of total magnetic energy (solid) and total kinetic energy (dashed), for a time long compared with the turnover and both diffusion timescales.

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Here R_e and R_m are the kinetic and magnetic Reynolds numbers. The above formulae are derived in Galloway (2003), but related results for other flow types were known well before then (Vainshtein & Cattaneo 1992, Brummell, Cattaneo & Tobias 2001). The bad news about all this is that at high R_e the relative fraction of magnetic energy is very small, in contrast with the substantial fractions which are observed in many astrophysical objects. This has lead since 1990 to much questioning of how dynamos, mean field or otherwise, can be viable in astrophysics. The convective, intermittent, numerical photospheric dynamo of Cattaneo (1999) is a good illustration of this. This debate has continued until the present day, and in my opinion has yet to be satisfactorily resolved.

3. Non-filamentary dynamos

For a long time, numerical dynamos based on the full nonlinear equations rather than mean-field parametrisations seemed to suggest that all such dynamos were filamentary. The first example where this was not the case appeared in the PhD thesis of Archontis (2000), who studied a problem similar to the ABC-forced dynamo, but with the cosines omitted. The corresponding velocity field was known to give a very effective and probably fast kinematic dynamo (Galloway & Proctor 1992). However, this velocity field is *not* a solution to the Navier-Stokes equation with the forcing term $\mathbf{F} = \nu(\sin z, \sin x, \sin y)$, because the nonlinear term is now unbalanced. With no magnetic field, the flow initially adopts the right form if started from rest, but then quickly evolves into something quite different as soon as the nonlinear term starts to bite.

Once a divergence-free random seed magnetic field is added, something quite remarkable happens. Figure 3 shows the time evolution of the total magnetic and kinetic energies over a long time. The "something quite different" referred to above is apparently an effective dynamo and leads to a time-dependent state which nonetheless looks quasi-steady. Presumably this dynamo is filamentary and resembles the ABC dynamo discussed in the last section. However, after several diffusion times spent in this state, a transition occurs to another state where the energies are almost equal to one another. Inspection of this state shows that $\mathbf{u}$ and $\mathbf{B}$ are in fact *locally* almost equal to one another throughout nearly all of the domain, when expressed in Alfvénic units. The technicalities of the limiting behaviour as both diffusivities tend to zero are described in Cameron & Galloway (2006a). The fields $\mathbf{u}$ and $\mathbf{B}$ tend to exactly the same thing, and in the limit both are close, *but not equal*, to $0.5(\sin z, \sin x, \sin y)$. The discrepancy is of the order of a few percent in a selection of Fourier modes, but why these modes are present and need to be there has so far defied understanding—they are clearly the result of some unknown solvability condition. Also amazing is that the solution is steady and stable—we have tested this

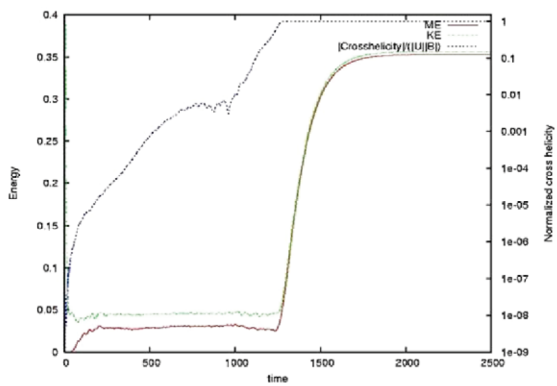

Figure 3. The evolution of total magnetic energy (red) and total kinetic energy (green), for the Archontis dynamo described in the text, over a time long compared with the turnover and both diffusion timescales. Also shown is the evolution of the normalised cross-helicity, better referred to as the magnetic alignment.

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down to diffusivities of 1/2000, and Gilbert, Ponty & Zheligovsky (2010) report that the stability also persists when they are 1/10000. This latter paper represents the current state of the art in understanding the Archontis dynamo; there is also an extensive description in Archontis, Dorch & Nordlund (2007). Note that these last authors use a fully compressible code (which seems to make little difference), and they employ a feedback mechanism which modulates the forcing term in an (unnecessary) attempt to keep the energies at a fixed level. In fact, the feedback appears to destabilise things because it acts on the Alfvénic timescale and can cause resonances. (We have reproduced this by using their feedback term in our incompressible code.)

To understand what causes the transition from the early quasi-steady behaviour to the evolved $\mathbf{u} \simeq \mathbf{B}$ state, it is useful to plot the integral of $\mathbf{u}.\mathbf{B}$ divided by the square roots of the integrals of u^2 and B^2 over the periodicity cube. This quantity is sometimes called the normalised cross-helicity, though the terminology is unfortunate and and it is more informative to regard it as a measure of the degree to which $\mathbf{u}$ and $\mathbf{B}$ are aligned—it is the average of $\cos\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{B}$. When the alignment is 1, the two fields are perfectly lined up, and then the nonlinear $\mathbf{u} \times (\nabla \times \mathbf{u})$ and $\mathbf{J} \times \mathbf{B}$ terms can cancel one another out in the momentum equation. The topmost curve in Figure 3 shows the evolution of the alignment; all through the dormant apparently quasi-steady phase it is steadily growing, and once its value is appreciable it triggers a transition to a completely aligned state. By adding $\mathbf{B}$ dotted with the momentum equation to $\mathbf{u}$ dotted with the induction equation and integrating over the box, one finds that the time rate of change of the alignment has a source term equal to the integral of $\nu\mathbf{F} \cdot \mathbf{B}$ together with viscous and magnetic diffusive terms. Given the form of $\mathbf{F}$, this shows that it is the amplitude of the $(\sin z, \sin x, \sin y)$ mode in the magnetic field which causes the alignment either to grow towards $+1$ if it is positive, or to decrease towards -1 if it is negative. (Note there is a symmetry between solutions with $\mathbf{u} = \pm\mathbf{B}$). It also explains why the alignment is a slow process taking place on a diffusive timescale—the initial dynamo may be fast but the final dynamo is slow! However, alignment is a well-known process in MHD turbulence, and in that context it may not always be slow—see Matthaeus *et al.* (2008).

The fact that both fields are close to $0.5(\sin z, \sin x, \sin y)$ means that the solution has a very regular, non-turbulent form This is shown in Figure 4, which plots the stable and unstable manifolds connecting some of the stagnation points.

Can further solutions with $\mathbf{u} \simeq \pm\mathbf{B}$ be constructed? In fact, if one is not fussy about how unrealistic a force is to be employed, there is a simple recipe by which such dynamos can be constructed to order. Take any neutrally stable solution $\mathbf{B_0}$ to the kinematic problem with prescribed velocity $\mathbf{u_0}$ and magnetic diffusivity η_0. We can now generate an equilibrium solution to the full dynamo problem (including the momentum equation) for $\eta = \epsilon\eta_0$. This is $\mathbf{u} = \epsilon\mathbf{u_0} + \mathbf{B_0}$, with $\mathbf{B} = \mathbf{B_0}$, and $\mathbf{F} =$ whatever it has to be in order that the momentum equation

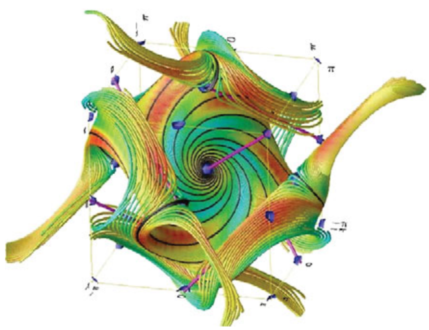

Figure 4. A plot showing the beautiful structure of the velocity and magnetic fields for the final steady state of the Archontis dynamo; both fields are almost indistinguishable from one another. Plotted are trajectories of the solution which form heteroclinic orbits connecting some of the stagnation points.

is satisfied. (The actual formula is given in Cameron & Galloway (2006a), along with a slightly more general scaling transformation.) As $\epsilon \to 0$ we end up with an Alfvénic solution.

Variations on this theme were followed up in Cameron & Galloway (2006b), particularly for the Gibson (1968) three-sphere rotor dynamo. We also looked there at a family of forcings

$$\mathbf{F} = \nu(\sin z + \epsilon \cos y, \sin x + \epsilon \cos z, \sin y + \epsilon \cos x)$$

where $\epsilon = 0$ gives the Archontis dynamo and $\epsilon = 1$ gives the (1,1,1) *ABC* dynamo. This enables the study of the transition from the non-filamentary to the filamentary cases, as well as a few other issues not dealt with here.

4. Discussion

These non-filamentary dynamos are fascinating objects, but the obvious question is whether they are isolated freaks or whether such behaviour (with $\mathbf{u} \simeq \mathbf{B}$) can be expected to occur as a typical feature of an astrophysical dynamo, at least over some fraction of the flow domain. When the forcing and both diffusivities are formally absent, any steady $\mathbf{u}, \mathbf{B}$ pair with $\mathbf{u} = \pm\mathbf{B}$ (in Alfvénic units) satisfies the induction and momentum equations; these are often referred to as Alfvénic solutions. This is because the nonlinearities in the momentum equation cancel one another out exactly. Friedlander & Vishik (1995) proved the remarkable result that *all* these solutions are neutrally stable. This is in complete contrast to the non-magnetic case, where one finds that typical solutions to the Euler equation are unstable. When small diffusivities are added, one can conjecture that some fraction of these Alfvénic solutions end up being stable attractors. The challenge is to ascertain which ones emerge in practice—we have seen how hard it is to determine the solvability conditions that enable the Archontis dynamo to home in on one particular solution which is closely related, but not identical, to the forcing function.

The natural next step is to examine the numerical calculations done to date and see whether there is any evidence of alignment processes at work. The best evidence I have seen occurs in the ASH-code simulations of faster-rotating Suns due to Brown, Browning, Brun, Miesch and Toomre, which are found elsewhere in these proceedings (see also Brown *et al.* 2010; the solutions are referred to as wreath-building dynamos, and are clearly strongly influenced by differential rotation). There are many hints in other calculations, but nothing as convincing as what is seen in the "toys". One important point about the latter is that they have boundary conditions which are compatible with an aligned outcome; the Archontis dynamo for instance is periodic. Any real astrophysical object will have boundary conditions which are *not* compatible, and it is not clear to what extent boundary effects can wreck any alignment mechanism. In the Sun, the non-compatible boundary conditions are enough to ensure that the magnetic field seen at the photosphere is highly intermittent. The question is whether this persists throughout the Sun or whether somewhere, deeper down, there may be a region where alignment mechanisms are significant for field generation. The tachocline is the obvious place to consider, and Robert Cameron and myself have been trying to construct an aligned dynamo which works there (see the discussion item by Proctor).

This has been a personal view of what I think are some interesting issues in nonlinear dynamo theory. I have not discussed several topics to which other authors would attach great importance, such as scale separation or the significance of various kinds of helicity—they understand these much better than I do. I would like to end by thanking Robert Cameron for his conviction that non-filamentary dynamos are possible, and Juri Toomre for his example in formulating and fostering the solution of problems which address fundamental issues and propel the subject forward rather than spread confusion.

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Discussion

Brandenburg: When you allow for scale separation (bigger box than the size of one cell), you do not get flux alignment that can lead to a Yoshizawa effect; see Sur & Brandenburg 2009 (MNRAS 299, 273)

Galloway: I have deliberately chosen to stay off the subject of scale separation in this talk but I realise it is another interesting aspect and I refer readers to the paper you cite above.

Pouquet: Why do you say that the growth of $V - B$ alignment is a slow (diffusion time) process?

Galloway: One can write down an equation for the evolution of the integral of $\mathbf{u}.\mathbf{B}$ over the fundamental periodic cube. After converting various terms to surface integrals which vanish because of the periodicity, one is left on the right hand side just with terms which are proportional to the diffusivities. Thus the total must evolves slowly. This does not preclude the idea that their could be cancelling patches with opposite sign which evolve on the fast timescale (see the next question).

Proctor: Do you find Archontis type dynamos with “shocks” across which $\mathbf{u}.\mathbf{B}$ changes sign?

Galloway: We have not seen such a phenomenon in any of the relatively small number of calculations conducted so far. However, Robert Cameron and myself have a proposal for a tachocline dynamo which needs what you refer to in order that the magnetic field changes sign every 11 years while the differential rotation does not. So we very much hope that what you suggest is possible! (This work is being submitted to MNRAS.)

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