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# GEODESIC CORRESPONDENCE IN THE BRANS-DICKE THEORY

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In a recent article [1] vacuum field solutions of the Brans-Dicke [2] field equations were found, the space-time metric in each solution being of the Friedmann type. Most of these solutions existed only for specific values of the parameter $\omega$ and, in particular, the two largest sets of solutions corresponded to the values $\omega=-\frac{3}{2}$ and $\omega=-\frac{4}{3}$. Peters [3, 4] has shown that when $\omega=-\frac{3}{2}$ all solutions of the Brans-Dicke vacuum equations are conformal to space-times with vanishing Ricci tensor. The purpose of this note is to investigate the possible geometric consequences of the value $\omega=-\frac{4}{3}$.

When $\omega=-\frac{4}{3}$ the field equations for vacuum in the Brans-Dicke theory may be written in the form

(1) $$R_{\mu\nu}+\frac{1}{\phi}\,\phi_{;\mu\nu}-\frac{4}{3\phi^2}\,\phi_{,\mu}\phi_{,\nu}=0$$

and

(2) $$\phi^{;\alpha}_{;\alpha}=0.$$

Consider two Riemannian spaces $V_n$, $\bar{V}_n$ with respective fundamental forms

$$ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu$$

and

$$d\bar{s}^2=\bar{g}_{\mu\nu}\,dx^\mu\,dx^\nu.$$

If the geodesics in both spaces are expressed in terms of the same arbitrary parameter $\lambda$ and if the resulting geodesic equations in $V_n$ are identical with those in $\bar{V}_n$, the spaces are said to be in geodesic correspondence or projectively related [5, 6]. A necessary and sufficient condition for $V_n$, $\bar{V}_n$ to be in geodesic correspondence is that their respective Christoffel symbols are related by

(3) $$\overline{\begin{Bmatrix}\sigma\\ \mu\nu\end{Bmatrix}}=\begin{Bmatrix}\sigma\\ \mu\nu\end{Bmatrix}+\delta^\sigma_\mu\psi_{,\nu}+\delta^\sigma_\nu\psi_{,\mu}$$

where $\psi$ is a scalar function of the co-ordinates. The Ricci tensors in the two spaces satisfy [5]

$$\bar{R}_{\mu\nu}=R_{\mu\nu}+(n-1)(\psi_{;\mu\nu}-\psi_{,\mu}\psi_{,\nu})$$

where the semi-colon denotes covariant differentiation with respect to the $\{^{\sigma}_{\nu\mu}\}$. Thus two space-times $V_4$, $\bar{V}_4$ with corresponding geodesics will satisfy

(4) $$\bar{R}_{\mu\nu}=R_{\mu\nu}+3(\psi_{;\mu\nu}-\psi_{,\mu}\psi_{,\nu}).$$



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By defining a new variable

$$\psi = \tfrac{1}{3} \log \phi$$

equations (1) and (2) become

$$R_{\mu\nu}+3(\psi_{;\mu\nu}-\psi_{,\mu}\psi_{,\nu}) = 0 \tag{5}$$

and

$$\psi^{\alpha}_{;\alpha}+3\psi_{,\alpha}\psi^{\alpha}_{,} = 0. \tag{6}$$

Comparing equations (4) and (5) it is seen that the equations are identical if $\bar{R}_{\mu\nu}=0$. Hence it follows that when $\omega=-\frac{4}{3}$ the space-time solutions of the Brans-Dicke vacuum field equations are in geodesic correspondence with the space-time solutions of the Einstein vacuum field equations provided that equation (3) is satisfied.

It is known [5] that spaces of constant curvature can be in geodesic correspondence only with spaces of constant curvature; the values of the curvatures of the corresponding spaces are not necessarily the same. In particular it follows from a result due to Petrov [6] that if a space-time $\bar{V}_4$, with vanishing Ricci tensor, is in geodesic correspondence with another space-time $V_4$ then $V_4$ and $\bar{V}_4$ must be of constant curvature which implies that $\bar{V}_4$ is necessarily flat space-time.

Hence we have the following result: the Brans-Dicke vacuum field equations admit solutions which are spaces of constant curvature for arbitrary values of the parameter $\omega$ [1]. These spaces are in geodesic correspondence with other spaces of constant curvature. The special case when the solutions are in geodesic correspondence with Minkowski flat space-time occurs if $\omega=-\frac{4}{3}$. If the solutions with $\omega=-\frac{4}{3}$ are not of constant curvature then they are not in geodesic correspondence with any other space-time and nothing more can be said.

Although the value $\omega=-\frac{4}{3}$ is sufficient to ensure the geodesic correspondence with flat space-time, it is not a necessary condition. Consider the Brans-Dicke vacuum equations for general $\omega$ and define a new variable

$$\psi = k^{-1} \log \phi.$$

The equations become

$$R_{\mu\nu}+k\psi_{;\mu\nu}+k^2(1+\omega)\psi_{,\mu}\psi_{,\nu} = 0 \tag{7}$$

and

$$\psi^{\alpha}_{;\alpha}+k\psi_{,\alpha}\psi^{\alpha}_{,} = 0 \tag{8}$$

where equation (8) exists only if $\omega\neq-\frac{3}{2}$. Equation (7) is identical with equation (5) if

$$\omega k^2+3k+3 = 0 \tag{9}$$

and

$$(k-3)(\psi_{;\mu\nu}+k\psi_{,\mu}\psi_{,\nu}) = 0. \tag{10}$$

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If $k=3$, which from (9) corresponds to $\omega=-\frac{4}{3}$, equation (10) is satisfied but if $k\neq 3$ (i.e. $\omega\neq-\frac{4}{3}$) then geodesic correspondence with flat space-time occurs only if $\psi$ satisfies the additional condition

$$\psi_{;\mu\nu}+k\psi_{,\mu}\psi_{,\nu} = 0$$

which is equivalent to

$$\phi_{;\mu\nu} = 0. \tag{11}$$

Equation (9) has real roots provided that $\omega\leq\frac{3}{4}$; hence for these values of $\omega$ it is possible to find Brans-Dicke vacuum solutions which are in geodesic correspondence with flat space-time provided that the scalar $\phi$ satisfies equation (11) rather than the weaker condition (2). Only when $\omega=-\frac{4}{3}$ is equation (2) sufficient.

Finally we note that the solutions found in [1] corresponding to $\omega=-\frac{4}{3}$ were all different forms of the de Sitter universe, each associated with a different scalar function $\phi$. The de Sitter universe is a space of constant curvature and so, from the argument above, it follows that it is in geodesic correspondence with flat space-time. On the other hand the well-known Brans-Dicke solutions with $\omega=-\frac{4}{3}$ is not a space of constant curvature and so is not in geodesic correspondence with any other space-time.

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