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Numerical Semigroups Having a Toms Decomposition

J. C. Rosales and P. A. García-Sánchez

Abstract. We show that the class of system proportionally modular numerical semigroups coincides with the class of numerical semigroups having a Toms decomposition.

Let $\mathbb{N}$ be the set of nonnegative integers. A *submonoid* M of $\mathbb{N}$ is a subset of $\mathbb{N}$ closed under addition and such that $0 \in M$. A *numerical semigroup* S is a submonoid of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite. This last condition is equivalent to $\gcd(S) = 1$, where gcd stands for greatest common divisor.

Let M be a submonoid of $\mathbb{N}$ and let d be a positive integer. Then

$$\frac{M}{d} = \{n \in \mathbb{N} \mid dn \in M\}$$

is again a submonoid of $\mathbb{N}$, called the *quotient of M by d*.

Let S be a numerical semigroup. According to [3], we say that S has a *Toms decomposition* if there exist $q_1, \dots, q_n, m_1, \dots, m_n$ and L such that

(i) $\gcd(\{q_i, m_i\}) = \gcd(\{L, q_i\}) = \gcd(\{L, m_i\}) = 1$ for all $i \in \{1, \dots, n\}$,
(ii) $S = \frac{1}{L} \bigcap_{i=1}^{n} \langle q_i, m_i \rangle$.

Let a, b and c be positive integers. We say that the monoid $\frac{\langle a, b\rangle}{c}$ is a *Toms block* if $\gcd(\{a, b\}) = \gcd(\{a, c\}) = \gcd(\{b, c\}) = 1$. As we are imposing the condition $\gcd(\{a, b\}) = 1$, every Toms block is a numerical semigroup. Observe that $\frac{1}{L} \bigcap_{i=1}^{n} \langle q_i, m_i \rangle = \bigcap_{i=1}^{n} \frac{\langle q_i, m_i\rangle}{L}$. So a numerical semigroup admits a Toms decomposition if and only if it can be expressed as an intersection of finitely many Toms blocks with the same denominator.

We show that Toms blocks are tightly related to the class of numerical semigroups studied in [1]. Let α and β be two positive real numbers such that $\alpha < \beta$. Let T be the (additive) submonoid of $\mathbb{R}_0^+$ generated by $[\alpha, \beta]$. Then $T \cap \mathbb{N}$ is a numerical semigroup. We denote this numerical semigroup by $\mathrm{S}([\alpha, \beta])$. A numerical semigroup is *proportionally modular* if it is of this form. Theorem 13 in [1] states that a numerical semigroup S is proportionally modular if and only if there exist positive integers a, b and c such that $c < a < b$ and $S = \{x \in \mathbb{Z} \mid ax \bmod b \le cx\}$, (where by $a \bmod b$ we mean the remainder of the division of a by b, with a an integer and b a positive integer). Moreover, from [1, Corollary 9], one can deduce that in this case $S = \mathrm{S}\big(\big[\frac{b}{a}, \frac{b}{a-c}\big]\big)$ In this way we obtain the following result, already implicit in [1],

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which allows us a particular choice of the endpoints of the intervals used to define a proportionally modular numerical semigroup.

Lemma 1 *Let S be a proportionally modular numerical semigroup other than $\mathbb{N}$. Then there exist two rational numbers α and β such that $1 < \alpha < \beta$ and $S = \mathrm{S}([\alpha, \beta])$.*

A numerical semigroup is *system proportionally modular* if it is the intersection of finitely many proportionally modular numerical semigroups. In view of the above characterization of proportionally modular numerical semigroups, this means that there exist $a_1, \dots, a_r$, $b_1, \dots, b_r$, $c_1, \dots, c_r$ positive integers such that S is the set of integer solutions to the system of inequalities

$$a_1 x \bmod b_1 \leq c_1 x, \dots, a_r x \bmod b_r \leq c_r x.$$

Proportionally modular numerical semigroups can be characterized as those numerical semigroups that are quotients of numerical semigroups generated by two elements [2, Theorem 5]. Hence every numerical semigroup having a Toms decomposition is system proportionally modular. We consider the converse. Does every system proportionally modular numerical semigroup S have a Toms decomposition? In other words if, according to [2, Theorem 5], S can be expressed as $S = \langle n_1, m_1 \rangle / d_1 \cap \dots \cap \langle n_r, m_r \rangle / d_r$, then can we simultaneously have the d_is equal and each $\langle n_i, m_i \rangle / d_i$ a Toms block? The answer to this question is affirmative, and it is given in Theorem 10.

The idea of the proof of Theorem 10 relies on the following result, which follows from the proof of [2, Theorem 5].

Lemma 2 *Let a_1, a_2, b_1 and b_2 be positive integers such that $1 < \frac{a_1}{b_1} < \frac{a_2}{b_2}$. If* $\gcd(\{a_1, a_2\}) = 1$, *then*

$$\mathrm{S}\Big(\Big[\frac{a_1}{b_1}, \frac{a_2}{b_2} \Big] \Big) = \frac{\langle a_1, a_2 \rangle}{a_2 b_1 - a_1 b_2}.$$

In this result, a condition on the greatest common divisor of the numerators of the fraction defining the interval is needed. This condition, as we see next, is not relevant. Given a proportionally modular numerical semigroup defined by an interval, we are going to show how to perturb the endpoints of this interval so that the resulting numerical semigroup remains the same. By perturbing the left endpoint of the interval, we will be able to obtain intervals whose endpoints fulfill the desired gcd condition. From this we prove that for every proportionally modular numerical semigroup there are infinitely many Toms blocks equal to it. If we are given a finite family of proportionally modular numerical semigroups, by perturbing the right endpoint of the intervals defining them, we will show how the denominators in the obtained Toms block can be chosen to be the same.

Remark 3 For the sake of simplicity, we will allow fractions with a positive integer x as a numerator and denominator zero. We make the convention that $y < \frac{x}{0}$ for any integer y.

Membership in a proportionally modular numerical semigroup is easily characterized once one has an interval defining the semigroup. This is made explicit in the following result, which is easy to prove and can be understood as a reformulation of [1, Lemma 1].

Lemma 4 *Let α and β be two positive real numbers, and let x be a positive integer. Then $x \in \mathrm{S}([\alpha,\beta])$ if and only there exist a positive integer k_x such that $x/k_x \in [\alpha,\beta]$. Thus, $x \in \mathbb{N} \setminus \mathrm{S}([\alpha,\beta])$ if and only if there exists a nonnegative integer n_x such that $\frac{x}{n_x+1} < \alpha < \beta < \frac{x}{n_x}$.*

The next lemma shows how we can modify the left endpoint of the interval defining a proportionally modular numerical semigroup, in a way that the resulting semigroup stays the same.

Lemma 5 *Let a_1, a_2, b_1 and b_2 be positive integers such that $1 < \frac{a_1}{b_1} < \frac{a_2}{b_2}$. Then there exist positive integers a_0 and b_0 such that $1 \le b_0 < a_0$ and for all x, y positive integers such that $\frac{a_0}{b_0} \le \frac{x}{y} \le \frac{a_1}{b_1}$, one gets that $\mathrm{S}\big(\big[\frac{a_1}{b_1},\frac{a_2}{b_2}\big]\big) = \mathrm{S}\big(\big[\frac{x}{y},\frac{a_2}{b_2}\big]\big)$.*

Proof Let $S = \mathrm{S}\big(\big[\frac{a_1}{b_1},\frac{a_2}{b_2}\big]\big)$. From Lemma 4, we know that if $h \in \mathbb{N}\setminus S$, then there exists $n_h \in \mathbb{N}$ such that $\frac{h}{n_h+1} < \frac{a_1}{b_1} < \frac{a_2}{b_2} < \frac{h}{n_h}$. Set $\alpha = \max\{\frac{h}{n_h+1} \mid h \in \mathbb{N}\setminus S\}$ (this maximum exists, since the complement of S in $\mathbb{N}$ is finite). Let a_0 and b_0 be positive integers such that $\alpha < \frac{a_0}{b_0} < \frac{a_1}{b_1}$. Now take any positive integers x, y such that $\frac{a_0}{b_0} \le \frac{x}{y} \le \frac{a_1}{b_1}$. From the choice of α and Lemma 4, it follows easily that $S = \mathrm{S}\big(\big[\frac{x}{y},\frac{a_2}{b_2}\big]\big)$. ■

With this idea we can achieve endpoints of the interval fulfilling the gcd condition of Lemma 2. For a rational number x, we use $\lfloor x \rfloor$ to denote the largest integer less than or equal to x.

Lemma 6 *Let a_1, a_2, b_1 and b_2 be positive integers such that $1 < \frac{a_1}{b_1} < \frac{a_2}{b_2}$. Then there exist positive integers a_0, b_0 and N such that $b_0 < a_0$ and for every integer $x \ge N$ with $\gcd(\{x, a_2\}) = 1$, one has that*

$$\mathrm{S}\Big(\Big[\frac{a_1}{b_1},\frac{a_2}{b_2}\Big]\Big) = \frac{\langle x,\ a_2\rangle}{a_2\lfloor \frac{b_0 x}{a_0}\rfloor - b_2 x}.$$

Proof Let $S = \mathrm{S}\big(\big[\frac{a_1}{b_1},\frac{a_2}{b_2}\big]\big)$. By Lemma 5, we know that there are positive integers a_0 and b_0, such that $b_0 < a_0$, $\frac{a_0}{b_0} < \frac{a_1}{b_1}$ and if $\frac{a_0}{b_0} \le \frac{x}{y} \le \frac{a_1}{b_1}$ for some positive integers x and y, then we have that $S = \mathrm{S}\big(\big[\frac{x}{y},\frac{a_2}{b_2}\big]\big)$. Note that $\frac{a_0}{b_0} \le \frac{x}{y} \le \frac{a_1}{b_1}$ if and only if $\frac{b_1}{a_1}x \le y \le \frac{b_0}{a_0}x$. As $\frac{b_1}{a_1} < \frac{b_0}{a_0}$, there exists a positive integer N such that if $x \ge N$, then $x\big(\frac{b_0}{a_0} - \frac{b_1}{a_1}\big) > 1$. Hence if $x \ge N$, we obtain that $\frac{b_1}{a_1}x \le \lfloor \frac{b_0 x}{a_0} \rfloor \le \frac{b_0}{a_0}x$, and thus

$$\frac{a_0}{b_0} \le \frac{x}{\lfloor \frac{b_0 x}{a_0} \rfloor} \le \frac{a_1}{b_1}.$$

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By Lemma 5,

$$S = \mathrm{S}\Big(\Big[\frac{x}{\lfloor \frac{b_0 x}{a_0} \rfloor}, \frac{a_2}{b_2}\Big]\Big).$$

By taking x such that $\gcd(\{x, a_2\}) = 1$ we have, in view of Lemma 2, that

$$S = \frac{\langle x,\ a_2\rangle}{a_2\lfloor \frac{b_0 x}{a_0} \rfloor - b_2 x}. \qquad \blacksquare$$

From this result, we show next that there are infinitely many Toms blocks representing the same proportionally modular numerical semigroup.

Lemma 7 *Let a_1, a_2, b_1 and b_2 be positive integers such that $1 < \frac{a_1}{b_1} < \frac{a_2}{b_2}$ and $\gcd(\{a_2, b_2\}) = 1$. Then there exist positive integers a_0, b_0 and N such that $b_0 < a_0$ and for every integer $k \geq N$, one has that*

$$\mathrm{S}\Big(\Big[\frac{a_1}{b_1}, \frac{a_2}{b_2}\Big]\Big) = \frac{\langle k a_0 b_0 a_2 + 1,\ a_2\rangle}{k b_0 a_2(b_0 a_2 - b_2 a_0) - b_2}.$$

Moreover, this is a Toms block.

Proof Let $S = \mathrm{S}\big(\big[\frac{a_1}{b_1}, \frac{a_2}{b_2}\big]\big)$. By Lemma 6, we know that there exist positive integers $b_0 < a_0$ and N such that for all $x \geq N$ with $\gcd(\{x, a_2\}) = 1$, one has that $S = \frac{\langle x,\, a_2\rangle}{a_2\lfloor \frac{b_0 x}{a_0}\rfloor - b_2 x}$. Let $k \geq \frac{N-1}{a_0 b_0 a_2}$. Then $x = k a_0 b_0 a_2 + 1$ is greater than or equal to N, $\gcd(\{x, a_2\}) = 1$, and since $b_0 < a_0$,

$$\Big\lfloor \frac{b_0 x}{a_0} \Big\rfloor = \Big\lfloor \frac{k a_0 b_0^2 a_2}{a_0} + \frac{b_0}{a_0} \Big\rfloor = k a_2 b_0^2.$$

Hence

$$S = \frac{\langle k a_0 b_0 a_2 + 1,\ a_2\rangle}{k a_2^2 b_0^2 - b_2(k a_0 b_0 a_2 + 1)} = \frac{\langle k a_0 b_0 a_2 + 1,\ a_2\rangle}{k a_2 b_0(a_2 b_0 - a_0 b_2) - b_2}.$$

Next we show that this representation is a Toms block.

- $\gcd(\{k a_0 b_0 a_2 + 1, a_2\}) = 1$,
- $\gcd(\{k a_2 b_0(a_2 b_0 - a_0 b_2) - b_2, a_2\}) = \gcd(\{b_2, a_2\}) = 1$,
- $\gcd(\{k a_0 b_0 a_2 + 1, k a_2^2 b_0^2 - b_2(k a_0 b_0 a_2 + 1)\}) = \gcd(\{k a_0 b_0 a_2 + 1, k a_2^2 b_0^2\}) = 1$.

$\blacksquare$

We show how to perturb the right endpoint of the interval.

Lemma 8 *Let a_1, a_2, b_1 and b_2 be positive integers such that $1 < \frac{a_1}{b_1} < \frac{a_2}{b_2}$. Then there exists a nonnegative integer N such that for every integer k greater than or equal to N, one has that*

$$\mathrm{S}\Big(\Big[\frac{a_1}{b_1}, \frac{a_2}{b_2}\Big]\Big) = \mathrm{S}\Big(\Big[\frac{a_1}{b_1}, \frac{k a_2 + 1}{k b_2}\Big]\Big).$$

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Proof Let $S = \mathrm{S}\big(\big[\frac{a_1}{b_1}, \frac{a_2}{b_2}\big]\big)$. In view of Lemma 4, for every $h \in \mathbb{N} \setminus S$, there exists $n_h \in \mathbb{N}$ with $\frac{h}{n_h+1} < \frac{a_1}{b_1} < \frac{a_2}{b_2} < \frac{h}{n_h}$. Let $\alpha = \min\{\frac{h}{n_h} \mid h \in \mathbb{N} \setminus S\}$ (as we are allowing to divide by zero, this coud be infinite). The sequence $\{\frac{ka_2+1}{kb_2}\}_{k\in\mathbb{N}\setminus\{0\}}$ is strictly decreasing and converges to $\frac{a_2}{b_2}$. Thus, there exists $N \in \mathbb{N}$ such that if $k \geq N$, we have that $\frac{a_2}{b_2} < \frac{ka_2+1}{kb_2} < \alpha$ and arguing as in Lemma 5, we conclude that $S = \mathrm{S}\big(\big[\frac{a_1}{b_1}, \frac{ka_2+1}{kb_2}\big]\big)$. ■

With this result, we can show that for a finite family of proportionally modular numerical semigroups, the right endpoint of the interval can be chosen to be reduced fractions with the same denominator.

Lemma 9 *Let* $S_i = \mathrm{S}\big(\big[\frac{a_{i,1}}{b_{i,1}}, \frac{a_{i,2}}{b_{i,2}}\big]\big)$ *with* $a_{i,1}$, $a_{i,2}$, $b_{i,1}$ *and* $b_{i,2}$ *positive integers with* $1 < \frac{a_{i,1}}{b_{i,1}} < \frac{a_{i,2}}{b_{i,2}}$, *and* $i \in \{1, \dots, r\}$. *Then there exist positive integers* $c_1, \dots, c_r$ *and* d *such that for all* $i \in \{1, \dots, r\}$,

- $S_i = \mathrm{S}\big(\big[\frac{a_{i,1}}{b_{i,1}}, \frac{c_i}{d}\big]\big)$ *and*
- $\gcd(\{c_i, d\}) = 1$.

Proof From Lemma 8, we know that for every $i \in \{1, \dots, r\}$, there exists $N_i \in \mathbb{N}$ so that if k_i is a positive integer greater than N_i, one has that $S_i = \mathrm{S}\big(\big[\frac{a_{i,1}}{b_{i,1}}, \frac{k_i a_{i,2}+1}{k_i b_{i,2}}\big]\big)$. For every $i \in \{1, \dots, r\}$, set $k_i = t\frac{b_{1,2}^2 \cdots b_{r,2}^2}{b_{i,2}}$, with t a positive integer large enough to ensure that $k_i \geq N_i$ for all i. Then

$$S_i = \mathrm{S}\left(\left[\frac{a_{i,1}}{b_{i,1}}, \frac{t\frac{b_{1,2}^2 \cdots b_{r,2}^2}{b_{i,2}} a_{i,2} + 1}{t b_{1,2}^2 \cdots b_{r,2}^2}\right]\right).$$

Clearly, $\gcd(\{t\frac{b_{1,2}^2 \cdots b_{r,2}^2}{b_{i,2}} a_{i,2} + 1, t b_{1,2}^2 \cdots b_{r,2}^2\}) = 1$ for all $i \in \{1, \dots, r\}$. ■

We are now ready to prove the main result.

Theorem 10 *Every system proportionally modular numerical semigroup admits a Toms decomposition.*

Proof Let S be a system proportionally modular numerical semigroup. If $S = \mathbb{N}$, then $\mathbb{N} = \frac{\langle 2, 3\rangle}{5}$ suits our needs. Otherwise, $S = S_1 \cap \cdots \cap S_r$ for some $S_1, \dots, S_r$ proportionally modular numerical semigroups different from $\mathbb{N}$. In view of Lemmas 1 and 9, there exist some positive integers $a_1, \dots, a_r, b_1, \dots, b_r, c_1, \dots, c_r$ and d such that $S_i = \mathrm{S}\big(\big[\frac{a_i}{b_i}, \frac{c_i}{d}\big]\big)$ with $\gcd(\{c_i, d\}) = 1$ for all $i \in \{1, \dots, r\}$. From Lemma 7, we know that there exist positive integers $b_{i_0} < a_{i_0}$ and $N_i \in \mathbb{N}$ such that for all $k_i \geq N_i$ one obtains that

$$\frac{\langle k_i a_{i_0} b_{i_0} c_i + 1,\ c_i\rangle}{k_i c_i b_{i_0}(c_i b_{i_0} - a_{i_0} d) - d}$$

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is a Toms block equal to S_i. Let $m_i = c_i b_{i_0}(c_i b_{i_0} - a_{i_0} d)$. Let $t = \max\{N_1, \dots, N_r\}$. Then setting $k_i = t\frac{m_1 \cdots m_r}{m_i}$ one concludes that

$$S = \bigcap_{i=1}^{r} \frac{\langle k_i a_{i_0} b_{i_0} c_i + 1,\ c_i \rangle}{t m_1 \cdots m_r - d}$$

is a Toms representation for S. ∎

Toms wondered [3] if every numerical semigroup could be expressed in this way. Since every numerical semigroup having a Toms decomposition is system proportionally modular, one can use [1, Algorithm 27] to decide whether or not a given numerical semigroup has Toms decomposition, and one can also find in that paper numerical semigroups not having such a decomposition.

In view of Toms result [3], we obtain the following corollary of Theorem 10.

Corollary 11 *Every ordered group of the form* $(\mathbb{Z}, S)$*, where* S *is a system proportionally modular numerical semigroup, occurs as the ordered* K_0*-group of a simple, separable, and nuclear* C^**-algebra.*

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Departamento de Álgebra
Universidad de Granada
E-18071 Granada
Spain
e-mail: jrosales@ugr.es
pedro@ugr.es

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