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PRIME IDEALS IN REGULAR SELF-INJECTIVE RINGS

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1. Introduction. Although the notion of the maximal quotient ring of a nonsingular ring has been around for some time, not much is known about its structure in general beyond the important theorems of Johnson and Utumi [**4**; **11**] that it is von Neumann regular and self-injective. The purpose of this paper is to study the structure of such a regular, self-injective ring R by looking at its prime ideals. Initially, we show that the primes of R separate into two types, called "essential" and "closed", and that for any prime P, the two-sided ideals in the ring R/P are linearly ordered. In the case of a closed prime P, this result is strengthened to the effect that the two-sided ideals in R/P are well-ordered. We consider the Krull dimension of R in the case when R is a prime ring, and prove that this dimension is almost always at least one less than the global dimension. Finally, R is shown to be a direct product (possibly infinite) of prime rings if and only if it has enough minimal two-sided ideals.

Note. We assume throughout this paper that R is an associative ring with unit, that R is von Neumann regular, and that R is injective as a right module over itself. The term "prime ideal" is used in its standard two-sided sense, to refer to a proper two-sided ideal P such that R/P is a prime ring, i.e., such that in the ring R/P, products of nonzero two-sided ideals are nonzero. For alternative characterizations of prime ideals, see [**7**, pp. 54, 55]. As advance warning, we mention that throughout most of sections 3 and 4 it is assumed that R is a prime ring. Finally, unless otherwise noted, all modules used in this paper are unital right R-modules.

2. General primes. This section is concerned with the classification of prime ideals into essential primes and closed primes, and with some other results about arbitrary primes. Here and elsewhere in this paper, we use the notions of *essential*, *singular*, and *closed* submodules, and of the *closure* of a submodule: the reader is referred to [**2**] for the definitions and basic properties. Due to our assumption that R is regular, we know that the module R_R is nonsingular [**2**, Proposition 3, p. 70].

If a prime ideal P of R is essential as a right ideal of R, then we shall call P an *essential prime*. At the other extreme, when P is closed as a right ideal, we shall call P a *closed prime*. After disposing of one preliminary result, we show that all primes in R fall into one or the other of these cases.

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LEMMA 1. *Let H be a two-sided ideal of R which is closed as a right ideal. If K is the left annihilator of H in R, then $R = H \oplus K$.*

Proof. Since R is a semiprime ring, K is also the right annihilator of H [**7**, p. 111], from which it follows easily that $(R/K)_R$ is nonsingular. Thus H_R and K_R are each closed in R_R, hence they both are injective. Inasmuch as $H \cap K = (H \cap K)^2 \leqq KH = 0$, we obtain $R = H \oplus K \oplus I$ for some right ideal I. If $I \neq 0$, then since $I \nleqq K$ we obtain $IH \neq 0$ and $I \cap H \neq 0$, which is a contradiction.

PROPOSITION 2. *Any prime ideal P in R is either an essential prime or a closed prime.*

Proof. Letting H be the closure of P in R and K the left annihilator of H in R, we obtain $R = H \oplus K$ from Lemma 1. Since $HK = 0 \leqq P$, we must have either $H \leqq P$ or $K \leqq P$. If $H \leqq P$, then $P = H$ and P is a closed prime, while if $K \leqq P$, then $K = 0$ and $H = R$, whence P is an essential prime.

At this point we can give a criterion for 0 to be a prime ideal, i.e., for R to be a prime ring.

PROPOSITION 3. *R is a prime ring if and only if it is indecomposable as a ring.*

Proof. Necessity is clear, for all prime rings are indecomposable. Conversely, assuming that R is indecomposable, we infer from Lemma 1 that all nonzero two-sided ideals of R are essential as right ideals. Inasmuch as R_R is nonsingular, it follows that R is prime.

We next turn to showing that the two-sided ideals above some fixed prime are linearly ordered, i.e., that the two-sided ideals in any factor of R by a prime ideal are linearly ordered. We proceed via two lemmas, which are proved somewhat more generally than needed here in order to make them usable in later sections. The following terminology is used in the first lemma: A module A is *subisomorphic* to a module B provided A is isomorphic to a submodule of B. Also, the following observation is used: Due to the regularity of R, any finitely generated right ideal of R is a direct summand of R, and hence is projective and injective.

LEMMA 4. *Let A be a right ideal of R, and let $x \in R$. Then $x \in RA$ if and only if xR is subisomorphic to a finite direct sum of copies of A.*

Proof. If $x \in RA$, then $x \in r_1A + \ldots + r_nA$ for some elements $r_i \in R$. Letting B be the direct sum of n copies of A, there is an epimorphism of B onto $r_1A + \ldots + r_nA$. Since xR is projective, it follows that xR is subisomorphic to B.

Conversely, assume that xR is subisomorphic to $A_1 \oplus \ldots \oplus A_n$, where each $A_i = A$. Inasmuch as xR is injective, there must exist an epimorphism

$$f: A_1 \oplus \ldots \oplus A_n \rightarrow xR.$$

From the injectivity of R_R, we infer that each of the maps

$$A_i \to A_1 \oplus \ldots \oplus A_n \to xR$$

is given by left multiplication by some $r_i \in R$. Thus there exist elements $a_i \in A$ such that $x = f(a_1, \ldots, a_n) = r_1a_1 + \ldots + r_na_n$.

We have already observed that every finitely generated right ideal of R is nonsingular, projective, and injective. According to [**10**, Theorem 2.7], we also have that all finitely generated nonsingular modules are projective and injective. Thus, for ease of terminology, we shall state results in terms of nonsingular injective modules whenever possible.

LEMMA 5. *Let P be any prime ideal in R. Given nonsingular injective modules A and B, there exist decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that*

(a) $A_1 \cong B_1$;

(b) *either* $A_2 = A_2P$ *or* $B_2 = B_2P$.

Proof. Let $\mathscr{A}$ denote the collection of all pairs (C, D), where $C \leqq A, D \leqq B$, and $C \cong D$. We say that a family $\{(C_i, D_i)\} \subset \mathscr{A}$ is *independent* whenever $\{C_i\}$ is an independent family of submodules of A and $\{D_i\}$ is an independent family of submodules of B. Choosing a maximal independent family $\{(C_i, D_i)\} \subset \mathscr{A}$, we obtain decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that A_1 is an injective hull for $\bigoplus C_i$ and B_1 is an injective hull for $\bigoplus D_i$. Inasmuch as $C_i \cong D_i$ for all i, we obtain $A_1 \cong B_1$. In view of the maximality of $\{(C_i, D_i)\}$, we note that it is not possible for any nonzero submodule of A_2 to be subisomorphic to B_2.

Suppose that $A_2 \neq A_2P$ and $B_2 \neq B_2P$. Choosing some $a \in A_2 \backslash A_2P$, we see that aR is finitely generated and nonsingular, hence projective. Thus $aR \cong eR$ for some idempotent $e \in R$, and since $a \notin A_2P$, we infer that $e \notin P$. Likewise, choosing $b \in B_2 \backslash B_2P$, there must be an idempotent $f \in R \backslash P$ such that $bR \cong fR$. Inasmuch as P is prime, $eRf \not\subset P$, whence $erf \neq 0$ for some $r \in R$. The projectivity of $erfR$ ensures that $erfR$ is subisomorphic to fR, hence aR has a nonzero submodule which is subisomorphic to bR. This is a contradiction.

THEOREM 6. *If P is any prime ideal in R, then the two-sided ideals in the ring R/P are linearly ordered.*

Proof. If not, then we can choose two-sided ideals H and K in R, both containing P, along with elements $x \in H \backslash K$ and $y \in K \backslash H$. According to Lemma 5, there exist decompositions $xR = A_1 \oplus A_2$ and $yR = B_1 \oplus B_2$ such that $A_1 \cong B_1$ and either $A_2 = A_2P$ or $B_2 = B_2P$, say $A_2 = A_2P$. Now A_1 is subisomorphic to K and $A_2 \leqq P \leqq K$, whence xR is subisomorphic to $K \oplus K$. According to Lemma 4 we must have $x \in K$, which is impossible.

COROLLARY 7. *If P is any prime ideal in R, then every proper two-sided ideal in R which contains P is also prime.*

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Proof. Let Q be a proper two-sided ideal containing P, and let H and K be any two-sided ideals which properly contain Q. In view of Theorem 6, we may assume that $H \leqq K$, whence $H = H^2 \leqq HK$ and so $HK \nleqq Q$.

3. Closed primes. The simplest non-artinian example of a regular, self-injective ring is the ring of all linear transformations on an infinite-dimensional vector space, and it has long been known that the two-sided ideals in such a ring can be labelled by infinite cardinals in an order-preserving fashion [**3**, Theorem 5, p. 258]. Recalling that such a ring is also prime, and taking Theorem 6 into account, we ask whether the two-sided ideals in any factor of R by a prime ideal must be well-ordered. This may be false for essential primes (examples will be constructed in a sequel to this paper), but for closed primes the main theorem of this section provides an affirmative answer:

THEOREM 8. *If P is a closed prime ideal in R, then the two-sided ideals in the ring R/P are well-ordered.*

We postpone the proof of Theorem 8 in order to develop some intermediate results. First we make an easy normalization: Since there is a ring decomposition $R = P \times K$ (Lemma 1), we see that R/P is a regular, right self-injective ring, hence we may assume (without loss of generality) that $P = 0$. This assumption that R is a prime ring will be in force until Theorem 8 is proved. Second, we construct a two-sided ideal $H(\alpha)$ corresponding to each infinite cardinal α. Finally, we show that these ideals $H(\alpha)$ exhaust the nonzero two-sided ideals of R.

In view of the assumption that R is a prime ring, Lemma 5 takes on the following strengthened form:

LEMMA 5′. *Given any nonsingular injective modules A and B, either A is subisomorphic to B or B is subisomorphic to A.*

For any infinite cardinal α, we define

$$H(\alpha) = \{0\} \cup \{x \in R | xR \ncong E[\alpha(xR)]\}.$$

Here we are using $E[M]$ to denote the injective hull of a module M, and αA to denote the direct sum of α copies of a module A. Note that $H(\alpha) \subset H(\beta)$ whenever $\alpha \leqq \beta$: for if $x \in R\backslash H(\beta)$, then

$$xR \cong E[\beta(xR)] = E[\alpha\beta(xR)] \cong E[\alpha(E[\beta(xR)])] \cong E[\alpha(xR)]$$

and so $x \notin H(\alpha)$.

THEOREM 9 (*Bumby* [**1**]). *Any two injective modules which are subisomorphic to each other must be isomorphic.*

LEMMA 10. *Let B be a nonsingular injective module, α any cardinal. If B has a nonzero submodule C such that $C \cong E[\alpha C]$, then $B \cong E[\alpha B]$.*

Proof. We obviously may assume that $\alpha > 1$. Set $\mathscr{A} = \{A \leqq B | A \cong E[\alpha A]\}$, and expand $\{C\}$ to a maximal independent family $\{A_i\} \subset \mathscr{A}$. We have $B = E[\oplus A_i] \oplus D$ for some D, and the maximality of $\{A_i\}$ implies that D has no nonzero submodules which belong to $\mathscr{A}$. Inasmuch as C is a nonzero submodule of $E[\oplus A_i]$ which belongs to $\mathscr{A}$, it follows that $E[\oplus A_i]$ cannot be subisomorphic to D. According to Lemma 5′, D must thus be subisomorphic to $E[\oplus A_i]$, whence B is subisomorphic to $2E[\oplus A_i]$. Since $\alpha > 1$ we infer that $A_i \cong 2A_i$ for all i, hence $E[\oplus A_i] \cong 2E[\oplus A_i]$. Thus B is subisomorphic to $E[\oplus A_i]$, hence in view of Theorem 9 we obtain $B \cong E[\oplus A_i]$, from which we conclude that $B \cong E[\alpha B]$.

LEMMA 11. *Let A be a nonsingular injective module which is isomorphic to a proper submodule of itself. Then $A \cong 2A \cong E[\aleph_0 A]$.*

Proof. There exists a monomorphism $g : A \to A$ which is not epic. Since gA is isomorphic to A and so is injective, we obtain $A = gA \oplus B$ for some nonzero B, and then $g^n A = g^{n+1}A \oplus g^n B$ for all positive integers n. Now $\{gB, g^2 B, \ldots\}$ is a countably infinite independent sequence of pairwise isomorphic submodules of A, hence the submodule $G = \oplus g^n B$ satisfies $G \cong 2G \cong \aleph_0 G$. The module A contains an injective hull C for G, and we see that $C \cong 2C \cong E[\aleph_0 C]$, hence from Lemma 10 we obtain $A \cong 2A \cong E[\aleph_0 A]$.

Given any nonzero nonsingular injective module A, it follows easily from Lemma 11 that $A \cong 2A$ if and only if $A \cong E[\aleph_0 A]$. Therefore

$$H(\aleph_0) = \{0\} \cup \{x \in R | xR \not\cong 2(xR)\}.$$

The next lemma, and some of the ideas used in the proof of Proposition 13, are due to Professor Kaplansky (unpublished), and the author is grateful for his permission to include them here. The notation used in Lemma 12 is as follows: $(f, g) : K \oplus L \to M$ is the unique map induced by maps $f : K \to M$ and $g : L \to M$.

LEMMA 12 (Kaplansky). *Let $(f, g) : K \oplus L \to M$.*

(a) *Given any map $h : K \to L$, then $(f + gh, g)$ maps $K \oplus L$ into M, and* $\ker (f + gh, g) \cong \ker (f, g)$.

(b) *If g is an isomorphism of L onto M, then* $\ker (f, g) \cong K$.

Proof. (a) Let $p : K \oplus L \to K$ denote the natural projection, $j : L \to K \oplus L$ the natural injection. Checking that $(f + gh, g)(1 - jhp) = (f, g)$, we infer that $1 - jhp$ maps $\ker(f, g)$ into $\ker(f + gh, g)$. Likewise, $1 + jhp$ maps $\ker(f + gh, g)$ into $\ker(f, g)$. Inasmuch as

$$(1 - jhp)(1 + jhp) = (1 + jhp)(1 - jhp) = 1,$$

we conclude that $\ker(f, g) \cong \ker(f + gh, g)$.

(b) Setting $h = -g^{-1}f$, we see that $f + gh = 0$, hence $\ker(f + gh, g) \cong K$. According to (a), $K \cong \ker(f, g)$.

Before stating Proposition 13, we make an observation which can be formulated as saying that any homomorphism $f:A \to B$ between nonsingular injective modules must split. Namely, the nonsingularity of B implies that $\ker f$ is closed in A; thus $\ker f$ is injective and so is a direct summand of A.

PROPOSITION 13. *Let A, B, C be nonsingular injective modules such that $A \not\cong 2A$. If $A \oplus B \cong A \oplus C$, then $B \cong C$.*

Proof. Letting $(f, g):A \oplus B \to A$ denote the composition of the isomorphism $A \oplus B \to A \oplus C$ with the projection $A \oplus C \to A$, we note that (f, g) is an epimorphism and that $\ker(f, g) \cong C$.

Setting $A_1 = \ker f$, we have $A = A_1 \oplus A_2$ for some A_2. Inasmuch as fA_2 is isomorphic to A_2 and so is injective, we also get $A = fA_2 \oplus A_3$ for some A_3. Letting $k:B \to A_3$ denote the composition of $g:B \to A$ with the projection $A = fA_2 \oplus A_3 \to A_3$, and setting $B_1 = \ker k$, we obtain $B = B_1 \oplus B_2$ for some B_2. The surjectivity of (f, g) implies that $A = fA + gB = fA_2 + gB$, from which we infer that k is surjective. It follows that $B_2 \cong A_3$ and that $A = fA_2 + gB_2$.

We now have $A = A_1 \oplus A_2$ and

$$A = fA_2 \oplus A_3 \cong A_2 \oplus B_2.$$

According to Lemma 5′, either A_1 is subisomorphic to B_2, or vice versa. If A_1 is isomorphic to a proper submodule of B_2, then $A_1 \oplus A_2$ is isomorphic to a proper submodule of $A_2 \oplus B_2$, i.e., A is isomorphic to a proper submodule of itself. This is impossible by Lemma 11, and the same contradiction arises if B_2 is isomorphic to a proper submodule of A_1, hence the only possibility is $A_1 \cong B_2$.

Letting $h:A \to B$ denote the composition of the projection $A = A_1 \oplus A_2 \to A_1$, the isomorphism $A_1 \to B_2$, and the inclusion $B_2 \to B$, we see that $(f + gh)A = (f + gh)A_1 + (f + gh)A_2 = ghA_1 + fA_2 = gB_2 + fA_2 = A$. Setting $K = \ker(f + gh)$, we have $A = K \oplus L$ for some L, and since $f + gh$ is surjective it follows that the restriction of $f + gh$ to L is an isomorphism. We now have a map $(f + gh, g):K \oplus L \oplus B = A \oplus B \to A$, and the restriction of $(f + gh, g)$ to L is an isomorphism of L onto A, hence from Lemma 12 we obtain $\ker(f + gh, g) \cong K \oplus B$. On the other hand, Lemma 12 also says that $\ker(f + gh, g) \cong \ker(f, g) \cong C$, whence $K \oplus B \cong C$.

Thus B is subisomorphic to C. By symmetry, C is also subisomorphic to B, hence $B \cong C$ by Theorem 9.

PROPOSITION 14. *For any infinite cardinal α, $H(\alpha)$ is a two-sided ideal of R.*

Proof. Case I: $\alpha = \aleph_0$. If $H(\alpha)$ is not a two-sided ideal, then there exist $x, y \in H(\alpha)$ and $z \in (RxR + RyR)\backslash H(\alpha)$. According to Theorem 6 we may assume that $RxR \leqq RyR$, hence $z \in RyR$. Note that $z \neq 0$ and $y \neq 0$. Now $zR \cong 2(zR)$, and from Lemma 4 we see that zR is subisomorphic to $n(yR)$ for some positive integer n, whence Lemma 10 yields $n(yR) \cong 2n(yR)$. However, $yR \not\cong 2(yR)$, so by inducting on Proposition 13 we reach the contradiction $0 \cong n(yR)$.

Case II: $\alpha > \aleph_0$. Given $x, y \in H(\alpha)$ and $z \in RxR + RyR$, we must show that $z \in H(\alpha)$. According to Theorem 6 we may assume that $RxR \leqq RyR$, hence $z \in RyR$. If $y \in H(\aleph_0)$, then $z \in H(\aleph_0) \subset H(\alpha)$ by Case I, hence we need only consider the possibility $y \notin H(\aleph_0)$. Thus $y \neq 0$ and $yR \cong 2(yR)$. Inasmuch as zR is subisomorphic to $n(yR)$ for some positive integer n (Lemma 4), we infer from $yR \cong 2(yR)$ that zR is also subisomorphic to yR. Since $yR \ncong E[\alpha(yR)]$, Lemma 10 says that either $z = 0$ or else $zR \ncong E[\alpha(zR)]$, whence $z \in H(\alpha)$.

PROPOSITION 15. *Any nonzero two-sided ideal H of R must be equal to $H(\alpha)$ for some infinite cardinal α.*

Proof. If δ is an infinite cardinal strictly larger than the cardinality of R, then no right ideal of R can contain an independent family of δ nonzero right ideals. Thus $xR \ncong E[\delta(xR)]$ for all nonzero $x \in R$, whence $H(\delta) = R$. Since $H \leqq H(\delta)$, there must be a smallest infinite cardinal α for which $H \leqq H(\alpha)$, and we of course claim that $H = H(\alpha)$.

Suppose on the contrary that there exists a $y \in H(\alpha)\backslash H$. Choosing a nonzero $x \in H$, we see from Lemma 4 that yR cannot be subisomorphic to any finite direct sum of copies of xR. In particular, yR is not subisomorphic to xR, so by Lemma 5′, xR must be subisomorphic to yR. Choosing a maximal independent family $\{A_i\}$ from those submodules of yR which are isomorphic to xR, we get $yR = E[\oplus A_i] \oplus B$ for some B. The maximality of $\{A_i\}$ ensures that xR cannot be subisomorphic to B, hence by Lemma 5′, B must be subisomorphic to xR. Letting τ denote the cardinality of the set $\{A_i\}$, we see that $E[\tau(xR)]$ is subisomorphic to yR, while yR is subisomorphic to $E[(\tau + 1)(xR)]$. Inasmuch as yR is not subisomorphic to any finite direct sum of copies of xR, τ must be infinite. Thus $\tau = \tau + 1$, and with the help of Theorem 9 we conclude that $yR \cong E[\tau(xR)]$.

We now have an infinite cardinal τ, and we infer that $yR \cong E[\tau(yR)]$. Inasmuch as $yR \ncong E[\alpha(yR)]$, we obtain $\tau < \alpha$, and thus $H \nleqq H(\tau)$. Choosing a $z \in H\backslash H(\tau)$, we have $zR \cong E[\tau(zR)]$. Inasmuch as $xR \ncong yR \cong E[\tau(xR)]$, we see from Lemma 10 that zR cannot be subisomorphic to xR. Thus xR must be subisomorphic to zR (Lemma 5′), whence $E[\tau(xR)]$ is subisomorphic to $E[\tau(zR)]$. We infer from this that yR is subisomorphic to zR, from which it follows via Lemma 4 that $y \in H$, which is a contradiction.

Theorem 8 is now essentially proved. For, given any nonempty collection of nonzero two-sided ideals of R, there must be a smallest infinite cardinal α for which $H(\alpha)$ belongs to the collection, and then $H(\alpha)$ is the smallest ideal in the collection. Any other nonempty collection of two-sided ideals must contain 0, which is then the smallest ideal in the collection.

In view of Theorem 8, the referee has asked how far a prime, regular, right self-injective ring R can be from a *full linear ring*, i.e., the ring of all linear transformations on a vector space over a division ring. Using [**6**, Theorem 4.3], it is not hard to see that R is isomorphic to a full linear ring if and only if R has a minimal right ideal. In general, one might conjecture that R is isomorphic to a factor ring of a full linear ring. This fails, however, because of [**8**, Corollary to Theorem 3], which says that a factor of a full linear ring by a nontrivial two-sided ideal cannot be right self-injective. The only other possibility in sight seems to be that R might be isomorphic to the maximal right quotient ring of a factor of a full linear ring, but we see no clues to the probability of this.

With the help of Theorem 8, we can offer some information towards a query of Professor Kaplansky as to whether a prime regular ring must be primitive [**5**, p. 2]: In the self-injective case, the answer is yes.

COROLLARY 16. *R is prime if and only if it is primitive.*

Proof. Inasmuch as all primitive rings are prime, we need only consider the case when R is prime. The collection $\mathscr{P}$ of primitive ideals of R is nonempty (since $\mathscr{P}$ contains all the maximal two-sided ideals), hence by Theorem 8 $\mathscr{P}$ has a smallest element P. It is well-known that the Jacobson radical of a regular ring is 0, hence $\bigcap \mathscr{P} = 0$ and thus $P = 0$. Therefore 0 is a primitive ideal of R.

4. Krull dimension. From commutative ring theory, we borrow verbatim the concept of the *Krull dimension* of R: K.dim.(R) = the supremum of the lengths of all chains of prime ideals in R. The purpose of this section is to develop an inequality relating the Krull dimension of R to the global dimension in the case when R is a prime ring.

THEOREM 17. *If R is a prime ring, but not artinian, then* $1 + \text{K.dim.}(R) \leqq \text{r.gl.dim.}(R)$.

Proof. Since $\text{r.gl.dim.}(R) > 0$, it suffices to consider the case when $\text{K.dim.}(R) > 0$. Given any positive integer $n \leqq \text{K.dim.}(R)$, we shall prove that $\text{r.gl.dim.}(R) \geqq n + 1$.

The inequality $\text{K.dim.}(R) \geqq n$ implies that R must contain a chain of $n + 1$ prime ideals; since these ideals are all proper, R must have at least $n + 2$ distinct two-sided ideals. (Note that we can extract no more information from the Krull dimension than the number of two-sided ideals: for according to Corollary 7 and Theorem 8, K.dim.(R) is just the number of nontrivial two-sided ideals in R.) According to Proposition 15, there must exist $n + 1$ infinite cardinals $\alpha_0 < \alpha_1 < \ldots < \alpha_n$ such that the ideals $H(\alpha_i)$ are all distinct. We must have $\alpha_{n-1} \geqq \aleph_{n-1}$, hence $1 \notin H(\aleph_{n-1})$ and so $R \cong E[\aleph_{n-1}R]$. Thus R contains an independent family $\{A_j | j \in J\}$ of $\aleph_{n-1}$ nonzero principal right ideals.

For each $j \in J$, A_j is injective, $\bigoplus_{k \neq j} A_k$ has an injective hull E_j contained in R, and $A_j \cap E_j = 0$, hence we obtain a module decomposition $R = A_j \oplus E_j \oplus B_j$. If e_j is the idempotent corresponding to the first factor in this decomposition, then $e_jR = A_j$ and $e_jA_k = 0$ for all $k \neq j$. Collecting all the e_j, we obtain a set of $\aleph_{n-1}$ orthogonal idempotents.

According to [**9**, Propositions 7 and 5], R must now contain a set of "nice" idempotents [**9**, p. 641] whose cardinality is 2 to the power $\aleph_{n-1}$. Noting that a subset of a set of nice idempotents is also nice, we infer that R has a set N of nice idempotents of cardinality $\aleph_n$. Letting I denote the right ideal generated by N, we see from [**9**, Theorem A] that the projective dimension of I is exactly n. Therefore r.gl.dim.$(R) \geqq n + 1$.

In general, the inequality just proved can neither be strengthened nor be changed to equality, as the following two examples show.

First, let F be the field with two elements, V a vector space over F of dimension $\aleph_0$, and R the ring of all linear transformations on V (written on the left). Inasmuch as V is a nonsingular injective F-module, [**2**, Lemma G, p. 69] says that R is a regular, right self-injective ring. According to [**3**, Theorem 5, p. 258], R has exactly one nontrivial two-sided ideal, from which we infer that R is prime, and then, using Corollary 7, that K.dim.$(R) = 1$. Assuming the continuum hypothesis, [**9**, Remark 3.6] shows that r.gl.dim.$(R) = 2$, hence $1 +$ K.dim.$(R) =$ r.gl.dim.(R).

The second example is somewhat more involved, but does not use the continuum hypothesis. Given any positive integer n, choose a vector space V of dimension $\aleph_n$ and let P denote the ring of all linear transformations on V (written on the left). As in the example above, P is a regular, right self-injective ring. This time, [**3**, Theorem 5, p. 258] says that P has exactly $n + 1$ nontrivial two-sided ideals, arranged in a chain. We again infer that P is prime, and now that K.dim.$(P) = n + 1$. As in the proof of Theorem 17, we obtain $P \cong E[\aleph_n P]$, hence there is a monomorphism $\aleph_n P \to P$.

The ring P obviously has exactly one maximal two-sided ideal M, and $Q = P/M$ is a simple, regular ring. The regularity of P implies that ${}_PQ$ is flat, hence we obtain a monomorphism

$$(\aleph_n P) \oplus_P Q \to P \oplus_P Q,$$

i.e., a monomorphism $\aleph_n Q \to Q$.

Finally, define R to be the maximal right quotient ring of Q. The regularity of Q implies that Q is a nonsingular ring [**2**, Proposition 3, p. 70], hence [**2**, Theorems 1 and 2, p. 69] says that R is a regular, right self-injective ring. The intersection of Q with any nonzero two-sided ideal I of R yields a nonzero two-sided ideal $I \cap Q$ of Q; since Q is simple, we get $I \cap Q = Q$ and then $I = R$. Therefore R is a simple ring, whence R is prime and K.dim.$(R) = 0$.

As in the passage from P to Q, we obtain a monomorphism $\aleph_n R \to R$, hence $E[\aleph_n R]$ is subisomorphic to R. Inasmuch as R is subisomorphic to $E[\aleph_n R]$, we obtain from Theorem 9 that $R \cong E[\aleph_n R]$. Therefore $H(\aleph_n) < R$. Proceeding as in the proof of Theorem 17, we obtain

$$\text{r.gl.dim.}(R) \geqq n + 2 > 1 + \text{K.dim.}(R).$$

This example also shows that the $H(\alpha)$'s need not be either distinct or nonzero: since R is simple and $H(\aleph_n) < R$, we see that $H(\aleph_0), H(\aleph_1), \ldots, H(\aleph_n)$ are all zero.

5. Direct product decompositions. This section is devoted to characterizing when R is a direct product of prime rings. In view of Proposition 3 and Corollary 16, we obtain at the same time characterizations of when R is a direct product of either indecomposable or primitive rings.

THEOREM 18. *R is isomorphic to a direct product of prime rings if and only if every nonzero two-sided ideal of R contains a minimal two-sided ideal.*

Proof. Assume that $R = \prod R_i$, where each R_i is a prime ring. According to Theorem 8, the two-sided ideals in any R_i are well-ordered, hence every nonzero two-sided ideal in R_i contains a minimal two-sided ideal. It follows easily that the same property is satisfied in R.

Now suppose that the given condition on two-sided ideals holds, and let $\{M_i\}$ be a maximal independent family of minimal two-sided ideals. Setting K_i equal to the left annihilator of M_i, we obtain

$$(\bigoplus M_i) \cap (\bigcap K_i) = [(\bigoplus M_i) \cap (\bigcap K_i)]^2 \leqq (\bigcap K_i)(\bigoplus M_i) = 0.$$

Then from the maximality of $\{M_i\}$ we infer that $\bigcap K_i$ contains no minimal two-sided ideals, whence $\bigcap K_i = 0$. Thus we obtain an injective ring homomorphism

$$\phi : R \to P = \prod (R/K_i),$$

induced by the natural maps $\phi_i : R \to R/K_i$.

In order to show that R/K_i is a prime ring, consider any two-sided ideals I and J in R which properly contain K_i. Inasmuch as $IM_i \neq 0$ and $JM_i \neq 0$, it follows from the minimality of M_i that $IM_i = JM_i = M_i$, hence $M_i \leqq I \cap J$. Thus $M_i = M_i^3 \leqq IJM_i$, from which we conclude that $IJM_i \neq 0$ and $IJ \nleqq K_i$.

Next, we show that ϕR is an essential right (ϕR)-submodule of P. Given any nonzero $p \in P$, we have $p_j \neq 0$ for some j, i.e., $p_j = \phi_j x$ for some $x \in R \backslash K_j$. Then $xy \neq 0$ for some $y \in M_j$, and since $M_j \cap K_j = (M_j \cap K_j)^2 \leqq K_j M_j = 0$, we get $xy \notin K_j$. Therefore $[p(\phi y)]_j = \phi_j(xy) \neq 0$, hence $p(\phi y) \neq 0$. Note also that $[p(\phi y)]_j = [\phi(xy)]_j$. For any $i \neq j$, $y \in K_i$ because

$$yM_i \leqq M_j \cap M_i = 0,$$

whence

$$[p(\phi y)]_i = p_i(\phi_i y) = 0 = (\phi_i x)(\phi_i y) = [\phi(xy)]_i.$$

Therefore $p(\phi y) = \phi(xy)$, so that $p(\phi y)$ is a nonzero element of ϕR.

Inasmuch as ϕR is isomorphic to R and so is a right self-injective ring, the module $(\phi R)_{\phi R}$ cannot have any proper essential extensions. Thus $\phi R = P$ and $R \cong P$.

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