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ON CLOSURE CONDITIONS

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Quasigroups and groupoids with one or other of the Reidemeister or Thomsen closure conditions, the relationship among them with emphasis on their relationship to associativity viz groups, Abelian groups, have been investigated in [2], [3], [4], [5], [6], [12], and others. In [10] R- and T-groupoids, (that is, groupoids possessing one of the first two closure conditions mentioned above) which are generalizations of groups and Abelian groups were investigated. In this paper, we show that groupoids with the given identities may be described in terms of R- and T-groupoids. These results and others are used to give another proof of theorems given in [1], [7], and [5] describing the variety of all groups and Abelian groups defined by single laws.

1. In [8], [9] groupoids $G(\cdot)$ satisfying one or more of the identities

$$xz \cdot yz = xy \tag{1}$$

$$xy \cdot xz = zy \tag{2}$$

$$xy \cdot xz = yz \tag{3}$$

$$xy \cdot zy = zx \tag{4}$$

had been investigated. It was shown that a groupoid $G(\cdot)$ possessing an element $a \in G$ with the property $G \cdot a = G$ is an iso-group (i.e. a particular isotope of a group) if (3) holds in $G(\cdot)$. If, in addition (4) holds in $G(\cdot)$ or $G(\cdot)$ satisfies (4) with $G \cdot a = G$, for some $a \in G$, then $G(\cdot)$ is an iso-abelian group. The proof relies heavily upon the condition $G \cdot a = G$. Indeed the condition is essential for these results, because the groupoid with the multiplication table given in example 1 satisfies (1), (2), (3), and (4), but is not even a quasigroup:

	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_3	x_3
x_2	x_1	x_1	x_3	x_3
x_3	x_3	x_3	x_1	x_1
x_4	x_3	x_3	x_1	x_1

Example 1

We will show that groupoids with these given identities may be described in terms of R- and T-groupoids and use these results to characterize groups,

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Abelian groups, etc. By defining $x*y=z$ iff $y\cdot x=z$, it is easy to see that (1) and (3) and (2) and (4) are equivalent. So, we will consider only (1) and (2).

2. **Definitions.** A groupoid $G(\cdot)$ with the property that for all $x_i, y_i \in G$, $(i=1,2,3,4)$, the equations

$$x_1y_2=x_2y_1, \qquad x_1y_4=x_2y_3, \qquad x_4y_1=x_3y_2$$

imply

$$x_3y_4=x_4y_3,$$

is called an *R-groupoid*. This closure condition is the Reidemeister condition.

$G(\cdot)$ is called a *T-groupoid* if for all x_i, $y_i\in G$, $(i=1,2,3)$ the equations

$$x_1y_2=x_2y_1, \qquad x_1y_3=x_3y_1$$

imply

$$x_2y_3=x_3y_2.$$

This closure condition is the Thomsen condition.

Elements x_1, x_2 of a groupoid $G(\cdot)$ are said to be *left cancellative equivalent* (l.c.e.) if $yx_1=yx_2$ for all $y\in G$.

A groupoid is said to be *left cancellative equivalent* if $ax_1=ax_2$, $a\in G$, implies that x_1, x_2 are left concellative equivalent.

Right cancellative equivalent (r.c.e.) of both elements and groupoids is similarly defined.

For any groupoid $G(\cdot)$ we may define a congruence ρ by $x_1\rho x_2$ iff x_1, x_2 are l.c.e. and r.c.e. The quotient groupoid G/ρ is called the *reduction* of G. Let $h:G\to G/\rho$ be the canonical homomorphism then $h(x_1)=h(x_2)$, for $x_1, x_2\in G$ iff $x_1y=x_2y$, and $yx_1=yx_2$, for all $y\in G$.

Two groupoids $G(\cdot)$ and $H(*)$ are said to be isotopic, if there exist three one-one, onto mappings $\alpha, \beta, \gamma: G\to H$ such that $\gamma(x\cdot y)=\alpha x*\beta y$ holds for all $x, y\in G$.

3. Lemma 3.1. *If $G(\cdot)$ satisfies either* (1) *or* (2), *then the reduction of $G(\cdot)$ also satisfies* (1) *or* (2) *respectively.*

The proof of the lemma is straightforward.

Theorem 3.1. *Let $G(\cdot)$ be a groupoid which satisfies* (1). *Then $G(\cdot)$ is r.c.e. and the reduction $H(\circ)$ of G is an R-groupoid.*

Proof. First we show that $G(\cdot)$ is r.c.e. Suppose $x_1a=x_2a$, for $a, x_1, x_2\in G$. Then $x_1a\cdot xa=x_2a\cdot xa$, for all $x\in G$. Hence by (1), $x_1x=x_2x$ for all $x\in G$.

If x_1 and x_2 are r.c.e., then $xx\cdot x_1x=xx\cdot x_2x$, for all $x\in G$. It then follows by (1) that x_1, x_2 are l.c.e. Thus, because every pair of elements that are r.c.e. are l.c.e. and $G(\cdot)$ is itself l.c.e., its reduction $H(\circ)$ is right cancellative.

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Assume

(3.1) $$x_1\circ y_2=x_2\circ y_1,\qquad x_1\circ y_4=x_2\circ y_3\quad\text{and}\quad x_3\circ y_2=x_4\circ y_1,$$
$$\text{for}\quad x_i,\, y_i\in H\quad (i=1,2,3,4).$$

Since $H(\circ)$ satisfies (1) (by Lemma 3.1), from (3.1) we obtain $(x_3\circ y_2)\circ(x_1\circ y_2)=(x_4\circ y_1)\circ(x_2\circ y_1)$, that is, $x_3\circ x_1=x_4\circ x_2$. Using (1) again, $(x_3\circ y_4)\circ(x_1\circ y_4)=(x_4\circ y_3)\circ(x_2\circ y_3)$ which by the right cancellativity yields $x_3\circ y_4=x_4\circ y_3$. Thus $H(\circ)$ is an R-groupoid. This completes the proof of this theorem.

LEMMA 3.2. *If $G(\cdot)$ satisfies* (1) *and if $G(\cdot)$ is either l.c.e. or left cancellative* (*l.c.*) *or right cancellative* (*r.c.*) *in $G(\cdot)$ or $xx=$ constant holds for all x, then G is an R-groupoid.*

Proof. Let $G(\cdot)$ be a groupoid satisfying (1). Suppose $G(\cdot)$ is l.c.e. (l.c.). Assume (3.1) to hold in G for $x_i, y_i\in G$ $(i=1,2,3,4)$. From the last equality of (3.1) and (1) result $x_3x_3=x_4x_4$. Now (1) and (3.1) give, $x_1y_2\cdot x_3y_2=x_2y_1\cdot x_4y_1$, that is $x_1x_3=x_2x_4$. Since $x_1y_4=x_2y_3$, we have $x_1x_3\cdot y_4x_3=x_2x_4\cdot y_3x_4$. Thus, since G is l.c.e. (l.c.), $x_3x_3\cdot y_4x_3=x_4x_4\cdot x_3x_4$, that is $x_3y_4=x_4y_3$. Hence, R-condition holds in $G(\cdot)$.

The latter part of Theorem 3.1 shows that $G(\cdot)$ is an R-groupoid if $G(\cdot)$ is right cancellative.

Finally, suppose $xx=e$, for every $x\in G$. If $ab=cd$, then by (1), $bb\cdot ab=dd\cdot cd$ implies $ba=dc$. Thus, $x_1x_3=x_2x_4$ and $x_1y_4=x_2y_3$ imply $x_3x_1\cdot y_4x_1=x_4x_2\cdot y_3x_2$, that is, $x_3y_4=x_4y_3$. This proves Lermma 3.2.

THEOREM 3.2. *If $G(\cdot)$ satisfies both* (1) *and* (2), *then its reduction $H(\circ)$, is a cancellative T-groupoid.*

Proof. Under (1), $G(\cdot)$ is r.c.e. Suppose $ax_1=ax_2$, $a, x_1, x_2\in G$. Then $ax_1\cdot ax=ax_2\cdot ax$ and by (2), $xx_1=xx_2$. Thus $G(\cdot)$ is l.c.e. Hence the reduction $H(\circ)$ is cancellative.

Now, to prove that the T-condition holds in $H(\circ)$. Suppose

(3.2) $$x_1\circ y_2=x_2\circ y_1\quad\text{and}\quad x_1\circ y_3=x_3\circ y_1,$$

hold for $x_i, y_i\in H$, $(i=1,2,3)$.

Then $(x_1\circ y_2)\circ(x_1\circ y_3)=(x_2\circ y_1)\circ(x_3\circ y_1)$ by (1) and (2) yields $y_3\circ y_2=x_2\circ x_3$, which by using (1) and (2) again gives $(x_3\circ y_2)\circ(x_3\circ y_3)=(x_2\circ y_3)\circ(x_3\circ y_3)$. Cancellation gives the required implication $x_3\circ y_2=x_2\circ y_3$.

THEOREM 3.3. *If $G(\cdot)$ satisfies* (2), *then it is a T-groupoid.*

Proof. Suppose that $x_1y_2=x_2y_1$ and $x_1y_3=x_3y_1$ for all $x_i, y_i\in G$, $(i=1,2,3)$. The use of (2) and the hypothesis, yield

$$y_2y_3=x_1y_3\cdot x_1y_2=x_3y_1\cdot x_2y_1=(x_3y_1\cdot x_3x_3)(x_3y_1\cdot x_3x_2)=x_3x_2.$$

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Thus

$$x_2y_3 = x_3y_3 \cdot x_3x_2 = x_3y_3 \cdot y_2y_3 = (x_3y_3 \cdot x_3x_3) \cdot (x_3y_3 \cdot x_3y_2)$$
$$= x_3y_2 \cdot x_3x_3 = x_3y_2.$$

Consequently $G(\cdot)$ is a T-groupoid.

The groupoid given by example 1 satisfies (1), (2), (3), and (4) and its reduction is isomorphic to Z_2. The groupoid given by example 2 is isomorphic to its reduction and satisfies (3) but neither (1) nor (2) nor (4):

	x_1	x_2
x_1	x_1	x_2
x_2	x_1	x_2

Example 2

The groupoid given by example 3 satisfies (1) but neither (2) nor (3) nor (4):

	x_1	x_2
x_1	x_1	x_1
x_2	x_2	x_2

Example 3

4. **Characterizations.** In the sequel, we make use of the following results in [2]:

(RG). A quasigroup $G(\cdot)$ is isotopic to a group if and only if Reidemeister condition (R-condition) holds in $G(\cdot)$.

(TAG). A quasigroup $G(\cdot)$ is isotopic to an Abelian group iff Thomsen condition (T-condition) holds in $G(\cdot)$.

We give another proof of the following results [7], [5], [1] using the R-condition and T-condition.

THEOREM 4.1. *The variety of all groups is the variety of all groupoids $G(\cdot)$ satisfying the single law*

$$x \cdot [\{(xx \cdot y) \cdot z\} \cdot \{(xx \cdot x) \cdot z\}] = y, \quad \textit{for all} \quad x, y, z \in G. \tag{4.1}$$

Proof. First of all $G(\cdot)$ is a quasigroup [7, p. 21], [5, p. 30]. Putting $y = x$ in (4.1) and using z as a variable, we get $x \cdot (uu) = x = x \cdot vv$, for all $u, v \in G$, yielding $uu = \text{constant} = e$, for all $u \in G$ and $xe = x$ for all $x \in G$. Now (4.1) becomes

$$x \cdot [(ey \cdot z) \cdot (ex \cdot z)] = y, \quad \text{for all} \quad x, y, z \in G. \tag{4.3}$$
$$= x \cdot [(ey \cdot e) \cdot (ex \cdot e)] = x \cdot (ey \cdot ex),$$

which since x and y are arbitrary, results to

$$yz \cdot xz = yx, \quad \text{for all} \quad x, y, z \in G. \tag{1}$$

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Now Lemma 3.2 shows that R-condition holds in $G(\cdot)$. The use of (RG) yields the required result.

THEOREM 4.2. *If a groupoid $G(\cdot)$ satisfies* (1) (*known as transitivity equation*) *and* $(\cdot)$ *is left cancellative* (*l.c.*), *then $G(\cdot)$ is isotopic to a group* [1, p. 275].

Proof. Let $aa=e$ for some $a \in G$. Then $x=y=z=a$ in (1) gives $e \cdot e=e$. With $x=z=e$, (1) and l.c. imply $ye=y$ for all $y \in G$. Now putting $x=e$ and $z=y$ in (1) we have $ey \cdot yy=ey=ey \cdot e$, so that l.c. implies $yy=e$ for all $y \in G$.

If S and T are two mappings of G such that $ST=P$, where P is a permutation of G, then T is upon and S is one-to-one [5, p. 30]. Let L_x and R_x denote left and right multiplications of x respectively.

With $y=e$, (1) gives $R_zR_{ez}=I$. From this we see that R_z is one-to-one and R_{ez} is onto for every $z \in G$. This implies that R_{ez} is a permutation and consequently so is R_z. Again using (1), with $x=z$ and $y=e$ we get $L_eL_e=I$, that is, L_e is a permutation. Finally $x=z$ in (1) gives $R_xL_e=L_x$, showing thereby that L_x is a permutation. Thus $G(\cdot)$ is a quasigroup.

Use of Lemma 3.2 shows that R-condition holds in $G(\cdot)$. An application of (RG) shows that $G(\cdot)$ is isotopic to a group.

THEOREM 4.3. *The variety of Abelian groups is the variety of all groupoids $G(\cdot)$ satisfying the identity.*

$$x \cdot (yz \cdot yx)=z, \quad \text{for all} \quad x, y, z \in G. \tag{4.4}$$

Proof. First we note that $G(\cdot)$ is a quasigroup [7, p. 220]. In (4.4) taking $z=x$ and noting that y is a variable, we get $x \cdot uu=x=x \cdot vv$, for all $u, v \in G$ giving $uu=\text{constant}=e$ and $xe=x$, for all $x \in G$. With $y=x$ (4.4) gives $x \cdot xz=z$, so that, $x \cdot (yz \cdot yx)=z=x \cdot xz$ giving (2). Hence by Theorem 3.3, T-condition holds in $G(\cdot)$. Now applying (TAG), we obtain the sought for result.

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