CERTAIN REPRESENTATION ALGEBRAS

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(received 5 May 1964)

Introduction

Let Λ be the set of inequivalent representations of a finite group $\mathscr{G}$ over a field $\mathscr{F}$. Λ is made the basis of an algebra $\mathscr{A}$ over the complex numbers $\mathscr{C}$, called the representation algebra, in which multiplication corresponds to the tensor product of representations and addition to direct sum. Green [5] has shown that if char $\mathscr{F} \nmid |\mathscr{G}|$ (the non-modular case) or if $\mathscr{G}$ is cyclic, then $\mathscr{A}$ is semi-simple, i.e. is a direct sum of copies of $\mathscr{C}$. Here we consider two modular, non-cyclic cases, viz. where $\mathscr{G}$ is $\mathscr{V}_4 (= Z_2 \times Z_2)$ or $\mathscr{A}_4$ (alternating group) and $\mathscr{F}$ is of characteristic 2.

Finally we consider the analogous case where the multiplication is changed to the ordinary ring tensor product over the group algebra $\mathscr{F}(\mathscr{G})$.

I would like to thank Dr. G. E. Wall for indicating these problems.

1. Representation algebras of groups

Let $\mathscr{P}$ be an arbitrary commutative ring with a unity, and let $\mathscr{F}(\mathscr{G})$ be the group algebra of a group $\mathscr{G}$ over a field $\mathscr{F}$. The *representation algebra* $\mathscr{A}(\mathscr{P}, \mathscr{F}, \mathscr{G}) (= \mathscr{A})$ is defined as follows. It is the $\mathscr{P}$-module generated by the set of all isomorphism classes $\{\mathscr{M}\}$ of $\mathscr{F}(\mathscr{G})$-modules [1] $\mathscr{M}$, subject to the relations

$$\{\mathscr{M}\} = \{\mathscr{M}'\} + \{\mathscr{M}''\}, \tag{1}$$

for all $\mathscr{M}, \mathscr{M}', \mathscr{M}''$ such that $\mathscr{M} = \mathscr{M}' \oplus \mathscr{M}''$, and equipped with the bilinear multiplication given by

$$\{\mathscr{M}\}\{\mathscr{M}'\} = \{\mathscr{M} \times \mathscr{M}'\}. \tag{2}$$

Here $\mathscr{M} \times \mathscr{M}'$ is the module obtained from the tensor (Kronecker) product [2] of the representations afforded by $\mathscr{M}, \mathscr{M}'$. By the Krull-Schmidt theorem for $\mathscr{F}(\mathscr{G})$-modules, $\mathscr{A}$ is free as a $\mathscr{P}$-module and the $\mathscr{F}(\mathscr{G})$-indecomposable classes form a $\mathscr{P}$-basis. $\mathscr{A}$ is a commutative, associative algebra over $\mathscr{P}$,

[1] We consider only modules $\mathscr{M}$ of finite $\mathscr{F}$-dimension.

[2] See page 69 of [3] for the definition of *tensor product representation*.

and has identity element $\{\mathscr{F}_{\mathscr{G}}\}$, i.e., the class containing the trivial $\mathscr{F}(\mathscr{G})$-module.

When $\mathscr{P}$ is taken to be the field $\mathscr{C}$ of complex numbers, $\mathscr{A}(\mathscr{C}, \mathscr{F}, \mathscr{G})$ is semi-simple in the non-modular case, or when $\mathscr{G}$ is a cyclic group (Green [5]). In these cases there are only a finite number of different indecomposable classes and so $\mathscr{A}$ is a direct sum of copies of $\mathscr{C}$. In general the structure of $\mathscr{A}$ is more complicated, but we may still hope for semi-simplicity.

Green in [5] is more precise. When $\mathscr{P} = \mathscr{C}$, we define a *G-character* of $\mathscr{A}$ to be a non-zero algebra homomorphism $\phi : \mathscr{A} \to \mathscr{C}$. $\mathscr{A}$ is then *G-semisimple* if, given any non-zero element $A \in \mathscr{A}$, there exists some G-character ϕ of $\mathscr{A}$ such that $\phi(A) \neq 0$. We may define the *G-radical* of $\mathscr{A}$ to be the intersection $\cap \mathscr{M}_\alpha$ of all maximal ideals $\mathscr{M}_\alpha$ of $\mathscr{A}$, such that $\mathscr{A}/\mathscr{M}_\alpha \approx \mathscr{C}$. Then $\mathscr{A}$ is G-semisimpe if and only if the $\mathscr{G}$-radical $= (0)$.

Let $\mathscr{F}^*$ be an extension field of $\mathscr{F}$. Each $\mathscr{F}(\mathscr{G})$-module $\mathscr{M}$ gives rise to a $\mathscr{F}^*(\mathscr{G})$-module $\mathscr{M}^* = \mathscr{F}^* \otimes_{\mathscr{F}} \mathscr{M}$. We have the following

PROPOSITION 1.[3] *$\mathscr{M} \approx \mathscr{M}'$ if and only if $\mathscr{M}^* \approx \mathscr{M}'^*$.*

The mapping $\{\mathscr{M}\} \to \{\mathscr{M}^*\}$ gives rise to a natural homomorphism

(3) $$\mathscr{A}(\mathscr{C}, \mathscr{F}, \mathscr{G}) \to \mathscr{A}(\mathscr{C}, \mathscr{F}^*, \mathscr{G}).$$

From proposition 1 it follows that this is actually a monomorphism. In view of this natural embedding we shall use $\{\mathscr{M}\}$ to denote either $\{\mathscr{M}\}$ or $\{\mathscr{M}^*\}$; the interpretation will be clear from the context.

Let $\mathscr{H}$ be a subgroup of $\mathscr{G}$, let $\mathscr{L}$ be a $\mathscr{F}(\mathscr{H})$-module, and let $\mathscr{M}$ be a $\mathscr{F}(\mathscr{G})$-module. $\mathscr{L}^{\mathscr{G}}$ will denote the induced $\mathscr{F}(\mathscr{G})$-module

$$\mathscr{F}(\mathscr{G}) \otimes_{\mathscr{F}(\mathscr{H})} \mathscr{L},$$

while $\mathscr{M}_{\mathscr{H}}$ will denote the $\mathscr{F}(\mathscr{H})$-module obtained by restriction of the module multiplications to the subalgebra $\mathscr{F}(\mathscr{H})$ of $\mathscr{F}(\mathscr{G})$.[4]

PROPOSITION 2.[5] $\mathscr{L}^{\mathscr{G}} \times \mathscr{M} \approx (\mathscr{L} \times \mathscr{M}_{\mathscr{H}})^{\mathscr{G}}$.

Proposition 2 shows that the subspace spanned by all the $(\mathscr{G}, \mathscr{H})$-projective modules[6] is an ideal of $\mathscr{A}$.

In particular, taking $\mathscr{H} = \{E\}$, the trivial subgroup of $\mathscr{G}$, we have that the $(\mathscr{F}(\mathscr{G})$-$)$projective modules span an ideal $\mathscr{D}$ of $\mathscr{A}$, which we shall call the *projective ideal* of $\mathscr{A}$.

PROPOSITION 3. *The projective ideal $\mathscr{D}$ is semi-simple and finite dimensional.*

REMARK. In the non-modular case, $\mathscr{D} = \mathscr{A}$ and the proposition reduces

[3] See p. 200 of [3] for the proof.

[4] For further explanations, see [3].

[5] See, for instance, theorem 38.5 (ii), p. 268 of [3].

[6] See definition 63.1 (p. 427), and theorem 63.5 (p. 429) of [3].

to Green's result. We shall therefore assume in the proof that $\mathscr{F}$ is of characteristic $p \neq 0$.

PROOF. In proposition 1 take $\mathscr{F}^*$ to be algebraically closed. Then the restriction of the monomorphism (3) to $\mathscr{D}$, embeds $\mathscr{D}$ in the projective ideal $\mathscr{D}^*$ of $\mathscr{A}(\mathscr{C}, \mathscr{F}^*, \mathscr{G})$. It will therefore be sufficient to consider $\mathscr{F}$ algebraically closed.

Let $\mathscr{K}_1, \cdots, \mathscr{K}_r$ be the p-regular conjugacy classes of $\mathscr{G}$ and let $X_\nu \in \mathscr{K}_\nu$ $(\nu = 1, \cdots, r)$. Any $\mathscr{F}(\mathscr{G})$-module class $\{\mathscr{M}\}$ then defines a Brauer character χ, which is completely determined by the values [7] of the $\chi(X_\nu) \in \mathscr{C}$. Write

$$\beta_\nu(\mathscr{M}) = \chi(X_\nu).$$

β_ν can then be extended linearly over $\mathscr{C}$ to give a map $\beta_\nu : \mathscr{A} \to \mathscr{C}$. This is readily verified to be a $\mathscr{C}$-algebra homomorphism and so β_ν is a G-character of $\mathscr{A}$.

Consider now the restrictions γ_ν of the β_ν to $\mathscr{D}$. As $\mathscr{F}$ is algebraically closed, the number of different indecomposable projective modules (i.e. indecomposable summands of the regular module) is equal to the number of p-regular conjugacy classes [8], i.e. r. Let $\{\mathscr{P}_1\}, \cdots, \{\mathscr{P}_r\}$ be these different classes. The $\{\mathscr{P}_\mu\}$ are a basis of $\mathscr{D}$. We prove that $\mathscr{D}$ is semisimple by showing that $\bigcap_\nu \ker \gamma_\nu = (0)$. Now this last is so if and only if the matrix $(\gamma_\nu(\mathscr{P}_\mu))$ is non-singular. But this matrix is precisely the matrix H on p. 599 of [3], and is non-singular as the Cartan matrix C is non-singular.

COROLLARY 4. *If $\mathscr{F}$ is algebraically closed, $\mathscr{D}$ is isomorphic to the direct sum of r copies of $\mathscr{C}$.*

COROLLARY 5. *$\mathscr{D}$ is an ideal direct summand of $\mathscr{A}$.*

PROOF. This follows directly from the fact that $\mathscr{A}$, $\mathscr{D}$ have unit elements.

2. Representations of $\mathscr{V}_4$ over a field characteristic 2

All representations of the group $\mathscr{V}_4 (= Z_2 \times Z_2)$ over a field $\mathscr{F}$ of characteristic 2 have been essentially determined by two authors [1], [6]. Let $\mathscr{V}_4$ have generators X, Y satisfying $X^2 = Y^2 = E$, $XY = YX$, with E the identity element. In the group algebra $\mathscr{F}(\mathscr{V}_4)$ write

$$P = X + E, \qquad Q = Y + E.$$

Then $P^2 = Q^2 = 0$, $PQ = QP$ and

$$\mathscr{F}(\mathscr{V}_4) \approx \mathscr{F}[P, Q]/(P^2, Q^2) = \mathscr{R}, \text{ say,}$$

[7] See pages 588, 589 of [3] for the definition of Brauer characters, etc.

[8] See page 591 of [3].

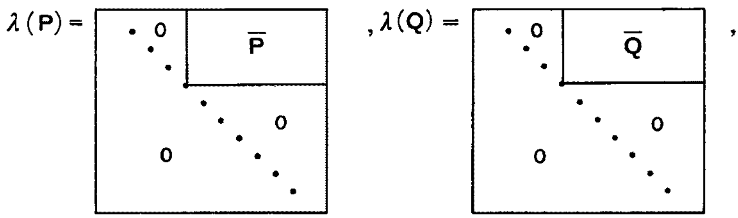
where (P^2, Q^2) denotes the ideal generated by P^2, Q^2 in the polynomial ring $\mathscr{F}[P, Q]$.

The following discussion of indecomposable $\mathscr{R}$-modules is independent of the characteristic of $\mathscr{F}$.

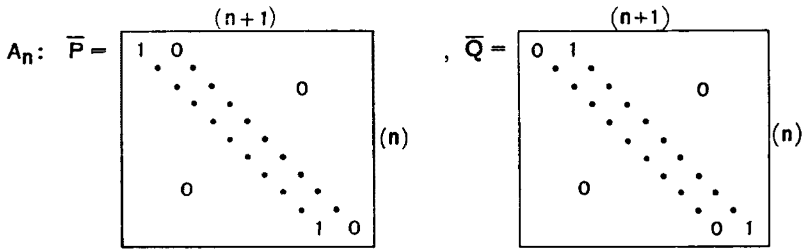
Let $A \to \lambda(A)$ $(A \in \mathscr{R})$ be a representation of $\mathscr{R}$, where the matrices $\lambda(A)$ have coefficients in $\mathscr{F}$. One indecomposable such representation is the regular representation whose $\mathscr{R}$-module class we denote by D. The underlying $\mathscr{R}$-module is both $\mathscr{R}$-projective and $\mathscr{R}$-injective and so is a direct summand of any module in which it occurs as a submodule. The remaining indecomposable representations all satisfy the condition

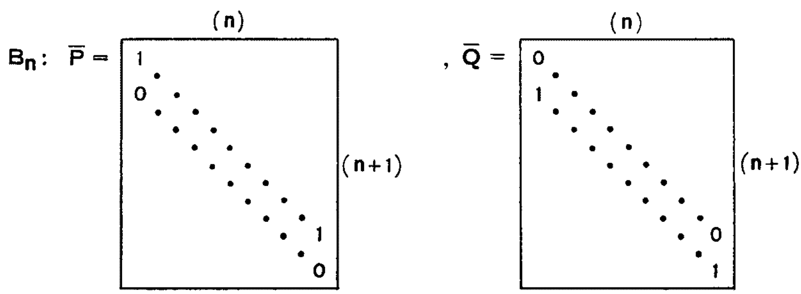
$$\lambda(P)\lambda(Q) = 0.$$

As $(\lambda(P))^2 = (\lambda(Q))^2 = 0$ in any case, by suitable choice of basis, $\lambda(P)$, $\lambda(Q)$ can be written in the form



and so it is sufficient in describing an indecomposable representation to give $\bar{P}$, $\bar{Q}$. The following cases A_n, B_n, $C_n(\pi)$, $C_n(\infty)$ arise.





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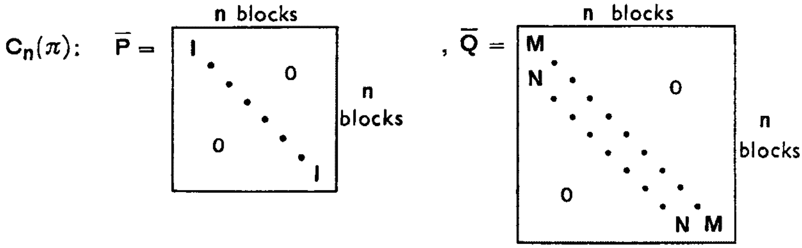
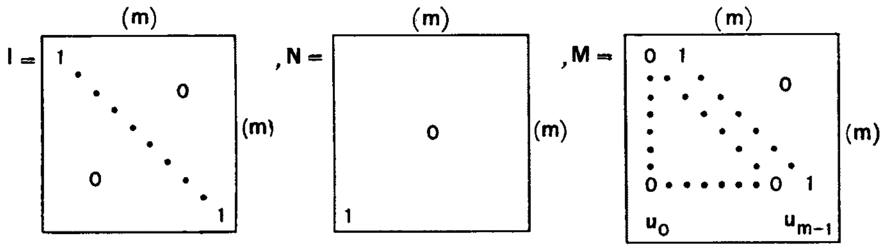
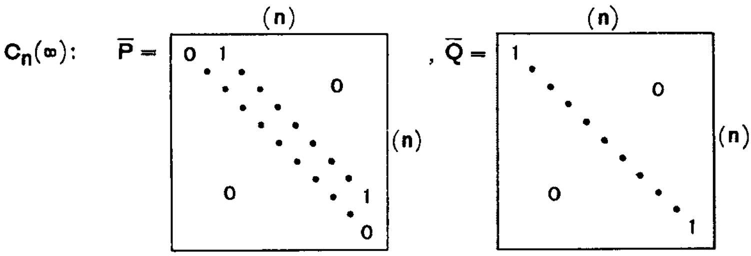
Both A_0, B_0 can be interpreted as the class of the trivial R-module.

Let $\pi = T^m - u_{m-1}T^{m-1} - \cdots - u_0$ be an irreducible polynomial in the indeterminate T over $\mathscr{F}$, with degree $\pi = m$. Thus we define



where





As a convention we shall say that the degree of ∞ is 1.

Here A_n, B_n, $C_n(\pi)$, D denote the module classes associated with the respective representations. With the above convention on deg(∞), π can be considered to range through all irreducible polynomials over $\mathscr{F}$, together with ∞. $\bar{Q}$ for $C_n(\pi)$ $(\pi \neq \infty)$ is an indecomposable Jordan block, with invariant factors $\pi^n, 1, 1, \cdots$. Indeed, once the A_n, B_n, D have been removed in the break-up of a given module into indecomposables, the decomposition of the remainder can be determined by elementary divisor techniques, suitable allowance being made for $C_n(\infty)$ [9].

If $\mathscr{F}^*$ is the algebraic closure of $\mathscr{F}$, the representation afforded by

[9] See § 5 of chapter II of [4].

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the class $C_n(\pi)$ may break up further over $\mathscr{F}^*$. Say $\mathscr{F}$ has characteristic p; let the degree of inseparability of π be t, and let the reduced degree of π be s, i.e. $m = sp^t$; let $a_1, \cdots, a_s$ be the different roots of π in $\mathscr{F}^*$. Then

$$\pi = \prod_{\alpha=1}^{s} (T - a_\alpha)^{p^t}. \tag{4}$$

The invariant factors for $\bar{Q}$ in $\mathscr{F}^*$ are

$$\prod_{\alpha} (T - a_\alpha)^{p^t}, 1, 1, \cdots,$$

and $\bar{Q}$ splits up into s different blocks each of size $n \cdot p^t$. We write

$$C_n(\pi) =_{\mathscr{F}^*} \sum_{\alpha=1}^{s} C_{n \cdot p^t}(T - a_\alpha), \tag{5}$$

implying that we are considering $\mathscr{F}^*(\mathscr{V}_4)$-module classes.

3. Tensor (Kronecker) products of the $\mathscr{F}(\mathscr{V}_4)$-module classes

Methods for calculating the tensor products of the $\mathscr{F}(\mathscr{V}_4)$-indecomposable modules have been given by Bašev in [1] when the field $\mathscr{F}$ is algebraically closed of characteristic 2. The author has found that Bašev's results are correct except for the following case. Let $a \in \mathscr{F}$, $a \neq 0, 1$ (or ∞). Then we have

$$C_1(T+a)C_1(T+a) = C_2(T+a), \tag{6}$$

$$C_n(T+a)C_n(T+a) = n(n-1)D + 2C_n(T+a) \qquad (n > 1). \tag{7}$$

Our results can be extended to the case where $\mathscr{F}$ is not algebraically closed by using proposition 1. Let $\mathscr{F}^*$ be the algebraic closure of $\mathscr{F}$. Consider, for instance, $C_n(\pi)C_n(\pi)$ $(n > 1)$, where π is given by (4) with $p = 2$.

$$\begin{aligned} C_n(\pi)C_n(\pi) &=_{\mathscr{F}^*} \sum_{\alpha,\beta=1}^{s} C_{n \cdot 2^t}(T+a_\alpha)C_{n \cdot 2^t}(T+a_\beta) \qquad \text{(by 5)}, \\ &=_{\mathscr{F}^*} \sum_{\alpha=1}^{s} C_{n \cdot 2^t}(T+a_\alpha)C_{n \cdot 2^t}(T+a_\alpha) \\ &\qquad + \sum_{\alpha \neq \beta}^{s} C_{n \cdot 2^t}(T+a_\alpha)C_{n \cdot 2^t}(T+a_\beta). \end{aligned}$$

But

$$C_{n \cdot 2^t}(T+a_\alpha)C_{n \cdot 2^t}(T+a_\beta) =_{\mathscr{F}^*} (n \cdot 2^t)(n \cdot 2^t)D \qquad (\alpha \neq \beta), \tag{8}$$

and so

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$$C_n(\pi)C_n(\pi) =_{\mathscr{F}^*} \sum_{\alpha=1}^{s} [n \cdot 2^t(n \cdot 2^t - 1)D + 2C_{n \cdot 2^t}(T+a_\alpha)] + s(s-1)n^2 2^{2t} D \quad \text{(by (7), (8))},$$
$$=_{\mathscr{F}^*} nm(nm-1)D + 2C_n(\pi),$$

where $m = \deg \pi$. Thus by proposition 1 we have

$$C_n(\pi)C_n(\pi) = nm(nm-1)D + 2C_n(\pi),$$

this being an equation in $\mathscr{F}(\mathscr{V}_4)$-module classes.

Let π_1 denote either T, $T+1$, ∞ or any inseparable irreducible polynomial over $\mathscr{F}$, let π_2 denote any other irreducible polynomial; let π denote the general irreducible polynomial of type π_1 or π_2.

The results are summarised in the following multiplication table.

$n \leqq n'$	A_n	B_n	$C_n(\pi)$, $\deg \pi = m$	D
$A_{n'}$	$nn'D + A_{n+n'}$	$n(n'+1)D + A_{n'-n}$	$nn'mD + C_n(\pi)$	$(2n'+1)D$
$B_{n'}$	$n(n'+1)D + B_{n'-n}$	$nn'D + B_{n+n'}$	$nn'mD + C_n(\pi)$	$(2n'+1)D$
$C_{n'}(\pi')$ $\deg \pi' = m'$	$nn'm'D + C_{n'}(\pi')$	$nn'm'D + C_{n'}(\pi')$	$nmn'm'D$, if $\pi \neq \pi'$; $nm(n'm-1)D + 2C_n(\pi)$, if $\pi = \pi'$, except that $C_1(\pi_2)C_1(\pi_2) = C_2(\pi_2)$	$2n'm'D$
D	$(2n+1)D$	$(2n+1)D$	$2nmD$	$4D$

4. The representation algebra for $\mathscr{V}_4$

We shall now look at $\mathscr{A}(\mathscr{P}, \mathscr{F}, \mathscr{V}_4) = \mathscr{A}$, where $\mathscr{F}$ has characteristic 2. We require that $\mathscr{P}$ should contain a subring isomorphic to $Z[2^{-\frac{1}{2}}]$.

$A_0 = B_0$ is the identity I of $\mathscr{A}$. Further $I_D = \frac{1}{4}D$ is an idempotent. Thus I_D generates the projective ideal which is an ideal direct summand with complement generated by $I - I_D$. Write

$$\bar{A}_n = A_n(I - I_D) = A_n - \frac{2n+1}{4} D,$$

$$\bar{B}_n = B_n(I - I_D) = B_n - \frac{2n+1}{4} D,$$

$$\bar{C}_n(\pi) = C_n(\pi)(I - I_D) = C_n(\pi) - \frac{nm_\pi}{2} D,$$

where $\deg \pi = m_\pi$. The multiplication table in the ideal $(I - I_D)$ is then as follows:

(9)

$n \leqq n'$	$\bar{A}_n$	$\bar{B}_n$	$\bar{C}_n(\pi)$
$\bar{A}_{n'}$	$\bar{A}_{n+n'}$	$\bar{A}_{n'-n}$	$\bar{C}_n(\pi)$
$\bar{B}_{n'}$	$\bar{B}_{n'-n}$	$\bar{B}_{n+n'}$	$\bar{C}_n(\pi)$
$\bar{C}_{n'}(\pi')$	$\bar{C}_{n'}(\pi')$	$\bar{C}_{n'}(\pi')$	0, if $\pi \neq \pi'$. $2\bar{C}_n(\pi)$, if $\pi = \pi'$, except that $\bar{C}_1(\pi_2)\bar{C}_1(\pi_2) = \bar{C}_2(\pi_2)$.

Let $X = \bar{A}_1$. Then X is invertible and

$$X^n = \begin{cases} \bar{A}_n, & n \geqq 0, \\ \bar{B}_n, & n < 0, \end{cases}$$

with $X^n X^m = X^{n+m}$, for all integers n, m.
Clearly

$$X^n \bar{C}_{n'}(\pi) = \bar{C}_{n'}(\pi),$$

for all n, π, and $n' > 0$. Put

$$\begin{aligned} I_{1,\pi_1} &= \tfrac{1}{2}\bar{C}_1(\pi_1), \\ I_{n,\pi_1} &= \tfrac{1}{2}(\bar{C}_n(\pi_1) - \bar{C}_{n-1}(\pi_1)) && (n > 1), \\ I_{1,\pi_2} &= \tfrac{1}{4}(\bar{C}_2(\pi_2) - \sqrt{2}\bar{C}_1(\pi_2)), \\ I_{2,\pi_2} &= \tfrac{1}{4}(\bar{C}_2(\pi_2) + \sqrt{2}\bar{C}_1(\pi_2)), \\ I_{n,\pi_2} &= \tfrac{1}{2}(\bar{C}_n(\pi_2) - \bar{C}_{n-1}(\pi_2)) && (n > 2). \end{aligned}$$

The $I_{n,\pi}$ are mutually orthogonal idempotents. Hence $\mathscr{A}$ can be written

$$\mathscr{A} \approx \left(\mathscr{P}\left[X, \frac{1}{X}\right] + \{\bigoplus_{n,\pi} \mathscr{P} I_{n,\pi}\}\right) \oplus \mathscr{P} I_D,$$

where $X^m I_{n,\pi} = I_{n,\pi}$ (all integers m) and where $\{\oplus_{n,\pi} \mathscr{P} I_{n,\pi}\}$ is the direct sum of ideals isomorphic to $\mathscr{P}$.

The structure of $\mathscr{A}$ is somewhat more complicated if $\mathscr{P}$ merely contains a subring isomorphic to $Z[\frac{1}{2}]$, or if $\mathscr{P} = Z$. It can be proved that $\mathscr{A}$ is semisimple in the Jacobson sense if $\mathscr{P}$ is a Jacobson ring (Noetherian ring in which every prime ideal is the intersection of maximal ideals), though the quotients $\mathscr{A}/\mathscr{M}$ ($\mathscr{M}$ a maximal ideal) may be very varied in nature.

THEOREM. *$\mathscr{A}(\mathscr{C}, \mathscr{F}, \mathscr{V}_4)$ is G-semisimple.*

PROOF. $\mathscr{C}[X, 1/X]$ is a principal ideal domain and the maximal ideals have the form $(X - a)$, $a \in \mathscr{C}$, $a \neq 0$. Clearly $\mathscr{C}[X, 1/X]$ is G-semisimple and so the G-radical of $\mathscr{A}$ is contained in

$$\{\bigoplus_{n,\pi} \mathscr{C}I_{n,\pi}\} \oplus \mathscr{C}I_D.$$

Write

$$\mathscr{M}_D = (I - I_D),$$
$$\mathscr{M}_{n,\pi} = (I - I_{n,\pi}).$$

Then $\mathscr{A}/\mathscr{M}_D$, $\mathscr{A}/\mathscr{M}_{n,\pi}$ are isomorphic to $\mathscr{C}$ and

$$\mathscr{M}_D \cap (\bigcap_{n,\pi} \mathscr{M}_{n,\pi}) \cap (\{\bigoplus_{n,\pi} \mathscr{C}I_{n,\pi}\} \oplus \mathscr{C}I_D) = (0),$$

and so $\mathscr{A}$ is G-semisimple.

Thus there exists a set of G-characters ϕ_α on $\mathscr{A}$. We may think of a set of coordinates $\{\phi_\alpha(\mathscr{M})\}$ of a $\mathscr{F}(\mathscr{V}_4)$-module class $\{\mathscr{M}\}$, which completely determine $\{\mathscr{M}\}$ and which are compatible with direct sum and tensor product of modules.

5. Representations of $\mathscr{A}_4$ over a field $\mathscr{F}$ of characteristic 2

We regard $\mathscr{A}_4$ (alternating group of 4 symbols) as being an extension of $\mathscr{V}_4$ by a cyclic group of order 3. Thus we take generators W, X, Y satisfying

$$W^3 = X^2 = Y^2 = E, \qquad XY = YX,$$
$$W^{-2}XW^2 = W^{-1}YW = XY,$$

where E is the identity element. $\mathscr{V}_4$ is the subgroup generated by X, Y.

Let $\mathscr{F}$ be an algebraically closed field of characteristic 2. By Higman's theorem 1 in [7], every indecomposable $\mathscr{F}(\mathscr{A}_4)$-module is a direct summand of the $\mathscr{F}(\mathscr{A}_4)$-module induced from an indecomposable $\mathscr{F}(\mathscr{V}_4)$-module. We now look at such induced $\mathscr{F}(\mathscr{A}_4)$-modules.

A $\mathscr{F}(\mathscr{V}_4)$-module $\mathscr{L}$ (and the corresponding representation of $\mathscr{F}(\mathscr{V}_4)$) will be called *stable* in $\mathscr{A}_4$ if the $\mathscr{F}(\mathscr{V}_4)$-submodule

$$W \otimes_{\mathscr{F}(\mathscr{V}_4)} \mathscr{L} \quad \text{of} \quad (\mathscr{L}^{\mathscr{A}_4})_{\mathscr{V}_4}$$

is isomorphic to $\mathscr{L}$. We now find which indecomposable $\mathscr{F}(\mathscr{V}_4)$-modules are stable in $\mathscr{A}_4$.

Let

$$\mathscr{G} \to \lambda(G), \quad G \to \bar{\lambda}(G) \qquad (G \in \mathscr{F}(\mathscr{V}_4))$$

be the representations afforded by the $\mathscr{F}(\mathscr{V}_4)$-modules $\mathscr{L}$ and $W \otimes \mathscr{L}$ respectively. Choosing bases appropriately, we can write

$$\bar{\lambda}(G) = \lambda(W^{-1}GW) \qquad (G \in \mathscr{F}(\mathscr{V}_4)).$$

If $P = X+E$ $Q = Y+E$, it is readily seen that

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$$\bar{\lambda}(P) = \lambda(Q)$$
$$\bar{\lambda}(Q) = \lambda(P)+\lambda(Q)+\lambda(PQ),$$

and $\mathscr{L}$ is stable in $\mathscr{A}_4$ if and only if the pair $(\bar{\lambda}(P), \bar{\lambda}(Q))$ is similar to $(\lambda(P), \lambda(Q))$.

Now $\bar{\lambda}(P)\bar{\lambda}(Q) = \lambda(P)\lambda(Q)$. For the representation afforded by the class D we have $\lambda(P)\lambda(Q) \neq 0$, and so $\bar{\lambda}(P)\bar{\lambda}(Q) \neq 0$ and D is stable in $\mathscr{A}_4$. If $\mathscr{R}$ is any module in the classes A_n, B_n, $C_n(\pi)$, then so is $W \otimes \mathscr{R}$, as $\lambda(P)\lambda(Q) = \lambda(PQ)$ remains 0. In this latter case we must compare the pair $(\lambda(Q), \lambda(P)+\lambda(Q))$ with $(\lambda(P), \lambda(Q))$ under similarity, or, using the notation of § 2, the pair $(\bar{Q}, \bar{P}+\bar{Q})$ with $(\bar{P}, \bar{Q})$ under independent non-singular transformations on both sides. This can be done using the invariants in § 5 of chapter II of [4]. Thus it can be shown that A_n, B_n are stable in $\mathscr{A}_4$. As $\mathscr{F}$ is algebraically closed, π (irreducible) has the form $T+a$, for $a \in \mathscr{F}$, or ∞. We write $C_n(a)$ for $C_n(\pi)$, where $a \in \mathscr{F} \cup \{\infty\}$. By elementary divisors (as mentioned in § 2 for $\bar{Q}$), we see that

$$\{W \otimes \mathscr{L}\} = C_n(\theta(a)),$$

where $\mathscr{L}$ is in the class of $C_n(a)$, and where

$$\theta(a) = \frac{1+a}{a},$$

with the obvious interpretation when $a = \infty$ or 0. Note that $\theta^3(a) = a$. Thus $C_n(a)$ is stable if and only if

$$\theta(a) = a,$$

i.e.

$$a^2+a+1 = 0,$$

or a is a primitive cube root ω of unity in $\mathscr{F}$. θ is a permutation on $\mathscr{F} \cup \{\infty\}$. We denote the typical class of transitivity by $\mu = \{a, \theta(a), \theta^2(a)\}$. However there are two additional classes, $\{\omega\}$ and $\{\omega^2\}$.

To obtain the indecomposable $\mathscr{F}(\mathscr{A}_4)$-modules we look at $\mathscr{L}^{\mathscr{A}_4}$, where $\mathscr{L}$ is an indecomposable $\mathscr{F}(\mathscr{V}_4)$-module. If $\mathscr{L}$ is not stable in $\mathscr{A}_4$, then $\mathscr{L}^{\mathscr{A}_4}$ is indecomposable by the theorem in § 2 of [2]. Thus we obtain indecomposable $\mathscr{F}(\mathscr{A}_4)$-modules $C_n^*(\mu)$ such that

$$(C_n^*(\mu))_{\mathscr{V}_4} = C_n(a)+C_n(\theta(a))+C_n(\theta^2(a)).$$

If $\mathscr{L}$ is stable in $\mathscr{A}_4$, then $\mathscr{L}^{\mathscr{A}_4}$ splits up into 3 indecomposable, non-isomorphic $\mathscr{F}(\mathscr{A}_4)$-modules $\mathscr{L}^\alpha$ (all superscripts will be considered to be integers modulo 3), such that $(\mathscr{L}^\alpha)_{\mathscr{V}_4} \approx \mathscr{L}$, as in proposition 3 of [2]. Thus we obtain classes

(10) $$A_0^\alpha, A_n^\alpha, B_n^\alpha, C_n^\alpha(\omega), C_n^\alpha(\omega^2), D^\alpha \qquad (n > 0).$$

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In particular A_0^α may be taken to be the class corresponding to the 1-dimensional representation

$$W \to \omega^\alpha \qquad (\alpha = 0, 1, 2),$$
$$X, Y \to 1.$$

Then we can suppose that $A_0^\alpha \times \{\mathscr{L}^\beta\} = \{\mathscr{L}^{\alpha+\beta}\}$. As the $\mathscr{L}^\alpha$ are extensions of $\mathscr{L}$, in the corresponding representations it is only necessary to assign a matrix $\lambda(W)$, to extend the matrix representations as detailed in § 2. If $\lambda(W)$ is assigned to the representation afforded by $\mathscr{L}^0$, then the corresponding matrix for $\mathscr{L}^\alpha$ is $\omega^\alpha\lambda(W)$. The author has constructed suitable matrices $\lambda(W)$ corresponding to classes A_n, B_n, D (all $n > 0$), but not for $C_n(\omega)$, $C_n(\omega^2)$ in general. However for $C_1^0(\omega)$ we take

$$\lambda(W) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix},$$

and for $C_2^0(\omega)$ we take

$$\lambda(W) = \left[\begin{array}{cc|cc} 1 & & 0 & \\ & \omega^2 & & \\ \hline & & \omega & \\ 0 & & \omega^2 & 1 \end{array}\right].$$

For $C_1^0(\omega^2)$, $C_2^0(\omega^2)$ we replace ω by ω^2 in these matrices. For A_1^0 we take

$$\lambda(W) = \left[\begin{array}{c|cc} 1 & 0 & \\ \hline 0 & 0 & 1 \\ & 1 & 1 \end{array}\right].$$

It should be noted that in general we still have not chosen which of the 3 extensions $\mathscr{L}^\alpha$ of $\mathscr{L}$ will be called $\mathscr{L}^0$. This choice will be exercised in the next section.

6. The representation algebra for $\mathscr{A}_4$

To obtain the structure of $\mathscr{A}(\mathscr{C}, \mathscr{F}, \mathscr{A}_4)$, where $\mathscr{F}$ is algebraically closed of characteristic 2, it is not necessary to find explicitly all tensor (Kronecker) products. By proposition 3 and its corollaries it will only be necessary to obtain the products of the $\mathscr{F}(\mathscr{A}_4)$-modules modulo the projective ideal $\mathscr{D} = (D^0, D^1, D^2)$, and all equations in this section will be taken to be modulo $\mathscr{D}$. Further by restricting the ring multiplications to $\mathscr{F}(\mathscr{V}_4)$ and considering the corresponding products of the $\mathscr{F}(\mathscr{V}_4)$-modules, we see that the multiplication table (9) must be valid on removing the superscripts α.

Now

$$C_n^*(\mu) = (C_n(a))^{\mathscr{A}_4}$$

when $a \neq \omega, \omega^2$ and $\mu = \{a, \theta(a), \theta^2(a)\}$, and so, using proposition 2, we quickly obtain all products involving $C_n^*(\mu)$. Thus

(11)
$$A_m^\alpha C_n^*(\mu) = C_n^*(\mu), \quad B_m^\alpha C_n^*(\mu) = C_n^*(\mu),$$
$$C_m^*(\mu)C_n^*(\mu') = \begin{cases} 0, & \text{if } \mu \neq \mu', \\ 2C^*_{\min(m,n)}(\mu), & \text{if } \mu = \mu', \end{cases}$$

except that

$$C_1^*(\mu)C_1^*(\mu) = C_2^*(\mu) \quad \text{for all} \quad \mu \neq \{1, 0, \infty\}.$$

Also

$$C_1^*(1, 0, \infty)C_1^*(1, 0, \infty) = 2C_1^*(1, 0, \infty).$$

We now choose $A_n^0(n > 1)$, $B_n^0(n > 0;)$ to satisfy

$$A_n^0 = (A_1^0)^n, \quad A_1^0 B_1^0 = A_0^0, \quad B_n^0 = (B_1^0)^n.$$

Thus we have

$$A_n^0 B_m^0 = \begin{cases} A_{n-m}^0, & \text{if } n \geqq m, \\ B_{m-n}^0, & \text{if } n < m, \text{ etc.} \end{cases}$$

A direct calculation shows that

(12i) $$C_1^0(\omega)C_1^0(\omega) = C_2^0(\omega),$$

(12ii) $$C_1^0(\omega)C_2^0(\omega) = 2C_1^0(\omega).$$

As yet $C^\alpha(\omega)$ $(n > 2)$ have not been specified. Say

$$C_1^0(\omega)C_n^\alpha(\omega) = C_1^\beta(\omega) + C_1^\gamma(\omega).$$

Then $\beta = \gamma$ or not. Choose $C_n^0(\omega)$ so that one of the following relations is true

(13i), (13ii)
$$C_1^0(\omega)C_n^0(\omega) = \begin{cases} 2C_1^0(\omega), & \text{or} \\ C_1^1(\omega) + C_1^2(\omega). \end{cases}$$

If $n(> 1)$ is such that (13i) is true then for $n \geqq m \geqq 1$, the associativity of multiplication implies that

(14i) $$C_m^0(\omega)C_n^0(\omega) = 2C_m^0(\omega),$$

while if (13ii) is true, then

(14ii) $$C_m^0(\omega)C_n^0(\omega) = C_m^1(\omega) + C_m^2(\omega).$$

Again a direct calculation shows that

$$A_1^0 C_1^0(\omega) = C_1^1(\omega).$$

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By associativity of multiplication we prove in succession that

$$(15)\qquad \begin{cases} A_1^0 C_n^0(\omega) = C_n^1(\omega), \\ A_m^0 C_n^0(\omega) = C_n^m(\omega), \\ B_m^0 C_n^0(\omega) = C_n^{-m}(\omega) \end{cases}$$

(superscripts are modulo 3).

Similarly

$$A_1^0 C_1^0(\omega^2) = C_1^2(\omega^2),$$

and so

$$(16)\qquad \begin{cases} A_m^0 C_n^0(\omega^2) = C_n^{2m}(\omega^2), \\ B_m^0 C_n^0(\omega^2) = C_n^{-2m}(\omega^2). \end{cases}$$

We now look at the structure of $\mathscr{A} = \mathscr{A}(\mathscr{C}, \mathscr{F}, \mathscr{A}_4)$. The projective ideal $\mathscr{D}$ is isomorphic to $\mathscr{C} \oplus \mathscr{C} \oplus \mathscr{C}$. The complement to $\mathscr{D}$ in $\mathscr{A}$ is isomorphic to $\mathscr{B} = \mathscr{A}/\mathscr{D}$, and so to find $\mathscr{B}$ we continue as above modulo $\mathscr{D}$.

A_0^0 is the identity element of $\mathscr{B}$. Let u be a primitive cube root of unity in $\mathscr{C}$, and write

$$J_\beta = \tfrac{1}{3}(A_0^0 + u^\beta A_0^1 + u^{2\beta} A_0^2) \qquad (\beta = 0, 1, 2).$$

Then

$$A_0^0 = J_0 + J_1 + J_2,$$

and the J_β are mutually orthogonal idempotents.

Write

$$(17)\qquad \begin{cases} A_{n\beta} = A_n^0 J_\beta, \quad B_{n\beta} = B_n^0 J_\beta \\ C_{n\beta}(\omega) = C_n^0(\omega) J_\beta, \quad C_{n\beta}(\omega^2) = C_n^0(\omega^2) J_\beta. \end{cases}$$

Then

$$A_n^\alpha J_\beta = u^{-\alpha\beta} A_{n\beta}, \quad \text{etc.}$$

and

$$A_{n\alpha} A_{m\beta} = \begin{cases} 0, & \text{if } \alpha \neq \beta, \\ A_{(m+n)\beta}, & \text{if } \alpha = \beta, \text{ etc.} \end{cases}$$

Further

$$(18)\qquad C_n^*(\mu) J_\beta = \begin{cases} C_n^*(\mu), & \text{if } \beta = 0, \\ 0, & \text{if } \beta \neq 0. \end{cases}$$

Finally the elements (17) and $C_n^*(\mu)$ together form a basis of $\mathscr{B}$ over $\mathscr{C}$.

We now look at the 3 ideal direct summands of $\mathscr{B}$ generated by the J_β. Set $Y_\beta = A_{1\beta}$, $1/Y_\beta = B_{1\beta}$; then $Y_\beta^m = A_{m\beta}$ etc., and the subalgebra of $\mathscr{B} J_\beta$ generated by $A_{n\beta}$, $B_{n\beta}$ may be written $\mathscr{C}[Y_\beta, 1/Y_\beta]$, Y_β being regarded as an indeterminate over $\mathscr{C}$. From (15) and (16)

$$Y_\beta^m C_{n\beta}(\omega) = u^{-\beta m} C_{n\beta}(\omega),$$
$$Y_\beta^m C_{n\beta}(\omega^2) = u^{\beta m} C_{n\beta}(\omega^2),$$

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for m any integer, and

$$C_{n\beta}(\omega)C_{n'\beta}(\omega^2) = 0,$$

for all positive n, n'. From (12i), (12ii), (14i),

$$\begin{aligned} C_{1\beta}(\omega)C_{1\beta}(\omega) &= C_{2\beta}(\omega), \\ C_{1\beta}(\omega)C_{2\beta}(\omega) &= 2C_{1\beta}(\omega) \\ C_{2\beta}(\omega)C_{2\beta}(\omega) &= 2C_{2\beta}(\omega). \end{aligned}$$

As in § 4, set

$$\begin{aligned} I_{1\beta}(\omega) &= \tfrac{1}{4}(C_{2\beta}(\omega)+\sqrt{2}\,C_{1\beta}(\omega)) \\ I_{2\beta}(\omega) &= \tfrac{1}{4}(C_{2\beta}(\omega)-\sqrt{2}\,C_{1\beta}(\omega)), \end{aligned}$$

and these are mutually orthogonal idempotents. For $n > 2$, if we have the situation of (13i), then

$$C_{n\beta}(\omega)C_{n\beta}(\omega) = 2C_{n\beta}(\omega),$$

and we write

$$\bar{C}_{n\beta}(\omega) = \tfrac{1}{2}C_{n\beta}(\omega).$$

In case (13ii), we have

$$C_{n\beta}(\omega)C_{n\beta}(\omega) = (u^{-\beta}+u^{-2\beta})C_{n\beta}(\omega),$$

and we write

$$\bar{C}_{n\beta}(\omega) = \frac{1}{u^{-\beta}+u^{-2\beta}}\,C_{n\beta}(\omega).$$

Then the $\bar{C}_{n\beta}(\omega)$ are idempotents. To obtain orthogonal idempotents we put

$$I_{3\beta} = \bar{C}_{3\beta}(\omega)-I_{1\beta}(\omega)-I_{2\beta}(\omega),$$

and for $n > 3$

$$I_{n\beta}(\omega) = \bar{C}_{n\beta}(\omega)-\bar{C}_{(n-1)\beta}(\omega).$$

Then all the $I_{n\beta}(\omega)$ are mutually orthogonal idempotents. $I_{n\beta}(\omega^2)$ are similarly defined. From (11), (18), we can proceed as in § 4 and $I_{n0}(\mu)$ are defined.

Hence $\mathscr{A}$ has the following structure.

$$\left\{\begin{aligned} &\left(\mathscr{C}\left[Y_0, \frac{1}{Y_0}\right] + \left\{\bigoplus_{\substack{n\geq 1\\ \phi=\omega,\,\omega^2,\,\mu}} \mathscr{C}I_{n0}(\phi)\right\}\right) \\ &\oplus \left(\bigoplus_{\beta=1,2}\left[\mathscr{C}\left[Y_\beta, \frac{1}{Y_\beta}\right] + \left\{\bigoplus_{\substack{n\geq 1\\ \phi=\omega,\,\omega^2}} \mathscr{C}I_{n\beta}(\phi)\right\}\right]\right) \\ &\oplus (\mathscr{C}\oplus\mathscr{C}\oplus\mathscr{C}), \end{aligned}\right.$$

where

$$\begin{aligned} Y_\beta^m I_n(\omega^\alpha) &= u^{-\alpha\beta m}I_{n\beta}(\omega^\alpha), \\ Y_0^m I_{n0}(\mu) &= I_{n0}(\mu); \end{aligned}$$

the last term is the projective ideal $\mathscr{D}$.

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As in § 4 this is G-semisimple. As far as G-semisimplicity is concerned we may now drop the restriction that $\mathscr{F}$ is algebraically closed. For, if not, let $\mathscr{F}^*$ be the algebraic closure of $\mathscr{F}$. Then, by (3), $\mathscr{A}(\mathscr{C}, \mathscr{F}, \mathscr{A}_4)$ can be regarded as embedded in $\mathscr{A}(\mathscr{C}, \mathscr{F}^*, \mathscr{A}_4)$. Thus the restriction of the G-characters to the subalgebra will ensure the G-semisimplicity of $\mathscr{A}(\mathscr{C}, \mathscr{F}, \mathscr{A}_4)$.

THEOREM. *$\mathscr{A}(\mathscr{C}, \mathscr{F}, \mathscr{A}_4)$ is G-semisimple for all fields $\mathscr{F}$ of characteristic* 2.

7. Ring-tensor-product representation algebras

Given a commutative ring $\mathscr{R}$ and two $\mathscr{R}$-modules $\mathscr{M}, \mathscr{M}'$ then the tensor product

$$\mathscr{M} \otimes_{\mathscr{R}} \mathscr{M}'$$

can also be defined to be an $\mathscr{R}$-module. This product is then commutative, associative and distributes over direct sum $\oplus$. If we now take the set of $\mathscr{R}$-modules which satisfy the ascending and descending chain conditions, this set is closed under $\oplus$, $\otimes$ and the Krull-Schmidt theorem is applicable. If $\mathscr{P}$ is any commutative ring with an identity element, then, as in § 1, we can define the representation algebra $\mathscr{A}(\mathscr{P}, \mathscr{R})$ to be the free $\mathscr{P}$-module generated by the set of all $\mathscr{R}$-indecomposable isomorphic classes $\{\mathscr{M}\}$, equipped this time with the multiplication

$$\{\mathscr{M}\}\{\mathscr{M}'\} = \{\mathscr{M} \otimes_{\mathscr{R}} \mathscr{M}'\}.$$

If $\mathscr{R}$ is a Dedekind domain, then the indecomposable $\mathscr{R}$-modules of finite length have the form

$$\mathscr{R}/\mathscr{Q}_\alpha^n,$$

where $\mathscr{Q}_\alpha$ is any non-zero prime ideal of $\mathscr{R}$. Further it is readily seen that

$$\mathscr{R}/\mathscr{Q}_\alpha^n \otimes_{\mathscr{R}} \mathscr{R}/\mathscr{Q}_\beta^m = \begin{cases} (0), & \text{if } \alpha \neq \beta, \\ \mathscr{R}/\mathscr{Q}_\alpha^{\min(n,m)}, & \text{if } \alpha = \beta. \end{cases}$$

Write then

$$I_{\alpha 1} = \{\mathscr{R}/\mathscr{Q}_\alpha\},$$
$$I_{\alpha n} = \{\mathscr{R}/\mathscr{Q}_\alpha^n\} - \{\mathscr{R}/\mathscr{Q}_\alpha^{n-1}\} \qquad (n > 1).$$

Then

$$\mathscr{A}(\mathscr{P}, \mathscr{R}) = \bigoplus_{\alpha, n \geq 1} \mathscr{P} I_{\alpha n}.$$

This algebra does not have an identity.

Another case which can readily be deduced from the above is that of the quotient of the Dedekind domain $\mathscr{R}$ by an ideal $\mathscr{I} = \Pi \mathscr{Q}_\alpha^{n_\alpha}$, where only a finite number of n_α are strictly positive ($n_\alpha > 0$). Then the indecomposable $\mathscr{R}/\mathscr{I}$-modules of finite length have the form

$$\mathscr{R}/\mathscr{Q}_\alpha^{m_\alpha}, \quad m_\alpha = 1, \cdots, n_\alpha, \quad \text{when} \quad n_\alpha \geqq 1.$$

Again

$$\mathscr{A}(\mathscr{P}, \mathscr{R}/\mathscr{I}) = \bigoplus_\alpha \Big(\bigoplus_{m_\alpha=1}^{n_\alpha} \mathscr{P} I_{\alpha m_\alpha} \Big).$$

This algebra has finite rank over $\mathscr{P}$ and has an identity.

We now take $\mathscr{R} = \mathscr{F}[P, Q]/(P^2, Q^2)$, as in § 2 ($\mathscr{F}$ of arbitrary characteristic). We assume for simplicity that $\mathscr{F}$ is algebraically closed. Then the different classes are A_n, B_n, $C_n(a)$, D, where $a \in \mathscr{F} \cup \{\infty\}$.

The multiplication table under $\otimes_{\mathscr{R}}$ is as follows.

$n \leqq m$	A_n	B_n	$C_n(a)$	D
A_m	$(n+1)(m+1)A_0$	$(m+2)(n-1)A_0 + A_{m-n+1}$	$(m+1)nA_0$	A_m
B_m	$m(n+1)A_0$ $(m > n)$ $(n+2)(n-1)A_0+A_1$ $(m = n)$	$(n-1)(m-1)A_0 + B_{m+n-1}$	$n(m-1)A_0 + C_n(a)$	B_m
$C_m(a')$	$(n+1)mA_0$	$m(n-1)A_0 + C_m(a')$	$n(m-1)A_0 + C_n(a)$ $(a = a')$ nmA_0 $(a \neq a')$	$C_m(a')$
D	A_n	B_n	$C_n(a)$	D

D is the identity element in $\mathscr{A} = \mathscr{A}(\mathscr{C}, \mathscr{R})$. A_0, B_1 are obvious idempotents and

$$D = A_0 + [B_1 - A_0] + [D - B_1]$$

is a splitting of the identity into mutually orthogonal idempotents. The elements A_0, $D - B_1$ generate ideal direct summands each isomorphic to $\mathscr{C}$. Write

$$\bar{A}_n = (B_1 - A_0)A_n,$$
$$\bar{B}_n = (B_1 - A_0)B_n,$$
$$\bar{C}_n(a) = (B_1 - A_0)C_n(a).$$

Then the multiplication table in the ideal $(B_1 - A_0)$ generated by $B_1 - A_0$ is as follows.

$n \leqq m$	$\bar{A}_n$	$\bar{B}_n$	$\bar{C}_n(a)$
$\bar{A}_m$	0	$\bar{A}_{m-n+1}$	0
$\bar{B}_m$	0 $(n > m)$ $\bar{A}_1$ $(m = n)$	$\bar{B}_{m+n-1}$	$\bar{C}_n(a)$
$\bar{C}_m(a')$	0	$\bar{C}_m(a')$	$\bar{C}_n(a)$ $(a = a')$ 0 $(a \neq a')$

Place $T = \bar{B}_2$. Then $\bar{B}_{n+1} = T^n$. Place $I_{1a} = \bar{C}_1(a)$, $I_{na} = \bar{C}_n(a) - \bar{C}_{n-1}(a)$ $(n > 1)$. Then the ideal generated by the $\{\bar{C}_n(a)\}$ is $\oplus_{a,n>0}\mathscr{C}I_{na}$. The subalgebra generated by the $\{\bar{B}_n\}$ may be written $\mathscr{C}[T]$, where the identity element is $\bar{B}_1$. Write $U_n = \bar{A}_n$. Then the structure of the ideal $(B_1 - A_0)$ may be written

$$\mathscr{C}[T] + (\bigoplus_{n>0}\mathscr{C}U_n) + (\bigoplus_{a,n>0}\mathscr{C}I_{na}),$$

where

$$U_n I_{ma} = 0, \qquad U_n U_m = 0,$$
$$TU_{m+1} = U_m, \qquad TI_{ma} = I_{ma},$$

and the I_{ma} are mutually orthogonal idempotents.

The Jacobson radical of this algebra is nonzero as it contains $U_1 (U_1^2 = 0)$. Hence, a fortiori, $\mathscr{A}(\mathscr{C}, \mathscr{R})$ is not G-semisimple. When the characteristic of $\mathscr{F}$ is 2, we get a direct comparison between the two kinds of representation algebras that can be formed from $\mathscr{F}(\mathscr{V}_4)$-modules.

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