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# CONDITIONAL PERCOLATION ON ONE-DIMENSIONAL LATTICES

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**Abstract**

Conditioning independent and identically distributed bond percolation with retention parameter $p$ on a one-dimensional periodic lattice on the event of having a bi-infinite path from $-\infty$ to $\infty$ is shown to make sense, and the resulting model exhibits a Markovian structure that facilitates its analysis. Stochastic monotonicity in $p$ turns out to fail in general for this model, but a weaker monotonicity property does hold: the average edge density is increasing in $p$.

*Keywords:* Conditional percolation; stochastic domination; one-dimensional lattice; Markov chain

2000 Mathematics Subject Classification: Primary 60K35; 82B43; 60J10

## 1. Introduction

In this paper we consider bond percolation on a class of one-dimensional periodic lattices. A basic example of such lattices is the 'ladder' graph in Figure 1, while the general definition will be given in Definition 4.1, below. In bond percolation on a graph $G = (V, E)$, the edges are randomly declared to be closed (i.e. deleted) or open (retained) according to some probability measure on $\{0, 1\}^E$, where 0 means closed and 1 means open.

The usual independent and identically distributed (i.i.d.) bond percolation model, in which each edge is independently retained with the same probability $p \in [0, 1]$, on one-dimensional periodic lattices has been mostly ignored, the reason being that the critical probability $p_c$ of having an infinite cluster equals 1, so that the event of having an infinite cluster has probability 0 for any nontrivial value of $p$ (this follows from an easy Borel–Cantelli argument). Here, we will liven up the model by conditioning on the existence of an infinite cluster reaching infinitely far in both directions. More precisely, we condition on the existence of an open path passing from one end to the other of a finite portion of the lattice, and take limits as the portion grows

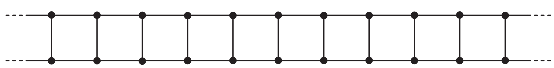

FIGURE 1: The ladder graph.

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towards the full lattice. Thus, our construction is somewhat reminiscent of the 'incipient infinite cluster' on the $\mathbb{Z}^d$ lattice, studied by Kesten [7], Jarai [6], and others. One difference, however, is that while these authors considered the incipient infinite cluster at criticality, we will start from i.i.d. percolation below criticality (simply because $p_c = 1$ in our setting).

We will show, for the case of the ladder graph in Figure 1, that the limiting measure is indeed well defined (Section 2), and derive a useful Markov representation of this measure (Section 3). These results are extended to general one-dimensional periodic lattices in Section 4. Finally, in Section 5, we study how the model depends on $p$. It turns out that stochastic monotonicity in $p$ fails in general, but a weaker property holds such that the 'edge density', suitably defined, is an increasing function of $p$.

In a companion paper [2], the Markov representation played a crucial role in the analysis of a biased random walk on the infinite cluster arising in our model. The bias is governed by a drift parameter $\beta \geq 1$, where $\beta = 1$, and the main result in [2] is the derivation of an explicit critical value $\beta_c = \beta_c(p)$ such that the asymptotic speed to the right is strictly positive for $\beta \in (1, \beta_c)$, and zero for $\beta \geq \beta_c$. This is motivated by the works of Berger *et al.* [3] and Sznitman [8] on a similar random walk on the infinite cluster of i.i.d. percolation on $\mathbb{Z}^d$, where zero speed and positive speed regimes are found, but where the monotonicity property needed to separate these by a unique critical value $\beta_c$ was left open.

## 2. Conditional percolation on the ladder: existence of limiting measure

Consider the ladder graph $\mathcal{L} = (V, E)$ in Figure 1 with vertex set $V = \mathbb{Z} \times \{0, 1\}$ and $E$ the set of edges connecting vertices at Euclidean distance 1 from each other. We wish to consider i.i.d. bond percolation on $\mathcal{L}$ with edge probability $p \in (0, 1)$ conditioned on the event $B$ of having an 'open path from $-\infty$ to $\infty$'. More precisely, $B$ is the event that, for any $N > 0$, there exists an open path from some vertex with $x$-coordinate $-N$ to some vertex with $x$-coordinate $N$. As noted in Section 1, the existence of an infinite cluster has probability 0 for any $p \in (0, 1)$, whence the same holds for the event $B$. Making sense of conditioning on $B$ is therefore a nontrivial issue, and we will approach the problem via a limiting procedure. Thus, for $N_1, N_2 > 0$, let $B_{N_1,N_2}$ be the event that there exists an open path from some vertex with $x$-coordinate $-N_1$ to some vertex with $x$-coordinate $N_2$, and, for $p \in (0, 1)$, let $\mathrm{P}_{p,N_1,N_2}$ be the probability measure on $\{0, 1\}^E$ that arises by conditioning i.i.d. bond percolation with parameter $p$ on the event $B_{N_1,N_2}$. We have the following result, where convergence is in the product topology, meaning that the probability of any cylinder event converges.

**Theorem 2.1.** *For any $p \in (0, 1)$, the probability measures $\mathrm{P}_{p,N_1,N_2}$ converge weakly to a probability measure $\mathrm{P}_p$ on $\{0, 1\}^E$ as $N_1, N_2 \to \infty$.*

The proof proceeds by mimicking the disagreement percolation technique of van den Berg [9].

*Proof of Theorem 2.1.* Let $A \subset \{0, 1\}^E$ be any cylinder event; we need to prove that

$$\lim_{N_1,N_2\to\infty} \mathrm{P}_{p,N_1,N_2}(A)$$

exists. To this end, it is enough to show that, for any $\varepsilon > 0$, there exists an $M > 0$ such that

$$|\mathrm{P}_{p,N_1,N_2}(A) - \mathrm{P}_{p,N_1',N_2'}(A)| < \varepsilon \tag{2.1}$$

whenever $N_1$, $N_2$, $N_1'$, and $N_2'$ all exceed $M$. Fix $a > 0$ such that the cylinder event $A$ is defined solely in terms of edges with endpoints in $[-a, a] \times \{0, 1\}$.

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Let $X, X' \in \{0, 1\}^E$ be two independent random edge configurations with respective distributions $\mathrm{P}_{p,N_1,N_2}$ and $\mathrm{P}_{p,N_1',N_2'}$ with $N_1$, $N_2$, $N_1'$, and $N_2'$ all exceeding $a$. We will reveal $X$ and $X'$ sequentially as follows. First we peek at the values in $X$ and $X'$ of all edges sitting at or to the left of $x$-coordinate $-(\min(N_1, N_1'))$, and all edges sitting to the right of $x$-coordinate $\min(N_2, N_2')$. Next, we start examining the $X$ and $X'$ values of the remaining edges one at a time systematically going from left to right, meaning that we always choose the edge whose midpoint has the smallest $x$-coordinate, with the $y$-coordinate acting as tie-breaker. We do this until we encounter a vertical edge $e$ (i.e. $e = \langle\{x, 0\}, \{x, 1\}\rangle$ for some $x \in \mathbb{Z}$) such that $X(e) = X'(e) = 1$, and define $-K_1$ as the $x$-coordinate of the vertical edge where this happens. We then stop and start going from $x$-coordinate $\min(N_2, N_2')$ towards the left in the same manner, again stopping as soon as we encounter a vertical edge $e$ such that $X(e) = X'(e) = 1$, defining $K_2$ as the $x$-coordinate of the vertical edge where this happens. At this point, we pause to reflect on the conditional distribution of $X$ and $X'$ restricted to the not-yet-examined edges, i.e. the set of edges having at least one endpoint with $x$-coordinate strictly between $-K_1$ and $K_2$. A moment's thought reveals that, for both $X$ and $X'$, the conditional distribution of what happens in this interval is precisely i.i.d. bond percolation with parameter $p$ conditioned on the event $B_{K_1,K_2}$ of having an open path between $x$-coordinates $-K_1$ and $K_2$. The key observation here is that this conditional distribution is identical for $X$ and for $X'$, so if we define $D$ to be the event $\{-K_1 < -a,\ a < K_2\}$, it is clear that

$$\Pr(X \in A \mid D) = \Pr(X' \in A \mid D),$$

where Pr is used as a generic symbol for probability, in this case denoting the probability measure on the probability space on which $X$ and $X'$ are jointly constructed. Hence,

$$\begin{aligned}
\mathrm{P}_{p,N_1,N_2}(A) - \mathrm{P}_{p,N_1',N_2'}(A) &= \Pr(X \in A) - \Pr(X' \in A) \\
&= \Pr(X \in A, D) + \Pr(X \in A, \neg D) - \Pr(X' \in A, D) \\
&\quad - \Pr(X' \in A, \neg D) \\
&= \Pr(X \in A, \neg D) - \Pr(X' \in A, \neg D),
\end{aligned}$$

so that

$$|\mathrm{P}_{p,N_1,N_2}(A) - \mathrm{P}_{p,N_1',N_2'}(A)| \le \Pr(\neg D).$$

So the key to establishing the desired inequality (2.1) is to show that $\Pr(\neg D)$ can be made arbitrarily small by letting $N_1$, $N_2$, $N_1'$, and $N_2'$ all be large.

In order to bound $\Pr(\neg D)$, the following observation is crucial. For any $e \in E$, the conditional probability that $X(e) = 1$ given the $X$-status of all other edges is either 1 or $p$ depending on whether $e$ is pivotal or not for the existence of an open path between $x$-coordinates $-N_1$ and $N_2$. Hence, conditioning on some partial information of $X(E \setminus \{e\})$ makes the conditional probability that $X(e) = 1$ at least $p$. The analogous statement for $X'$ holds as well. Since $X$ and $X'$ are independent, we can deduce that each time in the above procedure that we inspect an edge $e$ for its $X$ and $X'$ values, the conditional probability that $X(e) = X'(e) = 1$ given everything we have seen so far is at least $p^2$. Hence,

$$\begin{aligned}
\Pr(\neg D) &\le \Pr(K_1 < a) + \Pr(K_2 < a) \\
&\le (1 - p^2)^{\min(N_1, N_1') - a} + (1 - p^2)^{\min(N_2, N_2') - a} \\
&\le 2(1 - p^2)^{M - a},
\end{aligned}$$

which tends to 0 as $M \to \infty$, so (2.1) is established for large enough $M$, and the proof is complete.

Equipped with Theorem 2.1, we obtain two easy corollaries. The first verifies that the event $B$ of an open path from $-\infty$ to $\infty$ indeed happens almost surely (a.s.) under the limiting measure $\mathrm{P}_p$.

**Corollary 2.1.** *For any* $p \in (0, 1)$, $\mathrm{P}_p(B) = 1$.

*Proof.* For each $k > 0$, we have $\mathrm{P}_{p,N,N}(B_k) = 1$ whenever $N > k$. Hence,

$$\mathrm{P}_p(B_k) = \lim_{N\to\infty} \mathrm{P}_{p,N,N}(B_k) = 1,$$

and since $B = \bigcap_{k=1}^{\infty} B_k$, the result follows.

In the next corollary, we obtain translation invariance of $\mathrm{P}_p$, meaning that, for any $k \geq 1$, any

$$e_1 = \langle\{x_1, y_1\}, \{x_1', y_1'\}\rangle, \dots, e_k = \langle\{x_k, y_k\}, \{x_k', y_k'\}\rangle \in E,$$

any $z \in \mathbb{Z}$, and any $i_1, \dots, i_k \in \{0, 1\}$, an $\{0, 1\}^E$-valued random object $X$ with distribution $\mathrm{P}_p$ satisfies

$$\begin{aligned}\mathrm{P}_p(&X(\langle\{x_1, y_1\}, \{x_1', y_1'\}\rangle) = i_1, \dots, X(\langle\{x_k, y_k\}, \{x_k', y_k'\}\rangle) = i_k)\\ &= \mathrm{P}_p(X(\langle\{x_1+z, y_1\}, \{x_1'+z, y_1'\}\rangle) = i_1, \dots, X(\langle\{x_k+z, y_k\}, \{x_k'+z, y_k'\}\rangle) = i_k).\end{aligned}$$

**Corollary 2.2.** *For any* $p \in (0, 1)$, $\mathrm{P}_p$ *is translation invariant.*

*Proof.* Theorem 2.1 implies that $\lim_{N\to\infty} \mathrm{P}_{p,N-z,N+z}$ gives rise to the same limiting measure regardless of the choice of $z$. The corollary follows easily.

## 3. Conditional percolation on the ladder: Markov structure

Our task in this section will be to establish the Markov structure of $\mathrm{P}_p$ that will be crucial to the rest of our study. We begin with some lemmas that will be needed concerning stochastic domination and conditional probabilities.

We equip the edge configuration space $\{0, 1\}^E$ with the coordinatewise partial order '$\preceq$', meaning that, for $\xi, \xi' \in \{0, 1\}^E$, $\xi \preceq \xi'$ if $\xi(e) \leq \xi'(e)$ for every $e \in E$. A function $f\colon \{0, 1\}^E \to \mathbb{R}$ is said to be increasing if $f(\xi) \leq f(\xi')$ whenever $\xi \preceq \xi'$.

**Definition 3.1.** Let $\mu_1$ and $\mu_2$ be two probability measures on $\{0, 1\}^E$. We say that $\mu_1$ is *stochastically dominated* by $\mu_2$, written $\mu_1 \preceq_{\mathrm{d}} \mu_2$, if, for every bounded increasing function $f\colon \{0, 1\}^E \to \mathbb{R}$, we have $\mu_1(f) \leq \mu_2(f)$.

By the well-known Strassen's theorem (see, e.g. [4]), $\mu_1 \preceq_{\mathrm{d}} \mu_2$ is equivalent to the existence of a coupling of two $\{0, 1\}^E$-valued random objects $X_1$ and $X_2$, with distributions $\mu_1$ and $\mu_2$, respectively, such that $\Pr(X_1 \preceq X_2) = 1$.

**Lemma 3.1.** *For any* $p \in (0, 1)$, $\mathrm{P}_p$ *stochastically dominates i.i.d.* $(p)$ *bond percolation on* $\mathcal{L}$. *For any* $N$, *the same holds with* $\mathrm{P}_{p,N,N}$ *in place of* $\mathrm{P}_p$.

*Proof.* Write $\mu_p$ for the probability measure on $\{0, 1\}^E$ corresponding to i.i.d. bond percolation on $\mathcal{L}$. The Harris–FKG inequality (see, e.g. [4]) tells us that, for any two bounded increasing functions $f, g\colon \{0, 1\}^E \to \mathbb{R}$, we have $\mu_p(f)\mu_p(g) \leq \mu_p(fg)$. In particular, when $g$ is assumed to be $\{0, 1\}$-valued, we have $\mu_p(f) \leq \mu_p(f \mid g = 1)$. For the special case where $g$ is the indicator of the event $B_{N,N}$, this yields $\mu_p(f) \leq \mathrm{P}_{p,N,N}(f)$. Hence, $\mu_p \preceq \mathrm{P}_{p,N,N}$, and since stochastic domination is preserved under weak limits, we also have $\mu_p \preceq \mathrm{P}_p$.

For the next lemma, we define an edge $e \in E$ to be *pivotal*, relative to a configuration $\xi \in \{0,1\}^{E\setminus\{e\}}$ if (i) the set of open edges in $\xi$ does *not* contain an infinite path from $-\infty$ to $\infty$, while (ii) adding $e$ would create such a path. The configuration $\xi$ is said to be *permissible* if either it contains a path from $-\infty$ to $\infty$ or (ii) holds.

**Lemma 3.2.** $\mathrm{P}_p$ *admits conditional probabilities such that, for each* $e \in E$ *and each* $\xi \in \{0,1\}^{E\setminus\{e\}}$,

$$\mathrm{P}_p(e \text{ is open} \mid \xi) = \begin{cases} 1 & \text{if } e \text{ is pivotal}, \\ p & \text{otherwise}. \end{cases}$$

*Proof.* For $\xi \in \{0,1\}^{E\setminus\{e\}}$, let $A_{\xi,M}$ denote the event that the edge configuration on $\{0,1\}^{E\setminus\{e\}}$ restricted to $x$-coordinates between $-M$ and $M$ agrees with $\xi$. What we need to show is that, for $\mathrm{P}_p$-almost everywhere $\xi$, we have

$$\lim_{M\to\infty} \mathrm{P}_p(e \text{ is open} \mid A_{\xi,M}) = \begin{cases} 1 & \text{if } e \text{ is pivotal for } \xi, \\ p & \text{otherwise}. \end{cases} \tag{3.1}$$

Fix $e \in E$. If $e$ is a vertical edge, define $x$ to be the common $x$-coordinate of its two endpoints, while if $e$ is a horizontal edge, take $x$ to be the $x$-coordinate of its leftmost endpoint. For $\xi \in \{0,1\}^{E\setminus\{e\}}$ and $N > |x|$, define $N$*-permissibility* of $\xi$ and $N$*-pivotality* of $e$ (with respect to $\xi$) analogously to permissibility and pivotality but with a path between $x$-coordinates $-N$ and $N$ in place of one between $-\infty$ and $\infty$. It is immediate from the definition of $\mathrm{P}_{p,N,N}$ that

$$\mathrm{P}_{p,N,N}(e \text{ is open} \mid \xi) = \begin{cases} 1 & \text{if } e \text{ is } N\text{-pivotal}, \\ p & \text{otherwise}, \end{cases} \tag{3.2}$$

whenever $\xi$ is $N$-permissible. Next, define $*$*-permissibility* and $*$*-pivotality* analogously to permissibility and pivotality but with a path between the closest open (in $\xi$) vertical edge strictly to the right of $e$ and the closest open (in $\xi$) vertical edge strictly to the left of $e$ in place of one between $-\infty$ and $\infty$. For $N > |x| + 1$, define $D_{x,N}$ to be the event that at least one vertical edge with $x$-coordinate strictly between $-N$ and $x$ is open and at least one vertical edge with $x$-coordinate strictly between $x$ and $N$ is open. Also, define

$$D_{x,\infty} = \bigcup_{N=|x|+1}^{\infty} D_{x,N}.$$

By Lemma 3.1 and the Borel–Cantelli lemma,

$$\mathrm{P}_p(D_{x,\infty}) = 1. \tag{3.3}$$

Note, crucially, that on the event $D_{x,N}$, $\mathrm{P}_{p,N,N}$-a.s., $e$ is $N$-pivotal if and only if it is $*$-pivotal (we need look no further than to the nearest open vertical edge to the left and the nearest open vertical edge to the right to determine $N$-pivotality). Likewise, due to (3.3), $\mathrm{P}_p$-a.s., $e$ is pivotal if and only if it is $*$-pivotal. Hence, showing (3.1) is the same as showing, for $\mathrm{P}_p$-almost everywhere $\xi \in \{0,1\}^{E\setminus\{e\}}$ on the event $D_{x,\infty}$, that

$$\lim_{M\to\infty} \mathrm{P}_p(e \text{ is open} \mid A_{\xi,M}) = \begin{cases} 1 & \text{if } e \text{ is } *\text{-pivotal for } \xi, \\ p & \text{otherwise}. \end{cases} \tag{3.4}$$

Pick such a $\xi$, and pick $M = M(\xi) < \infty$ large enough so that $\xi \in D_{x,M}$. We can then learn from $A_{\xi,M}$ whether or not $e$ is $*$-pivotal for $\xi$. For any $N > M$, we thus find from (3.2) that

$$\mathrm{P}_{p,N,N}(e \text{ is open} \mid A_{\xi,M}) = \begin{cases} 1 & \text{if } A_{\xi,M} \text{ is such that } e \text{ is } *\text{-pivotal,} \\ p & \text{otherwise,} \end{cases}$$

and sending $N \to \infty$ yields

$$\mathrm{P}_{p}(e \text{ is open} \mid A_{\xi,M}) = \begin{cases} 1 & \text{if } A_{\xi,M} \text{ is such that } e \text{ is } *\text{-pivotal,} \\ p & \text{otherwise.} \end{cases}$$

Since this holds for all sufficiently large $M$ (depending on $\xi$), we obtain (3.4), and (3.1) follows as desired.

The next lemma will extend Lemma 3.2 from single-edge conditional probabilities to conditional distributions for finite edge sets. For a finite $F \subset E$ and two configurations $\xi \in \{0,1\}^{E\setminus F}$ and $\eta \in \{0,1\}^F$, write $(\xi \vee \eta)$ for the $\{0,1\}^E$-valued configuration that agrees with $\xi$ on $E \setminus F$ and with $\eta$ on $F$. We generalize the notion of permissibility by saying that $\xi \in \{0,1\}^{E\setminus F}$ is permissible if there exists some $\eta \in \{0,1\}^F$ such that $(\xi \vee \eta) \in B$, where $B$, as before, is the event that there exists an open path from $-\infty$ to $\infty$.

For $\xi \in \{0,1\}^{E\setminus F}$ permissible, define the measure $\mathrm{P}_{p,F,\xi}$ on $\{0,1\}^E$ by setting, for each $\eta \in \{0,1\}^F$,

$$\mathrm{P}_{p,F,\xi}(\eta) = \frac{\mathbf{1}_{\{(\xi\vee\eta)\in B\}}}{Z_{p,F,\xi}} \prod_{e\in F} p^{\eta(e)}(1-p)^{1-\eta(e)}, \tag{3.5}$$

where $\mathbf{1}$ denotes an indicator function and $Z_{p,F,\xi}$ is a normalizing constant, making $\mathrm{P}_{p,F,\xi}$ a probability measure. In other words, $\mathrm{P}_{p,F,\xi}$ means letting the edges in $F$ independently be open with probability $p$, conditional on $(\xi \vee \eta)$ containing an open path from $-\infty$ to $\infty$.

**Lemma 3.3.** $\mathrm{P}_p$ *admits conditional probabilities such that, for a corresponding* $\{0,1\}^E$*-valued random configuration* $X$*, we have, for any finite* $F \subset E$*, any permissible* $\xi \in \{0,1\}^{E\setminus F}$*, and any* $\eta \in \{0,1\}^E$*,*

$$\mathrm{P}_p(X(F) = \eta \mid X(E \setminus F) = \xi) = \mathrm{P}_{p,F,\xi}(\eta). \tag{3.6}$$

Note that, by Corollary 2.1, the restriction to $\xi$ being permissible involves no essential loss of generality. The lemma can be proved either by generalizing the argument in the proof of Lemma 3.2, or by the following argument, modeled after [5, Proof of Lemma 2.4].

*Proof of Lemma 3.3.* Pick $X \in \{0,1\}^E$ according to $\mathrm{P}_p$, set $\xi = X(E \setminus F)$, and define a $\{0,1\}^F$-valued Markov chain $\{X_0^\xi, X_1^\xi, \dots\}$ by setting $X_0^\xi = X(F)$ and giving it a transition mechanism as follows. At each time $n$, an edge $e \in F$ is chosen at random (uniform distribution) and $X_n^\xi(e')$ is set equal to $X_{n-1}^\xi(e')$ for all $e' \in F \setminus \{e\}$. If it turns out that setting $X_n^\xi(e) = 0$ would result in $(\xi \vee X_n^\xi) \notin B$ then we set $X_n^\xi(e) = 1$, while otherwise we set

$$X_n^\xi(e) = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1-p. \end{cases}$$

By Lemma 3.2, it is easy to see by checking local equilibrium that this transition mechanism preserves the initial distribution, so the Markov chain is stationary. By the same argument we see that $\mathrm{P}_{p,F,\xi}$ is a stationary distribution for the chain (Monte Carlo Markov chain aficionados may note that the chain we have defined is in fact a Gibbs sampler for $\mathrm{P}_{p,F,\xi}$). That the chain has a unique stationary distribution follows from the observation that the maximal state $\eta$, given by $\eta(e) = 1$ for all $e \in F$, can be reached in a finite number of steps from any other state (simply by turning on edges one at a time). Hence, the distribution of $X_n^\xi$ converges to $\mathrm{P}_{p,F,\xi}$ as $n \to \infty$, but since the chain was stationary, $X_0^\xi$ must therefore have distribution $\mathrm{P}_{p,F,\xi}$, and (3.6) follows.

After these preparatory lemmas, we are finally ready to approach the main issue of this section: Markov properties of $\mathrm{P}_p$. We begin with a negative observation. In order for $\mathrm{P}_p$ to exhibit Markovianness in the most direct sense, we would want the conditional distribution of everything to the right of $x$-coordinate 0 given everything to the left of (and including) $x$-coordinate 0 to depend on the latter only via the status of edges with $x$-coordinates in some fixed finite window $[-K, 0]$. The following example shows that this is not the case. Fix $K$, and let $A_K$ be the event that

$$\begin{gathered} X(\langle\{-(i+1), j\}, \{-i, j\}\rangle) = 1 \quad \text{for } i = 0, \dots, K \text{ and } j = 0, 1, \\ X(\langle\{-i, 0\}, \{-i, 1\}\rangle) = 0 \quad \text{for } i = 0, \dots, K, \\ X(\langle\{-(K+1), 0\}, \{-(K+1), 1\}\rangle) = 1. \end{gathered}$$

In other words, $A_K$ stipulates that we have two parallel open paths from $x$-coordinate 0 and backwards with no connection to each other until at $x$-coordinate $-(K+1)$. Also, let $A_K^*$ be the same event as $A_K$ except that we insist that the single edge $\langle\{-(K+1), 0\}, \{-K, 0\}\rangle$ takes value 0 rather than 1; see Figure 2. Note that $A_K$ and $A_K^*$ imply identical edge configurations on $[-K, 0]$. It follows easily from Lemma 3.3 that $\mathrm{P}_p(A_K)$ and $\mathrm{P}_p(A_K^*)$ are both strictly positive, while

$$\mathrm{P}(X\langle\{0, 1\}, \{1, 1\}\rangle = 0 \mid A_K) > 0 \tag{3.7}$$

and

$$\mathrm{P}(X\langle\{0, 1\}, \{1, 1\}\rangle = 0 \mid A_K^*) = 0. \tag{3.8}$$

Since $K$ was arbitrary, the discrepancy between the two conditional probabilities in (3.7) and (3.8) shows that the random edge configuration $X$ does not in itself exhibit any finite-range Markov property.

Not all is lost, however, and it turns out that a useful Markov chain representation can be obtained by encoding a little bit of nonlocal information at each $x$-coordinate. (The idea that such encodings may be useful in percolation theory goes back at least to [5].) For fixed $i \in \mathbb{Z}$, partition the edge set $E$ into $E = E^{i,-} \cup E^{i,+}$, where $E^{i,-}$ is the set of edges both of whose endpoints have $x$-coordinates not exceeding $i$, and $E^{i,+} = E \setminus E^{i,-}$. Given $X \in \{0, 1\}^E$, we say that a vertex $\{i, j\}$ is *backwards communicating* if it is connected to $-\infty$ via a path that is completely contained in $E^{i,-}$ and all of whose edges are open in $X$. Define the


FIGURE 2: The events $A_K$ (*left*) and $A_K^*$ (*right*). The edge $\langle\{0, 1\}, \{1, 1\}\rangle$ (*dashed line*) next to $A_K^*$ must be open for an open bi-infinite path to exist.

$\{\texttt{00}, \texttt{01}, \texttt{10}, \texttt{11}\}$-valued process $\{T_i\}_{i\in\mathbb{Z}}$ by setting, for each $i \in \mathbb{Z}$,

$$T_i = \begin{cases} \texttt{00} & \text{if neither } \{i,0\} \text{ nor } \{i,1\} \text{ is backwards communicating,} \\ \texttt{01} & \text{if } \{i,1\} \text{ but not } \{i,0\} \text{ is backwards communicating,} \\ \texttt{10} & \text{if } \{i,0\} \text{ but not } \{i,1\} \text{ is backwards communicating,} \\ \texttt{11} & \text{if both } \{i,0\} \text{ and } \{i,1\} \text{ are backwards communicating.} \end{cases} \tag{3.9}$$

Note that Corollary 2.1 implies that $\mathrm{P}_p(T_i = \texttt{00}) = 0$ for any $i$; the first line of (3.9) is included for completeness only.

The usefulness of $\{T_i\}_{i\in\mathbb{Z}}$ for studying the percolation process governed by $\mathrm{P}_p$ comes from the following result, and from Theorem 3.2, below.

**Theorem 3.1.** *$\{T_i\}_{i\in\mathbb{Z}}$ is a time-homogeneous Markov chain.*

*Proof.* Due to Corollary 2.2 on translation invariance, and the fact that $\{T_i\}_{i\in\mathbb{Z}}$ is obtained from $X$ in a way that commutes with translation, it is enough to show that $\{T_i\}_{i\in\mathbb{Z}}$ is a Markov chain; time homogeneity is then automatic. Thus, what we need to show is that $\mathrm{P}_p$ admits conditional probabilities such that, for any $\dots, T_{i-2}, T_{i-1}$ and $\texttt{cd} \in \{\texttt{01}, \texttt{10}, \texttt{11}\}$, we have

$$\mathrm{P}_p(T_i = \texttt{cd} \mid \dots, T_{i-2}, T_{i-1}) = \mathrm{P}_p(T_i = \texttt{cd} \mid T_{i-1}). \tag{3.10}$$

Define $E^i = E^{i,-} \setminus E^{i-1,-}$, so that in other words $E^i$ consists of the three edges $\langle\{i-1,0\},\{i,0\}\rangle$, $\langle\{i-1,1\},\{i,1\}\rangle$, and $\langle\{i,0\},\{i,1\}\rangle$. Note that, given $T_{i-1} \in \{\texttt{01}, \texttt{10}, \texttt{11}\}$ and the status $X(E^i)$ of these three edges, we can read off the value of $T_i$ from the formula

$$T_i = \texttt{cd},$$

where

$$\texttt{c} = \begin{cases} 1 & \text{if } X(E^i) \text{ connects } \{i,0\} \text{ to at least one vertex} \\ & v \in \{\{i-1,0\},\{i-1,1\}\} \text{ that is designated by } T_{i-1} \text{ to be} \\ & \text{backwards communicating,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\texttt{d} = \begin{cases} 1 & \text{if } X(E^i) \text{ connects } \{i,1\} \text{ to at least one vertex} \\ & v \in \{\{i-1,0\},\{i-1,1\}\} \text{ that is designated by } T_{i-1} \text{ to be} \\ & \text{backwards communicating,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, in order to establish (3.10), it is enough to show, for any $\eta \in \{0,1\}^{E^i}$, that

$$\mathrm{P}_p(X(E^i) = \eta \mid \dots, T_{i-2}, T_{i-1}) = \mathrm{P}_p(X(E^i) = \eta \mid T_{i-1}). \tag{3.11}$$

Since $\{\dots, T_{i-2}, T_{i-1}\}$ is a function of $X(E^{i-1,-})$, (3.11) follows if we can show that

$$\mathrm{P}_p(X(E^i) = \eta \mid X(E^{i-1,-})) = \mathrm{P}_p(X(E^i) = \eta \mid T_{i-1}). \tag{3.12}$$

Let us consider the case in which

$$T_{i-1} = \texttt{10} \tag{3.13}$$

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(the cases $T_{i-1} = \texttt{01}$ and $T_{i-1} = \texttt{11}$ follow analogously). Let $\xi, \xi' \in \{0,1\}^{E^{i-1,-}}$ be two configurations such that each of the statements $X(E^{i-1,-}) = \xi$ and $X(E^{i-1,-}) = \xi'$ imply that $T_{i-1} = \texttt{10}$. Lemma 3.1 implies that $X(E^{i-1,-})$ contains at least one open vertical edge $\mathrm{P}_p$-a.s., so there is no loss of generality in assuming that both $\xi$ and $\xi'$ contain such an edge, and we can find a $K$ such that both $\xi$ and $\xi'$ contain such an edge to the right of $x$-coordinate $K$.

For (3.12), what we need is to show that its left-hand side takes the same value $\mathrm{P}_p$-a.s. on the event $T_{i-1} = \texttt{10}$, which follows if we can show that

$$\mathrm{P}_p(X(E^i) = \eta \mid X(E^{i-1,-}) = \xi) = \mathrm{P}_p(X(E^i) = \eta \mid X(E^{i-1,-}) = \xi') \tag{3.14}$$

for any $\xi$ and $\xi'$ as above. Analogously to the reasoning in the proof of Lemma 3.2, define (for $M < i-1$) $A_{\xi,M,i-1}$ as the event that $X(E^{i-1,-} \setminus E^{M,-}) = \xi(E^{i-1,-} \setminus E^{M,-})$, and $A_{\xi',M,i-1}$ similarly. Showing (3.14) is tantamount to showing that

$$\lim_{M\to-\infty} \mathrm{P}_p(X(E^i) = \eta \mid A_{\xi,M,i-1}) = \lim_{M\to-\infty} \mathrm{P}_p(X(E^i) = \eta \mid A_{\xi',M,i-1}),$$

for which it is of course enough to show that

$$\mathrm{P}_p(X(E^i) = \eta \mid A_{\xi,M,i-1}) = \mathrm{P}_p(X(E^i) = \eta \mid A_{\xi',M,i-1}) \quad \text{for all } M < K. \tag{3.15}$$

So take $M < K$, and then $N > \max\{|M|, |i|\}$, and let us for the moment consider the measure $\mathrm{P}_{p,N,N}$ in place of $\mathrm{P}_p$. On the event $A_{\xi,M,i-1}$ we find (due to our choice (3.13)) that $\{i-1, 1\}$ but not $\{i-1, 0\}$ is connected to the first vertical open edge to the right of $x$-coordinate $i-1$, and, therefore, $\mathrm{P}_{p,N,N}$-a.s. on the same event that $\{i-1,1\}$ but not $\{i-1,0\}$ is connected to $x$-coordinate $-N$. It follows from the definition of $\mathrm{P}_{p,N,N}$ that its conditional distribution of $X(E^{i+1,+})$ given $A_{\xi,M,i-1}$ is simply i.i.d. $(p)$ percolation conditioned on the existence of an open path from $\{i-1,1\}$ to some vertex with $x$-coordinate $M$. The same argument holds with $A_{\xi',M,i-1}$ in place of $A_{\xi,M,i-1}$, so in particular $X(E^{i+1,+})$ has the same $\mathrm{P}_{p,N,N}$-conditional distribution regardless of which of the two events we condition on. Restricting to $E^i$ gives

$$\mathrm{P}_{p,N,N}(X(E^i) = \eta \mid A_{\xi,M,i-1}) = \mathrm{P}_{p,N,N}(X(E^i) = \eta \mid A_{\xi',M,i-1}). \tag{3.16}$$

Since all events in (3.16) are cylinder events, Theorem 2.1 allows us to pass to the limit with $N \to \infty$ and still have equality, obtaining (3.15). Backtracking our argument, we recall that this implies (3.14), which implies (3.12), which implies (3.11), which implies (3.10), so the proof is complete.

We will soon calculate the transition matrix for the chain $\{T_i\}_{i\in\mathbb{Z}}$. Together with the following result, which shows how to obtain the percolation process $X \in \{0,1\}^E$ from $\{T_i\}_{i\in\mathbb{Z}}$, this yields an efficient way to study the latter—both analytically and by way of computer simulation.

Recall that, given $T_{i-1}$ and $X(E^i)$, we know the value of $T_i$. Given $T_{i-1} = \texttt{ab} \in \{\texttt{01},\texttt{10},\texttt{11}\}$ and $T_i = \texttt{cd} \in \{\texttt{01},\texttt{10},\texttt{11}\}$, we call a configuration $\eta \in \{0,1\}^{E^i}$ $T_{i-1}, T_i$-*compatible* if $T_{i-1} = \texttt{ab}$ and $X(E^i) = \eta$ yields $T_i = \texttt{cd}$. Note that, for $T_{i-1} = \texttt{01}$ and $T_i = \texttt{10}$, or vice versa, there is no $T_{i-1}, T_i$-compatible $\eta \in \{0,1\}^{E^i}$. For all other choices, define, analogously to (3.5), the probability measure $\mathrm{P}_{p,i,T_{i-1},T_i}$ on $\{0,1\}^{E^i}$ by setting, for each $\eta \in \{0,1\}^{E^i}$,

$$\mathrm{P}_{p,i,T_{i-1},T_i}(\eta) = \frac{\mathbf{1}_{\{\eta \text{ is } T_{i-1},T_i\text{-compatible}\}}}{Z_{p,i,T_{i-1},T_i}} \prod_{e\in E^i} p^{\eta(e)}(1-p)^{1-\eta(e)},$$

where $Z_{p,i,T_{i-1},T_i}$ is a normalizing constant, making $\mathrm{P}_{p,i,T_{i-1},T_i}$ a probability measure.

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**Theorem 3.2.** *The conditional distribution of the percolation process $X \in \{0, 1\}^E$ given the Markov chain $\{T_i\}_{i\in\mathbb{Z}}$ is*

$$\prod_{j\in\mathbb{Z}} \mathrm{P}_{p,j,T_{j-1},T_j} . \tag{3.17}$$

In other words, given $\{T_i\}_{i\in\mathbb{Z}}$, it is for each $j$ the case that $X(E^j)$ has distribution $\mathrm{P}_{p,j,T_{j-1},T_j}$, with independence for different $j$s.

*Proof of Theorem 3.2.* To show that $X(E^j)$ has distribution $\mathrm{P}_{p,j,T_{j-1},T_j}$ conditional on $\{T_i\}_{i\in\mathbb{Z}}$, it is obviously enough to show that it has conditional distribution even if we condition further on $X(E \setminus E^j)$. But this is the same as conditioning only on $X(E \setminus E^j)$ and on $T_j$, because given these, the remaining $\{T_i\}_{i\in\mathbb{Z}\setminus\{j\}}$ can be reconstructed.

Now, if we had conditioned only on $X(E \setminus E^j)$ (and not on $T_j$ as well) then Lemma 3.3 would have given us the conditional distribution of $X(E^j)$ to be $\mathrm{P}_{p,E^j,\xi}$ (as defined in (3.5)) with $\xi \in \{0, 1\}^{E\setminus E^j}$. Conditioning further on $T_j$ means simply conditioning $\mathrm{P}_{p,E^j,\xi}$ on the event that $X(E^j)$ is $T_{j-1}, T_j$-compatible. This yields precisely the desired distribution $\mathrm{P}_{p,j,T_{j-1},T_j}$.

From here, the desired product form (3.17) follows using the fact that we obtained the same distribution $\mathrm{P}_{p,j,T_{j-1},T_j}$ even when conditioning further on $X(E \setminus E^j)$.

It remains to derive expressions for the elements of the transition matrix governing $\{T_i\}_{i\in\mathbb{Z}}$. The chain is $\{\mathtt{00}, \mathtt{01}, \mathtt{10}, \mathtt{11}\}$-valued, but since $\mathrm{P}_p(T_i = \mathtt{00}) = 0$ for any $i$, it is in effect a three-state Markov chain, and we may denote the $3 \times 3$ transition matrix by

$$\boldsymbol{p} = \begin{pmatrix} p_{\mathtt{01},\mathtt{01}} & p_{\mathtt{01},\mathtt{10}} & p_{\mathtt{01},\mathtt{11}} \\ p_{\mathtt{10},\mathtt{01}} & p_{\mathtt{10},\mathtt{10}} & p_{\mathtt{10},\mathtt{11}} \\ p_{\mathtt{11},\mathtt{01}} & p_{\mathtt{11},\mathtt{10}} & p_{\mathtt{11},\mathtt{11}} \end{pmatrix}.$$

We have already noted that $T_i = \mathtt{01}$ cannot be followed by $T_{i+1} = \mathtt{10}$ or vice versa; hence, $p_{\mathtt{01},\mathtt{10}} = p_{\mathtt{10},\mathtt{01}} = 0$. Furthermore, symmetry of the model under interchange of $y$-coordinate 0 and $y$-coordinate 1 yields $p_{\mathtt{01},\mathtt{01}} = p_{\mathtt{10},\mathtt{10}}$ and $p_{\mathtt{11},\mathtt{01}} = p_{\mathtt{11},\mathtt{10}}$. Hence,

$$\boldsymbol{p} = \begin{pmatrix} p_{\mathtt{01},\mathtt{01}} & p_{\mathtt{01},\mathtt{10}} & p_{\mathtt{01},\mathtt{11}} \\ p_{\mathtt{10},\mathtt{01}} & p_{\mathtt{10},\mathtt{10}} & p_{\mathtt{10},\mathtt{11}} \\ p_{\mathtt{11},\mathtt{01}} & p_{\mathtt{11},\mathtt{10}} & p_{\mathtt{11},\mathtt{11}} \end{pmatrix} = \begin{pmatrix} 1 - p_{\mathtt{01},\mathtt{11}} & 0 & p_{\mathtt{01},\mathtt{11}} \\ 0 & 1 - p_{\mathtt{01},\mathtt{11}} & p_{\mathtt{01},\mathtt{11}} \\ p_{\mathtt{11},\mathtt{01}} & p_{\mathtt{11},\mathtt{01}} & 1 - 2p_{\mathtt{11},\mathtt{01}} \end{pmatrix}.$$

(It may be noted that this chain is reversible, though we will not be using this fact.) It only remains to calculate $p_{\mathtt{01},\mathtt{11}}$ and $p_{\mathtt{11},\mathtt{01}}$. This is readily done by the following device, which is a variant of Theorem 3.2. Given $T_{i-1} = \mathtt{ab} \in \{\mathtt{01}, \mathtt{10}, \mathtt{11}\}$ and $T_{i+1} = \mathtt{cd} \in \{\mathtt{01}, \mathtt{10}, \mathtt{11}\}$, we call a configuration $\eta \in \{0, 1\}^{E^i\cup E^{i+1}}$ $T_{i-1}, T_{i+1}$-*dicompatible* if $T_{i-1} = \mathtt{ab}$ and $X(E^i \cup E^{i+1}) = \eta$ yields $T_{i+1} = \mathtt{cd}$.

**Lemma 3.4.** *The conditional distribution of $X(E^i \cup E^{i+1})$ given $T_{i-1}$ and $T_{i+1}$ equals the probability measure $\mathrm{P}_{p,i,2,T_{i-1},T_{i+1}}$ on $\{0, 1\}^{E^i\cup E^{i+1}}$ defined by setting, for each $\eta \in \{0, 1\}^{E^i\cup E^{i+1}}$,*

$$\mathrm{P}_{p,i,2,T_{i-1},T_{i+1}}(\eta) = \frac{\mathbf{1}_{\{\eta \text{ is } T_{i-1},T_{i+1}\text{-dicompatible}\}}}{Z_{p,i,2,T_{i-1},T_{i+1}}} \prod_{e\in E^i} p^{\eta(e)}(1 - p)^{1-\eta(e)},$$

*where, as usual, $Z_{p,i,2,T_{i-1},T_{i+1}}$ is a normalizing constant.*

*Proof.* This follows similarly to the proof of Theorem 3.2.

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The transition probabilities $p_{\texttt{01},\texttt{11}}$ and $p_{\texttt{11},\texttt{01}}$ can now be obtained as follows. Given $T_{i-1}$, $T_{i+1}$, and $X(E^i \cup E^{i+1})$, we can read off the value of $T_i$. Thus, Lemma 3.4 allows us to calculate $\mathrm{P}_p(T_i = \texttt{cd} \mid T_{i-1} = \texttt{ab},\ T_{i+1} = \texttt{ef})$ for any $\texttt{ab}, \texttt{cd}, \texttt{ef} \in \{\texttt{01}, \texttt{10}, \texttt{11}\}$. In particular, we obtain expressions (as a function of $p$) for the ratios

$$\frac{\mathrm{P}_p(T_i = \texttt{11} \mid T_{i-1} = \texttt{01},\ T_{i+1} = \texttt{01})}{\mathrm{P}_p(T_i = \texttt{01} \mid T_{i-1} = \texttt{01},\ T_{i+1} = \texttt{01})} \tag{3.18}$$

and

$$\frac{\mathrm{P}_p(T_i = \texttt{11} \mid T_{i-1} = \texttt{11},\ T_{i+1} = \texttt{01})}{\mathrm{P}_p(T_i = \texttt{01} \mid T_{i-1} = \texttt{11},\ T_{i+1} = \texttt{01})}. \tag{3.19}$$

The ratios in (3.18) and (3.19), which we may denote by $\alpha = \alpha(p)$ and $\beta = \beta(p)$, respectively, can alternatively be written as

$$\alpha = \frac{p_{\texttt{01},\texttt{11}}\, p_{\texttt{11},\texttt{01}}}{p_{\texttt{01},\texttt{01}}\, p_{\texttt{01},\texttt{01}}} = \frac{p_{\texttt{01},\texttt{11}}\, p_{\texttt{11},\texttt{01}}}{(1 - p_{\texttt{11},\texttt{01}})^2} \tag{3.20}$$

and

$$\beta = \frac{p_{\texttt{11},\texttt{11}}\, p_{\texttt{11},\texttt{01}}}{p_{\texttt{11},\texttt{01}}\, p_{\texttt{01},\texttt{01}}} = \frac{1 - 2p_{\texttt{11},\texttt{01}}}{1 - p_{\texttt{01},\texttt{11}}}. \tag{3.21}$$

These are two equations for the two unknowns $p_{\texttt{01},\texttt{11}}$ and $p_{\texttt{11},\texttt{01}}$. Solving for $p_{\texttt{11},\texttt{01}}$ in (3.20) and substituting into (3.21) gives a quadratic for $p_{\texttt{01},\texttt{11}}$, and ignoring false (negative) solutions gives

$$p_{\texttt{01},\texttt{11}} = \frac{1}{2p}\left(2p^2 - 1 + \sqrt{1 + 4p^2 - 8p^3 + 4p^4}\right)$$

and

$$p_{\texttt{11},\texttt{01}} = \frac{1}{4(1-p)}\left(2(1-p) - (3-2p)(1 + 2p - 2p^2 - \sqrt{1 + 4p^2 - 8p^3 + 4p^4}\right)).$$

It is readily checked that $p_{\texttt{01},\texttt{11}}$ and $p_{\texttt{11},\texttt{01}}$ are both strictly positive for $p \in (0, 1)$, making the chain irreducible. Its stationary distribution is readily calculated, and is given by

$$\pi = \{\pi_{\texttt{01}}, \pi_{\texttt{10}}, \pi_{\texttt{11}}\} = \left\{ \frac{p_{\texttt{11},\texttt{01}}}{2p_{\texttt{11},\texttt{01}} + p_{\texttt{01},\texttt{11}}}, \frac{p_{\texttt{11},\texttt{01}}}{2p_{\texttt{11},\texttt{01}} + p_{\texttt{01},\texttt{11}}}, \frac{p_{\texttt{01},\texttt{11}}}{2p_{\texttt{11},\texttt{01}} + p_{\texttt{01},\texttt{11}}} \right\}.$$

For later purposes (Section 5), we finally record that $\lim_{p\to 0} p_{\texttt{01},\texttt{11}} = 0$, $\lim_{p\to 0} p_{\texttt{11},\texttt{01}} = \frac{1}{2}$, and (consequently) $\lim_{p\to 0} \pi_{\texttt{01}} = \frac{1}{2}$.

## 4. More general graphs

The idea of conditioning i.i.d. percolation on the existence of a bi-infinite open path and finding a Markov chain representation for the resulting model can be generalized substantially beyond the infinite ladder $\mathcal{L}$ considered in previous sections. The natural generality, we believe, is to consider the class of graphs arising from the following definition (with only the additional assumption of connectivity).

**Definition 4.1.** A graph $\mathcal{G} = (V, E)$ is said to be a *one-dimensional periodic lattice* if

(i) $V = \mathbb{Z} \times \{1, \ldots, k\}$ for some finite $k$,

(ii) for any $x, x', i \in \mathbb{Z}$ and any $y, y' \in \{1, \ldots, k\}$, we have $\langle\{x, y\}, \{x', y'\}\rangle \in E$ if and only if $\langle\{x + i, y\}, \{x' + i, y'\}\rangle \in E$,

(iii) $\langle\{x, y\}, \{x', y'\}\rangle \in E$ implies that $|x - x'| \leq 1$.

Stochastic models defined on the same class of graphs have been considered by Alm and Janson [1] and others. Definition 4.1(iii) may seem somewhat restrictive, but it is easy to see, by means of a relabeling of $V$, that it involves no loss of generality compared to the (at first sight weaker-looking) condition

(iii′) $\mathcal{G}$ has bounded degree.

The structural property obtained in the following lemma will turn out to be useful. By a path from a vertex $v \in V$ to $-\infty$ we mean an infinite path starting at $v$ whose $x$-coordinate tends to $-\infty$. As in Section 3, let $E^{i,-}$ denote the set of edges $e \in E$ whose endpoints both have $x$-coordinates not exceeding $i$. Also, call a vertex $\{i, j\}$ *backwards well connected* if $\mathcal{G}$ contains a path from $\{i, j\}$ to $-\infty$ whose edges are all contained in $E^{i,-}$, and (for later purposes) call the same vertex *forwards well connected* if $\mathcal{G}$ contains a path from $\{i, j\}$ to $+\infty$ that never visits a vertex with $x$-coordinate strictly smaller than $i$.

**Lemma 4.1.** *Let $\mathcal{G} = (V, E)$ be a connected one-dimensional periodic lattice, let $\{i, j\}$ and $\{i, j'\}$ be two vertices with the same $x$-coordinate, and suppose that $\{i, j\}$ and $\{i, j'\}$ are both backwards well connected. Then there exists a path from $\{i, j\}$ to $\{i, j'\}$ whose edges are all contained in $E^{i,-}$.*

*Proof.* Note first that since $\mathcal{G}$ is connected, we have

$$\max\{\text{dist}_{\mathcal{G}}(\{i, l\}, \{i, m\}) : l, m \in \{1, \ldots, k\}\} < \infty, \tag{4.1}$$

where $\text{dist}_{\mathcal{G}}(u, v)$ denotes the length of the shortest path in $\mathcal{G}$ between $u$ and $v$. Let $\{i, j\}$ and $\{i, j'\}$ be as in the lemma, and assume for contradiction that

$$\text{there is no path between } \{i, j\} \text{ and } \{i, j'\} \text{ that is contained in } E^{i,-}. \tag{4.2}$$

The assumption that $\{i, j\}$ and $\{i, j'\}$ are backwards well connected guarantees that, for any $n \geq 0$, we can find a vertex $\{i - n, l\}$ on a path in $E^{i,-}$ from $\{i, j\}$ to $-\infty$, as well as a vertex $\{i - n, l'\}$ on a path in $E^{i,-}$ from $\{i, j'\}$ to $-\infty$. Assumption (4.2) implies that any path from $\{i - n, l\}$ to $\{i - n, l'\}$ has to pass through $x$-coordinate $i + 1$. Consequently, $\text{dist}_{\mathcal{G}}(\{i - n, l\}, \{i - n, l'\}) > 2n$. By Definition 4.1(ii), this implies that $\text{dist}_{\mathcal{G}}(\{i, l\}, \{i, l'\}) > 2n$, so that in particular

$$\max\{\text{dist}_{\mathcal{G}}(\{i, l\}, \{i, m\}) : l, m \in \{1, \ldots, k\}\} > 2n.$$

Since $n$ was arbitrary, this contradicts (4.1), so the proof is complete.

The assumption that $\mathcal{G} = (V, E)$ is a connected one-dimensional periodic lattice will henceforth be taken for granted. To introduce our conditional percolation model on $\mathcal{G}$, we proceed as with the ladder $\mathcal{L}$ in Section 2. Let $B_{N_1,N_2}$ be the event that there exists an open path from some vertex with $x$-coordinate $-N_1$ to some vertex with $x$-coordinate $N_2$, and, for $p \in (0, 1)$, let $\mathrm{P}_{\mathcal{G},p,N_1,N_2}$ be the probability measure on $\{0, 1\}^E$ that arises from i.i.d. bond percolation with parameter $p$ conditional on the event $B_{N_1,N_2}$. We have the following generalization of Theorem 2.1.

**Theorem 4.1.** *For any $p \in (0, 1)$, the probability measures $\mathrm{P}_{\mathcal{G},p,N_1,N_2}$ converge weakly to a probability measure $\mathrm{P}_{\mathcal{G},p}$ on $\{0, 1\}^E$ as $N_1, N_2 \to \infty$.*

Let us consider what happens when trying to imitate the proof of Theorem 2.1 in order to prove Theorem 4.1. The key use that the proof of Theorem 2.1 makes of the particular graph structure of $\mathcal{L}$ is the role played by open vertical edges. These serve as a kind of regeneration points: conditional on having vertical edges at two different $x$-coordinates $-K_1$ and $K_2$, whatever happens between these coordinates is conditionally independent of what happens on the outside. Or more simply, but equivalently, given an open vertical edge, the configurations to the left and to the right of it are conditionally independent. Achieving this kind of regeneration is obviously more complicated in the more general setting of Theorem 4.1, but using Lemma 4.1 it can be done, as follows.

Define the constant $R_{\mathcal{G}}$ as the smallest $R$ such that (i) any pair of vertices $\{0, j\}$ and $\{0, j'\}$ that are connected by a path in $E^{0,-}$ are also connected by a path in $E^{0,-} \setminus E^{-R,0}$, and (ii) no vertex $\{0, j\}$ that fails to have a path in $E^{0,-}$ to $-\infty$ has a path in $E^{0,-}$ to $x$-coordinate $-R$. Also, consider the lattice $\tilde{\mathcal{G}} = (\tilde{V}, \tilde{E})$ obtained by reversing left and right in $\mathcal{G}$, meaning more precisely that $\tilde{V} = V$ and

$$\tilde{E} = \{\langle \{i, j\}, \{i', j'\}\rangle : \langle \{i, j'\}, \{i', j\}\rangle \in E\},$$

and let $R^*_{\mathcal{G}} = \max\{R_{\mathcal{G}}, R_{\tilde{\mathcal{G}}}\}$. Next, for any $i$, let $D_i \subset \{0, 1\}^E$ denote the event that all edges with both $x$-coordinates in the interval $[i - R^*_{\mathcal{G}}, i]$ are open.

The point of the choice of $R^*_{\mathcal{G}}$ is the following. Pick $N_1, N_2$, and $i$ in such a way that $-N_1 < i - R^*_{\mathcal{G}} < i < N_2$, and suppose that the configuration $\eta \in \{0, 1\}^E$ also satisfies $\eta \in D_i \subset \{0, 1\}^E$. Knowing this, suppose that we wish to evaluate whether $\eta$ is also in the event $B_{N_1,N_2}$ that there exists an open path between $x$-coordinates $-N_1$ and $N_2$. For $B_{N_1,N_2}$ to happen, we claim that (on the event $D_i$) necessary and sufficient conditions are that

(a) there exists an open path in $E^{i,-}$ between some vertex with $x$-coordinate $-N_1$ and some forwards well-connected vertex with $x$-coordinate $i - R^*_{\mathcal{G}}$, and

(b) there exists an open path in $E^{i-R^*_{\mathcal{G}},+}$ between some backwards well-connected vertex with $x$-coordinate $i$ and some vertex with $x$-coordinate $N_2$.

To see this, note that each of (a) and (b) is clearly necessary for $B_{N_1,N_2}$, while, on the other hand, the definition of $D_i$ in conjunction with Lemma 4.1 guarantees the existence of an open path between any forwards well-connected vertex with $x$-coordinate $i - R^*_{\mathcal{G}}$ and any backwards well-connected vertex with $x$-coordinate $i$, so that the intersection of (a) and (b) implies $B_{N_1,N_2}$.

To evaluate whether (a) happens, we only need to look at edges in $E^{i-R^*_{\mathcal{G}},-}$, while to check if (b) happens, we only need to look at edges in $E^{i,+}$ (while it is true both events involve edges between $x$-coordinates $i - R^*_{\mathcal{G}}$ and $i$, these edges give no information since we have assumed $D_i$). It follows that the $\mathrm{P}_{\mathcal{G},p,N_1,N_2}$ conditioned on $D_i$ is tantamount to letting edges to the left of $x$-coordinate $i - R^*_{\mathcal{G}}$ be i.i.d. $(p)$ conditional on (a), while independently of this, edges to the right of $x$-coordinate $i$ are i.i.d. $(p)$ conditional on (b), and edges in between are open. The bottom line is that what happens to the left of $i - R^*_{\mathcal{G}}$ and what happens to the right of $i$ are conditionally independent given $D_i$. This turns $D_i$ into the kind of regeneration event we need to play the same role on $\mathcal{G}$ as open vertical edges play on $\mathcal{L}$. With this in mind, the proof of Theorem 2.1 is easily adapted to handle Theorem 4.1; we omit further details and consider Theorem 4.1 established.

At this point, the proofs of Corollaries 2.1 and 2.2 translate trivially to the $\mathcal{G}$ setting, and we may combine them in the following statement.

**Corollary 4.1.** *For any $p \in (0, 1)$, $\mathrm{P}_{\mathcal{G},p}$ is translation invariant and assigns probability 1 to the event $B = \bigcap_{N=1}^{\infty} B_{N,N}$ that a bi-infinite open path from $-\infty$ to $\infty$ exists.*

Moving on to the results for $\mathcal{L}$ in Section 3, these also admit translation to the more general context of connected one-dimensional periodic lattices. The proof of Lemma 3.1 on stochastic domination makes no reference to the graph structure and, therefore, generalizes immediately to the more general setting, described in the following lemma.

**Lemma 4.2.** *For any $p \in (0, 1)$, $\mathrm{P}_{\mathcal{G},p}$ stochastically dominates i.i.d. $(p)$ bond percolation on $\mathcal{G}$. For any $N$, the same holds with $\mathrm{P}_{\mathcal{G},p,N,N}$ in place of $\mathrm{P}_{\mathcal{G},p}$.*

The case of Lemma 3.2 concerning single-edge conditional probabilities involves the use of open vertical edges as regeneration events, but replacing these by the events $D_i$ discussed above gives an argument that goes through smoothly in the more general setting. The extension to finite edge sets in Lemma 3.3 uses nothing about the graph structure of $\mathcal{L}$, and, thus, generalizes immediately, giving us the following.

**Lemma 4.3.** *$\mathrm{P}_{\mathcal{G},p}$ admits conditional probabilities such that, for a corresponding $\{0, 1\}^E$-valued random configuration $X$, we have, for any finite $F \subset E$, any permissible $\xi \in \{0, 1\}^{E\setminus F}$, and any $\eta \in \{0, 1\}^E$,*

$$\mathrm{P}_{\mathcal{G},p}(X(F) = \eta \mid X(E \setminus F) = \xi) = \mathrm{P}_{\mathcal{G},p,F,\xi}(\eta),$$

*where $\mathrm{P}_{\mathcal{G},p,F,\xi}$ is defined by setting, for each $\eta \in \{0, 1\}^F$,*

$$\mathrm{P}_{\mathcal{G},p,F,\xi}(\eta) = \frac{\mathbf{1}_{\{(\xi\vee\eta)\in B\}}}{Z_{\mathcal{G},p,F,\xi}} \prod_{e\in F} p^{\eta(e)}(1-p)^{1-\eta(e)}. \tag{4.3}$$

Next comes the task of finding a Markov chain $\{T_i\}_{i\in\mathbb{Z}}$ that can represent the percolation process on $\mathcal{G}$ in the same way as was accomplished for $\mathcal{L}$ in Section 3. A first issue to decide is what a suitable state space $\mathcal{S}$ for the chain might be. The crucial property we are looking for is that $T_i$ should convey all the information of the percolation process $X(E^{i,-})$ to the left of $x$-coordinate $i$ that we need in order to determine which realizations to the right of $i$ will result in the event $B$ of having an open path between $-\infty$ and $\infty$. A first attempt—and the most obvious extension of the Markov chain we devised for $\mathcal{L}$—is to let $\mathcal{S} = \{0, 1\}^k$, where the $j$th bit of $T_i$ indicates whether $\{i, j\}$ is connected to $-\infty$ via an open path in $E^{i,-}$. This is not enough information, however, as the following example shows.

Take $\mathcal{G} = (V, E)$ with $V = \mathbb{Z} \times \{1, 2, 3\}$ and $E$ consisting of pairs of vertices that are either identical in the $x$-coordinate and differ by 1 in the $y$-coordinate, or vice versa. (Thus, $\mathcal{G}$ may be pictured as the ladder $\mathcal{L}$ with an extra rail.) Imagine that $\mathcal{S} = \{0, 1\}^k$, as suggested above, and that $T_i = (0, 0, 1)$, indicating that the vertex $\{i, 3\}$ is connected to $-\infty$ via an open path in $E^{i,-}$ while $\{i, 1\}$ and $\{i, 2\}$ are not. It may seem that (if we insist on the event $B$ happening) $T_i = (0, 0, 1)$ forces $X(E^{i,+})$ to supply an open path between $\{i, 3\}$ and $+\infty$. But in case $X(E^{i,-})$ happens to supply an open path between $\{i, 1\}$ and $\{i, 2\}$, then there is another possibility for $X(E^{i,+})$ to bring about $B$, namely to supply an open path between $\{i, 3\}$ and $\{i, 2\}$ as well as an open path between $\{i, 1\}$ and $+\infty$. Note also that this would not have been good enough if $\{i, 1\}$ and $\{i, 2\}$ had failed to connect in $X(E^{i,-})$.

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The bottom line of this example is that it is not enough for $T_i$ to encode which vertices at $x$-coordinate $i$ have an open path to $-\infty$ in $E^{i,-}$; it also has to encode which pairs among these vertices are connected to each other via open paths in $E^{i,-}$. To this end, we may take the state space $\mathcal{S}$ of the Markov chain to be the set of all binary symmetric $(k+1) \times (k+1)$ matrices $S = \{S_{j,j'}\}_{j,j' \in \{0,1,\dots,k\}}$ with diagonal elements equal to 1. When $T_i = S$, the matrix element $S_{j,j'}$ with $j$ and $j'$ nonzero indicates whether $\{i, j\}$ and $\{i, j'\}$ are connected by an open path in $E^{i,-}$, while, for $j' = 0$, it indicates whether $\{i, j\}$ is connected to $-\infty$ by an open path in $E^{i,-}$.

This defines the process $\{T_i\}_{i\in\mathbb{Z}}$. The proof of Theorem 3.1 is easily adapted, replacing as usual the use of open vertical edges as regeneration events by the events $D_i$ defined above, to establish the following.

**Theorem 4.2.** *For any connected one-dimensional periodic lattice $\mathcal{G}$, the process $\{T_i\}_{i\in\mathbb{Z}}$ as defined above becomes a time-homogeneous and stationary Markov chain.*

Similarly as for the case of the ladder $\mathcal{L}$ studied in Section 3, the relevance of $\{T_i\}_{i\in\mathbb{Z}}$ lies not only in its simple Markovian structure but also in the simple form of the distribution of the percolation configuration given the Markov chain; Theorem 3.2 has the following straightforward generalization to connected one-dimensional periodic lattices. As in the $\mathcal{L}$ case, we let $E^i$ denote the edge set $E^{i+1,-} \setminus E^{i,-}$.

**Theorem 4.3.** *Let $\mathcal{G} = (V, E)$ and $\{T_i\}_{i\in\mathbb{Z}}$ be as above. The conditional distribution of the percolation process $X \in \{0,1\}^E$ given $\{T_i\}_{i\in\mathbb{Z}}$ is*

$$\prod_{j\in\mathbb{Z}} \mathrm{P}_{\mathcal{G},p,j,T_{j-1},T_j},$$

*where the probability measure $\mathrm{P}_{\mathcal{G},p,i,T_{i-1},T_i}$ on $\{0,1\}^{E^i}$ is defined by setting, for each $\eta \in \{0,1\}^{E^i}$,*

$$\mathrm{P}_{\mathcal{G},p,i,T_{i-1},T_i}(\eta) = \frac{\mathbf{1}_{\{\eta \text{ is } T_{i-1},T_i\text{-compatible}\}}}{Z_{\mathcal{G},p,i,T_{i-1},T_i}} \prod_{e\in E^i} p^{\eta(e)}(1-p)^{1-\eta(e)}.$$

We remark that while in the case of the ladder in Section 3 the Markov chain $\{T_i\}_{i\in\mathbb{Z}}$ that we obtained turned out to be reversible (and the same thing would happen if we insisted on defining the chain formally as in the present section, which would in fact merely amount to a relabeling of the chain's state space), this feature does not carry through to the more general setting. To obtain a nonreversible counterexample, it suffices to add an edge of the form $\langle\{i,0\},\{i+1,1\}\rangle$ for each $i$ to the ladder $\mathcal{L}$; this leads to a Markov chain in which (with the state space terminology of Section 3) the event $\{T_0 = \mathtt{01},\ T_1 = \mathtt{10}\}$ has positive probability while $\{T_0 = \mathtt{10},\ T_1 = \mathtt{01}\}$ has zero probability, implying nonreversibility.

Not all states $S$ in the state space $\mathcal{S}$ defined above appear in the chain $\{T_i\}_{i\in\mathbb{Z}}$ with positive probability. For instance, transitivity of connectivity implies, for any nonzero $j, j', j''$, that only states $S$ such that

$$S_{j,j''} \geq S_{j,j'} S_{j',j''} \tag{4.4}$$

appear. In fact, (4.4) holds also when the indices $j, j', j''$ are allowed to take value 0; this follows from the observation (due to a Borel–Cantelli argument) that, for any $i \in \mathbb{Z}$, the regeneration event $D_{i'}$ happens for some $i' < i$, which causes any two open paths from $x$-coordinate $i$ to $-\infty$ to connect to each other. Yet another restriction on $S$ is that there must be a $j > 0$ such that $S_{j,0} = 1$, as otherwise the event $B$ would not happen. Further restrictions on $S$ may be implied by the particular graph structure of $\mathcal{G}$. A precise condition for which $S \in \mathcal{S}$ appears in the Markov chain is the following.

**Definition 4.2.** Fix $\mathcal{G}$, and let $\mathcal{S}$ be as above. Define $\mathcal{S}^* \subset \mathcal{S}$ as the set of states $S$ for which there exists an $\eta \in \{0, 1\}^E$ such that

$$\eta \in B, \qquad \eta \in \bigcup_{i=1}^{\infty} D_{-i}, \qquad \eta \in \bigcup_{i=R^*_{\mathcal{G}}}^{\infty} D_i, \tag{4.5}$$

and such that taking $X = \eta$ gives $T_0 = S$.

**Lemma 4.4.** *For any $S \in \mathcal{S}$, write* $\text{count}(S)$ *for the number of times that $S$ occurs in the sequence $\{T_i\}_{i\in\mathbb{Z}}$. Here* $\text{count}(S)$ *is an almost-sure constant which equals $\infty$ if $S \in \mathcal{S}^*$ and which equals* 0 *otherwise.*

*Proof.* Pick $X \in \{0, 1\}^E$ according to $\mathrm{P}_{\mathcal{G},p}$. The familiar Borel–Cantelli argument shows that $X \in D_i$ for some $i < 0$ a.s., and that $X \in D_i$ for some $i \geq R^*_{\mathcal{G}}$ a.s. We may also recall from Corollary 4.1 that $X \in B$ a.s. Hence, $X$ satisfies the conditions imposed on $\eta$ in (4.5). Thus, $T_0 \in S$ a.s., but since the chain is stationary, we have $T_i \in S$ a.s. for any $i$, so that $\text{count}(S) = 0$ for any $S \notin \mathcal{S}^*$.

Suppose on the other hand that $S \in \mathcal{S}^*$. Pick an $\eta \in \{0, 1\}^E$ satisfying (4.5) and such that taking $X = \eta$ gives $T_0 = S$. Given such an $\eta$, we can pick $i' < 0$ and $i'' \geq R^*_{\mathcal{G}}$ such that $\eta \in D_{i'}$ and $\eta \in D_{i''}$. It follows from the definition of the regeneration events $D_i$ that no matter what happens with $X$ outside of the interval $[i' - R^*_{\mathcal{G}}, i'']$, taking $X$ equal to $\eta$ on this interval cannot cause the event $B$ to fail. Hence, Lemma 4.3 tells us that $T_0 = S$ with positive probability, and in fact gives a nonzero lower bound on the conditional probability that $T_0 = S$ given what happens outside of $[i' - R^*_{\mathcal{G}}, i'']$. Applying the same argument to an infinite collection of disjoint intervals of length $i'' - i' + R^*_{\mathcal{G}}$ and using a Borel–Cantelli argument shows that $\text{count}(S) = \infty$ a.s.

It may be noted that taking $\eta \in \{0, 1\}^E$ in Definition 4.2 to be the configuration where *all* edges $e \in E$ are open yields a state $S_{\max} \in \mathcal{S}^*$ which is maximal in the following sense. The event $T_i = S_{\max}$ indicates that all vertices with $x$-coordinate $i$ that can be connected to each other in $E^{i,-}$ indeed are, and likewise for all vertices with $x$-coordinate $i$ that can be connected to $-\infty$ in $E^{i,-}$. Note that the regeneration event $D_i$ implies that $T_i = S_{\max}$.

Our final result in this section is the following.

**Theorem 4.4.** *The chain $\{T_i\}_{i\in\mathbb{Z}}$, viewed as having state space $\mathcal{S}^*$, is irreducible and aperiodic.*

*Proof.* For any $S \in \mathcal{S}^*$, the final step in the proof of Lemma 4.4 shows not only that $\text{count}(S) = \infty$, but also that $T_i = S$ happens for arbitrarily large $i$ a.s. Hence, for any $S, S' \in \mathcal{S}^*$, there exist a.s. $i, i' \in \mathbb{Z}$ such that $i < i'$, $T_i = S$, and $T_{i'} = S'$. Hence, the chain moves with positive probability from state $S$ to state $S'$ in a finite number of steps. But since $S, S' \in \mathcal{S}^*$ were arbitrary, it follows that the chain is irreducible.

For aperiodicity, a sufficient condition is that there exists a state $S \in S^*$ such that the chain jumps from $S$ to $S$ with positive probability. Lemma 4.3 guarantees that, with positive probability, all edges in the interval $[-R^*_{\mathcal{G}}, 1]$ are open. But on this event, both $D_0$ and $D_1$ happen, so that $T_0 = T_1 = S_{\max}$, and aperiodicity is established.

## 5. Stochastic monotonicity

The following monotonicity property of i.i.d. percolation is well known (and trivial). Writing $\mu_p$ for the probability measure on $\{0, 1\}^E$ (where $G = (V, E)$ is any graph) corresponding to i.i.d. bond percolation with parameter $p$, we have

$$\mu_{p_1} \preceq_{\mathrm{d}} \mu_{p_2}$$

whenever $p_1 \leq p_2$. It is natural to ask whether the analogous statement holds for our conditional percolation model on $\mathcal{L}$. The answer is no, however, as indicated by the following result.

**Proposition 5.1.** *For the conditional percolation model on $\mathcal{L}$, we have, for any fixed $p \in (0, 1)$, an $\varepsilon \in (0, p)$ such that*

$$\mathrm{P}_{p'} \not\preceq_{\mathrm{d}} \mathrm{P}_p \tag{5.1}$$

*for all $p' \in (0, \varepsilon)$.*

*Proof.* Fix $p \in (0, 1)$. In order to establish (5.1) for some given $p'$, we need to find an increasing event $A$ (i.e. an event $A \subset \{0, 1\}^E$ whose indicator $\mathbf{1}_A$ is an increasing function of the edge configuration) such that $\mathrm{P}_{p'}(A) > \mathrm{P}_p(A)$. To this end, let $A_n$ be the event that all horizontal edges in the upper layer of $\mathcal{L}$ between $x$-coordinates 0 and $n$ are open:

$$A_n = \{\eta \in \{0, 1\}^E : \eta(\langle\{i, 1\}, \{i + 1, 1\}\rangle) = 1 \text{ for } i = 0, \dots, n - 1\}.$$

By the usual Borel–Cantelli argument,

$$\mathrm{P}_p(X(\langle\{i, 1\}, \{i + 1, 1\}\rangle) = 1 \text{ for all } i \geq 0) = 0,$$

whence $\lim_{n\to\infty} \mathrm{P}_p(A_n) = 0$. Fix an $n$ such that

$$\mathrm{P}_p(A_n) < \tfrac{1}{2}. \tag{5.2}$$

Note now that if $T_i = T_{i+1} = \mathtt{01}$ then the edge $\langle\{i, 1\}\{i + 1, 1\}\rangle$ is forced to be open. Hence,

$$\mathrm{P}_p(A_n) \geq \mathrm{P}_p(T_0 = T_1 = \cdots = T_n = \mathtt{01}) = \pi_{\mathtt{01}}(p_{\mathtt{01},\mathtt{01}})^n. \tag{5.3}$$

Recall from the final lines of Section 3 that

$$\lim_{p\to 0} \pi_{\mathtt{01}} = \tfrac{1}{2} \tag{5.4}$$

and that $\lim_{p\to 0} p_{\mathtt{01},\mathtt{11}} = 0$. The latter observation implies that

$$\lim_{p\to 0} p_{\mathtt{01},\mathtt{01}} = 1. \tag{5.5}$$

Sending $p \to 0$ in (5.3) and substituting in (5.4) and (5.5) yields $\lim_{p\to 0} \mathrm{P}_p(A_n) \geq \frac{1}{2}$. Combining this with (5.2) yields $\mathrm{P}_{p'}(A_n) > \mathrm{P}_p(A_n)$ for all sufficiently small $p' > 0$, and the lemma is established.

We mention that Proposition 5.1 does not extend to arbitrary one-dimensional periodic lattices $\mathcal{G}$. A trivial counterexample is the usual $\mathbb{Z}^1$ lattice, but other, slightly less trivial, examples can be found. We will make no attempt at a general condition on $\mathcal{G}$ that decides whether the analogue Proposition 5.1 holds.

Another question that we leave unanswered is whether for $\mathcal{L}$ there are other choices of $p$ and $p'$ such that $\mathrm{P}_p \preceq_{\mathrm{d}} \mathrm{P}_{p'}$ can be established. Instead, we consider monotonicity with respect to a weaker property than stochastic domination, namely the following natural quantity.

**Definition 5.1.** For our one-dimensional conditional percolation model on $\mathcal{L}$ with parameter $p \in (0, 1)$, we define the *edge density* $\theta_{\mathcal{L}} = \theta_{\mathcal{L}}(p)$ as

$$\theta_{\mathcal{L}}(p) = \mathrm{P}_p(\langle\{0, 0\}, \{0, 1\}\rangle \text{ is open}) + \mathrm{P}_p(\langle\{0, 1\}, \{1, 1\}\rangle \text{ is open}) \\ + \mathrm{P}_p(\langle\{1, 0\}, \{1, 1\}\rangle \text{ is open}).$$

More generally, for our model on a connected one-dimensional periodic lattice $\mathcal{G} = (V, E)$ at parameter $p$, define the edge density as

$$\theta_{\mathcal{G}}(p) = \sum_{e \in E_0} s P_{\mathcal{G},p}(e \text{ is open}),$$

where, as before, $E_i = E^{i+1,-} \setminus E^{i,-}$ is the set of edges which has either two endpoints with $x$-coordinate $i+1$, or one endpoint with $x$-coordinate $i$ and the other endpoint with $x$-coordinate $i+1$.

By translation invariance, the edge density is the expected number of open edges per length unit. We also have the following ergodicity result. Given $\mathcal{G} = (V, E)$ and $X \in \{0, 1\}^E$, define $Y_n$ as the number of open edges with both endpoints having $x$-coordinates in the interval $[-n, n]$.

**Lemma 5.1.** *For any connected one-dimensional periodic lattice $\mathcal{G}$ and any $p \in (0, 1)$, we have, with $\mathrm{P}_{\mathcal{G},p}$-probability 1,*

$$\lim_{n\to\infty} \frac{Y_n}{2n} = \theta_{\mathcal{G}}(p). \tag{5.6}$$

*Proof.* Since the percolation process is translation invariant (Corollary 4.1), almost-sure existence of a limit in (5.6) follows from the pointwise ergodic theorem. It remains to show that the limit is an almost-sure constant and equals the value predicted by expectations, and, for this, it suffices to show that the percolation process is ergodic. By Theorem 4.4, the Markov chain $\{T_i\}_{i\in\mathbb{Z}}$ is ergodic. It is intuitively clear that the noise generated in the manner described in Theorem 4.3 on top of the Markov chain does not mess up ergodicity, and one way to see it is as follows. For each $i$, define $\hat{T}_i = (T_i, X(E^i))$. Theorem 4.3 implies that $\{\hat{T}_i\}_{i\in\mathbb{Z}}$ inherits the Markov chain property of $\{T_i\}_{i\in\mathbb{Z}}$, as well as the irreducibility and aperiodicity properties from Theorem 4.4. The new Markov chain is therefore ergodic. But this implies that the percolation process, which is a function of $\{\hat{T}_i\}_{i\in\mathbb{Z}}$, is also ergodic.

Despite Proposition 5.1, we may hope that $\theta_{\mathcal{G}}(p)$ is increasing in $p$. For the case $\mathcal{G} = \mathcal{L}$, it is possible to derive an explicit expression for $\theta_{\mathcal{L}}(p)$ using our explicit computation of the transition matrix $\boldsymbol{p}$ for $\{T_i\}_{i\in\mathbb{Z}}$ and its stationary distribution $\pi$, together with Theorem 3.2, and verify that $\theta_{\mathcal{L}}(p)$ is indeed increasing in $p$. Such an approach is however unsatisfactory for two reasons. First, it is not very instructive—it conveys little or no understanding as to *why* $\theta_{\mathcal{L}}(p)$ is increasing. Second, although the method may extend to other graphs on a case-by-case basis, it offers no clue as to how to extend it to *all* connected one-dimensional periodic lattices. We shall instead settle on a different approach, based on a coupling similar to the one employed in the proof of Theorem 2.1, to establish the following general result.

**Theorem 5.1.** *For any one-dimensional periodic lattice $\mathcal{G}$ and any $p_1$, $p_2$ such that $0 < p_1 < p_2 < 1$, we have*

$$\theta_{\mathcal{G}}(p_1) \le \theta_{\mathcal{G}}(p_2).$$

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For the proof, we will need the following minor extension of Lemma 5.1. For $n_1, n_2 \geq 0$, define $Y_{n_1,n_2}$ as the number of open edges with both endpoints having $x$-coordinates in the interval $[-n_1, n_2]$.

**Lemma 5.2.** *For any connected one-dimensional periodic lattice $\mathcal{G}$ and any $p \in (0, 1)$, we have, with $\mathrm{P}_{\mathcal{G},p}$-probability* 1,

$$\lim_{n_1,n_2\to\infty} \frac{Y_{n_1,n_2}}{n_1+n_2} = \theta_{\mathcal{G}}(p).$$

*Proof.* This is a standard consequence of ergodicity of the percolation process.

*Proof of Theorem 5.1.* We may assume that $\theta_{\mathcal{G}}(p_1) \neq \theta_{\mathcal{G}}(p_2)$, since otherwise we are done. Our coupling will be to pick $X, X' \in \{0,1\}^E$ independently with distributions $\mathrm{P}_{\mathcal{G},p_1}$ and $\mathrm{P}_{\mathcal{G},p_2}$. This is the simplest possible coupling of $\mathrm{P}_{\mathcal{G},p_1}$ and $\mathrm{P}_{\mathcal{G},p_2}$; what will give it life is the way in which we will reveal information about $X$ and $X'$ sequentially.

As in Section 4, define $D_i$ as the event that all edges with both endpoints having $x$-coordinates in the interval $[i - R^*_{\mathcal{G}}, i]$ are open in $X$ (recall from Section 4 the definition of the constant $R^*_{\mathcal{G}}$). Define $D'_i$ analogously but with respect to $X'$ rather than $X$. By the stipulated independence of $X$ and $X'$ we have

$$\Pr(D_i \cap D'_i) = \Pr(D_i)\Pr(D'_i) > 0. \tag{5.7}$$

By the proof of Lemma 5.1, $X$ and $X'$ are both ergodic, and the proof easily extends to show that they are jointly ergodic. Hence, (5.7) implies that, with probability 1, $D_i \cap D'_i$ happens for infinitely many $i$, and moreover that, for any $\varepsilon > 0$, there exists an $l > 0$ such that, for any interval $I$ of length $l$, we have

$$\mathrm{P}(\text{there exists } i \in I \text{ such that } D_i \cap D'_i \text{ happens}) > \tfrac{9}{10}. \tag{5.8}$$

Now pick an $\varepsilon > 0$ small enough so that $|\theta_{\mathcal{G}}(p_2) - \theta_{\mathcal{G}}(p_1)| > 3\varepsilon$. Let $Y_{n_1,n_2}$ be as in Lemma 5.1, and define $Y'_{n_1,n_2}$ analogously but with respect to $X'$. Pick an $N$ large enough so that

(i) $N = l$ for some $l$ chosen so that (5.8) holds,

(ii) $\Pr\left(\dfrac{Y_{n_1,n_2}}{n_1+n_2} \in (\theta_{\mathcal{G}}(p_1) - \varepsilon, \theta_{\mathcal{G}}(p_1) + \varepsilon) \text{ for all } n_1, n_2 \geq N\right) > \dfrac{9}{10}$, and

(iii) $\Pr\left(\dfrac{Y'_{n_1,n_2}}{n_1+n_2} \in (\theta_{\mathcal{G}}(p_2) - \varepsilon, \theta_{\mathcal{G}}(p_2) + \varepsilon) \text{ for all } n_1, n_2 \geq N\right) > \dfrac{9}{10}$

(that (ii) and (iii) are satisfied for large $N$ follows from Lemma 5.2). Consider the following sequential procedure for revealing $X$ and $X'$, reminiscent of the procedure in the proof of Theorem 2.1. First reveal everything that happens outside the interval $[-N, N]$. Then, starting at $x$-coordinate $-2N$, scan $X$ and $X'$ from left to right in a synchronized fashion until we encounter an $x$-coordinate $-K_1$ such that $D_{-K_1} \cap D'_{-K_1}$ happens. Then scan from right to left starting at $x$-coordinate $2N$ until we encounter an $x$-coordinate $K_2$ such that $D_{K_2+R^*_{\mathcal{G}}} \cap D'_{K_2+R^*_{\mathcal{G}}}$. Note that by condition (i) in the choice of $N$ we have

$$\mathrm{P}(-K_1 < -N < N < K_2) > \tfrac{8}{10}. \tag{5.9}$$

Suppose that this event happens, and define $F$ as the (random) set of edges that have at least one endpoint between $-K_1$ and $K_2$. Write $\mathcal{F}$ for the $\sigma$-field of everything we know about $X$ and $X'$ at this stage, and note that what remains to be revealed is exactly $X(F)$ and $X'(F)$.

Let $\xi \in \{0,1\}^{E\setminus F}$ be the configuration that sets all edges in $E \setminus F$ to be open. Recall definition (4.3) of $\mathrm{P}_{\mathcal{G},p,F,\xi}$, and note that, by the defining properties of $D_i$, the conditional distribution of $X(F)$ given $\mathcal{F}$ is $\mathrm{P}_{\mathcal{G},p_1,F,\xi}$ and the conditional distribution of $X'(F)$ given $\mathcal{F}$ is $\mathrm{P}_{\mathcal{G},p_2,F,\xi}$.

Imagine now that we send a friend to look at $X(F)$ and $X'(F)$, and ask her to inform us about what these configurations are without telling us which is which. Write $\eta^+, \eta^- \in \{0,1\}^F$ for the two configurations, where $\eta$ is taken to be the one with the largest number of edges (with an arbitrary tie-breaking convention). We claim that

$$\mathrm{P}(X(F) = \eta^- \mid \mathcal{F}, \eta^+, \eta^-) \geq \tfrac{1}{2}. \tag{5.10}$$

Before proving this, let us first show how it implies the theorem. Combining (5.10) with (5.9) yields

$$\mathrm{P}(\text{there exist } n_1, n_2 \geq N \text{ such that } Y'_{n_1,n_2} \geq Y_{n_1,n_2}) > \tfrac{1}{2}\tfrac{8}{10} = \tfrac{4}{10}. \tag{5.11}$$

But if it were the case that $\theta_{\mathcal{G}}(p_1) > \theta_{\mathcal{G}}(p_2)$, then conditions (ii) and (iii) in the choice of $N$ would show that

$$\mathrm{P}(\text{there exist } n_1, n_2 \geq N \text{ such that } Y'_{n_1,n_2} \geq Y_{n_1,n_2}) < \bigl(1 - \tfrac{9}{10}\bigr) + \bigl(1 - \tfrac{9}{10}\bigr) = \tfrac{2}{10},$$

contradicting (5.11). Hence, $\theta_{\mathcal{G}}(p_1) \leq \theta_{\mathcal{G}}(p_2)$, as desired.

It remains to establish (5.10). If $\eta^+ = \eta^-$ then the probability in (5.10) equals 1, so we may safely assume that $\eta^+ \neq \eta^-$. We write $m$ for the number of edges in $F$, and also write $m(\eta^+)$ and $m(\eta^-)$ for the number of open edges in $\eta^+$ and $\eta^-$, respectively. Note that $m(\eta^+) \geq m(\eta^-)$ and that (since $p_1 < p_2$) $p_2(1-p_1)/p_1(1-p_2) > 1$. We calculate

$$\begin{aligned}
&\frac{\mathrm{P}(X(F) = \eta^- \mid \mathcal{F}, \eta^+, \eta^-)}{\mathrm{P}(X(F) = \eta^+ \mid \mathcal{F}, \eta^+, \eta^-)} \\
&\quad = \frac{\mathrm{P}_{\mathcal{G},p_1,F,\xi}(\eta^-)\,\mathrm{P}_{\mathcal{G},p_2,F,\xi}(\eta^+)}{\mathrm{P}_{\mathcal{G},p_1,F,\xi}(\eta^+)\,\mathrm{P}_{\mathcal{G},p_2,F,\xi}(\eta^-)} \\
&\quad = \biggl(\frac{\mathbf{1}_{\{(\xi\vee\eta)\in B\}}}{Z_{\mathcal{G},p_1,F,\xi}} p_1^{m(\eta^-)}(1-p_1)^{m-m(\eta^-)} \frac{\mathbf{1}_{\{(\xi\vee\eta)\in B\}}}{Z_{\mathcal{G},p_2,F,\xi}} p_2^{m(\eta^+)}\biggr)(1-p_2)^{m-m(\eta^+)} \\
&\qquad \times \biggl(\frac{\mathbf{1}_{\{(\xi\vee\eta)\in B\}}}{Z_{\mathcal{G},p_1,F,\xi}} p_1^{m(\eta^+)}(1-p_1)^{m-m(\eta^+)} \frac{\mathbf{1}_{\{(\xi\vee\eta)\in B\}}}{Z_{\mathcal{G},p_2,F,\xi}} p_2^{m(\eta^-)}(1-p_2)^{m-m(\eta^-)}\biggr)^{-1} \\
&\quad = \biggl(\frac{p_2(1-p_1)}{p_1(1-p_2)}\biggr)^{m(\eta^+)-m(\eta^-)} \\
&\quad \geq 1,
\end{aligned}$$

and (5.10) follows.

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