OBITUARY

A. J. MACINTYRE, F.R.S.E.

The death on 4th August 1967, at the age of 59, of Archibald James Macintyre, Research Professor of Mathematics at the University of Cincinnati, after a brief illness, came as a severe shock to all who knew him. His good nature and kindness, his subtle but delightful sense of humour, the patience with which he explained mathematical problems to those less quick than he to see the point, his high standard, his humility—all these are but a fraction of the fine attributes which explain the esteem and popularity accorded to him, by both staff and students, wherever he went.

Elsewhere† I shall give a full account of Macintyre's career. Here I present only a short version, which will include what I consider to be the highlights of his work.

Macintyre entered Magdalene College, Cambridge as a Scholar in 1926 and became a Wrangler with distinction in 1929. He undertook research, latterly part-time, on meromorphic functions under Dr (now Sir) Edward Collingwood, which earned him his Cambridge Ph.D. in 1933. He was assistant lecturer at Swansea University College (1930-31), assistant lecturer and then lecturer at Sheffield University (1931-36), lecturer and later senior lecturer at Aberdeen University (1936-59) and finally, from 1959, research professor in Mathematics at Cincinnati, where he was also elected a Fellow of the Graduate School. He became a Fellow of the Royal Society of Edinburgh in 1946. He was for nearly 40 years a member of several mathematical societies, including the Edinburgh Mathematical Society.

Macintyre's family life was a happy one. In 1940 he married Sheila Scott, of Edinburgh, a Girtonian and herself a Wrangler in Mathematics. She too lectured in Mathematics at Aberdeen and Cincinnati, where she was both popular and highly regarded. She died in America in 1960 at the early age of 50. Both the surviving children, Allister and Susan, have settled in America.

Before discussing Macintyre's research in pure mathematics I ought to mention that he had deep interests elsewhere, especially in aerodynamics, fluid mechanics and related fields. He conducted with enthusiasm his experiments with model aeroplanes, was awarded D.S.I.R. grants for his experiments and corresponded widely with experts on these topics. He believed, for example, that the rudder and the elevator of the aeroplane should be located at the front and not at the back, and could give sound answers to the objections raised by experts, as well as sound objections to the present practice in these matters in the aircraft industry. At the time the special electronic equipment he needed

† *Bull. London Math. Soc.* **1** (1969), 368-381.

was almost unobtainable, and later supervision of an increasing number of research students in pure mathematics occupied much of his time. He was thus prevented from pushing ahead further with these ideas, although his interest was maintained throughout his life.

A. J. MACINTYRE'S WORK IN] PURE MATHEMATICS

As a classical analyst, Macintyre considered a diversity of problems, throughout many of which runs a strong thread, his keen interest in overconvergence. Amongst his 43 papers we find such topics as asymptotic paths, the flat regions of meromorphic functions, interpolation problems based on the Laplace transform and other formulae for regular functions, Tauberian theorems in connection with certain canonical products, and numerous problems, many studied jointly with R. Wilson, in the theory of the singularities of $f(z) = \sum_{n=0}^{\infty} c_n z^n$ on the circle of convergence.

A full bibliography is not given here, as it will appear elsewhere. However the books (**1**)-(**4**) contain proofs and references for many of Macintyre's results.

A significant part of Macintyre's work concerns the so-called flat regions of integral and meromorphic functions $f(z)$. J. M. Whittaker, who published important theorems on the subject [see references (3), (4) of (**4**), p. 106], had defined R to be a flat region if

$$H^{-1} < \{\log |f(z_1)| / \log |f(z_2)|\} < H \tag{1}$$

for all $z_1, z_2 \in R$, where $H > 1$ is a constant. Ostrowski had given a sharp form of Schottky's theorem which implies that if $q(z)$ ($\neq 0, 1$) is regular in $|z| < s$ then it is flat in $|z| < \theta s$ ($\theta \leqq \frac{1}{2}$) if $|q(0)|$ is large enough. In his first paper on flat regions (1935) Macintyre, using a dissection due to Valiron of the complex plane into curvilinear quadrilaterals, shows that a meromorphic function $f(z)$ of order $\rho > 0$ whose Nevanlinna characteristic $T(r)$ satisfies $T(r) = o(r^{\rho'}/\log r)$ where $\rho' > \rho$ and whose poles are of deficiency $\delta > 0$, differs from 0, 1, ∞ in a selection of these quadrilaterals and hence, applying Ostrowski's result, that $f(z)$ is flat, with $H - 1 < \varepsilon_\nu$, in a sequence of circles whose centres $\xi_\nu \to \infty$, where $\varepsilon_\nu \to 0$. In this way a new proof of a sharper version of Whittaker's results was obtained.

Later an entirely different approach gave sharper and more general theorems. Let $f(z)$ be regular in the circle γ: $|z| < R$, and let $z = z_s$ be the zeros of $f(z)$ in γ. The results referred to are

$$\log |f(z)| > -KT(kr), \tag{2}$$

$$\left| \frac{d^q}{dz^q} \log f(z) \right| < Kr^{-q}[T(kr)]^q \tag{3}$$

which are valid p.p. in $|z| < r$, with $k > 1$ and $q \geqq 1$ an integer, where $T(kr)$ denotes the Nevanlinna characteristic for $f(z)$ on $|z| = kr (< R)$. (2) was proved by Macintyre in 1938; in (**3**), pp. 72-77, Cartwright gives another proof for functions of finite order. (3) was obtained by Macintyre and Wilson in 1942. Nevanlinna had given a formula for $\log f(z)$ whose real part (the Poisson-Jensen formula) and qth derivative are used to prove (2) and (3) respectively. A lower bound for $|p(z)|$, where $p(z) = \prod_s (z-z_s)$, and an upper bound for $|\sum_s (z-z_s)^{-q}|$, outside the immediate neighbourhood N consisting of small circles γ_s about the z_s, are provided respectively by Cartan's lemma [(**4**), p. 46] and theorems due to Macintyre and Fuchs (1940). An additional factor $\log T(kr)$ appears on the right-hand side of (3) when $q = 1$, if the relative magnitude of N is measured by comparing the sum of the radii of the γ_s with r. However even in this case, which is of considerable importance in the coefficient theory (see below), an ingenious argument due to Macintyre and Fuchs shows that (3) holds if the z_s are all on a radius from the origin. Flat region results follow by integrating (3) with $q = 1$.

The methods apply equally well when $f(z)$ is meromorphic, in which case (2) has also a left-hand inequality $KT(kr) > \log |f(z)|$. Analogous results when $f(z)$ is meromorphic merely in an angle are obtained by using the above in suitable circles or by starting with the appropriate Nevanlinna formula for $\log f(z)$ with $f(z)$ defined only in the upper half-plane.

In another approach to problems of this nature Macintyre (1938) investigated the extent of the regions in which an integral function $f(z)$ remains large in the neighbourhood of a point $z = re^{i\theta}$ at which $|f(z)|$ takes its maximum value $M(r)$. By a complicated process Wiman had shown that the usual behaviour of $f(z)$ for such z is something like z^N where N is a constant, in simple cases the rank of the maximum term of the Maclaurin series for $f(z)$. Macintyre states that this is equivalent to saying that $\phi(T) = e^{-NT} f(ze^T)/f(z)$ is practically constant for such z, and shows by a simple argument that N is real and lies between the left and right derivatives $d\{\log M(r)\}/d\{\log r\}$. He obtains his results by applying Schwarz's lemma to $\phi(T) - 1$ in a suitable circle, having first proved boundedness of $\phi(T)$ by using the convexity property of $\log M(r)$ and a lemma of E. M. Wright on convex functions. He thus sharpened considerably the best theorems, due to Valiron, then known in this direction. A further consideration of $\phi(T)$ led him (1938) to the simplest proof then known, and to new generalizations, of Bloch's theorem.

In 1939, under the title " Laplace's transformation and integral functions ", Macintyre published a remarkable paper which not only gave new proofs and extensions of theorems of M. L. Cartwright [(**3**), p. 98; p. 129 (10)] as well as other interpolation results, but also was to lead to new results in the coefficient theory of power series.

Theorem. *Let the integral function* $F(z) = \sum_{n=0}^{\infty} a_n z^n$ *satisfy*

$$|F(z)| < Me^{k|z|} (k < \pi).$$

Then

(i) $|F(x)|<K(k)A$ *for all real x and an explicit* $K(k)$ *if* $|F(n)|<A$ $(n = 0, \pm 1, \pm 2, \ldots)$, *and*

(ii) $F(x)\to 0$ *as* $x\to +\infty$ *if* $F(n)\to 0$ *as* $n\to +\infty$ $(n = 1, 2, \ldots)$.

M. L. Cartwright had obtained these results by using the Lagrange interpolation formula and the Phragmén-Lindelöf principle. Macintyre, inspired by Pólya's well-known article [Math. Zeitschrift **29** (1929), 549-640] on gaps and singularities of power series, shows how to use the Borel-Laplace transform for these problems. Setting $f(z) = \sum_{n=0}^{\infty} a_n n! z^{-n-1}$ and $F(z) = (2\pi i)^{-1}\int_C e^{z\zeta}f(\zeta)d\zeta$ with a suitable contour C, he replaces $f(\zeta)$ by a second associated function $\psi(z)$ expressed as power series in $e^{\mp z}$ with coefficients $F(\pm n)$ in $\operatorname{Re} z>k$ and $\operatorname{Re} z< -k$, whence term-by-term integration leads to the interpolation formula

$$F(z) = \lim_{\delta\to 0} \pi^{-1} \sum_{-\infty}^{\infty} F(n)e^{-\delta|n|} \sin \omega(z-n)/(z-n)$$

if $\overline{\lim_{n\to\infty}} |F(\pm n)|^{1/n} \leqq 1$, $k<\omega<2\pi-k$. From this result a whole family of interpolation formulae, as well as (i) and (ii) of the Theorem, follow. Analogous theorems, both for integral functions of any finite order ρ, and for functions regular and of finite order in an angle, are obtained by similar methods.

We turn now to the coefficient theory of the Taylor series. In the same paper Macintyre made several remarks which supplemented the discussion given by Pólya (*loc. cit.*) on the connection between integral functions and the singularities of power series. Considerable advances in this theory were soon to be made, both individually and jointly by Macintyre and Wilson. The discussion here is a summary of an account kindly prepared for me by Professor Wilson. For a comprehensive account of developments in the theory up to 1955 see **(1)**.

Suppose that $|z| = 1$ is the circle of convergence of $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and, for the first set of results, that there is only one singularity S, namely an essential point at $z = 1$. The results are

$$|c_n| > e^{-\gamma n}, \tag{4}$$

$$|c_{n+1}/c_n - 1| < \varepsilon \tag{5}$$

($\gamma>0$, $\varepsilon>0$ arbitrary) for a common sequence $\{n'\}$ of n. (4) had already been given by Pólya, the upper density, maximal density or density of $\{n'\}$ being unity, according to the nature of the singularity S which he had defined [almost isolated (a.i.); easily approachable or approachable; a.i. of finite exponential order, respectively]. Flat region techniques of Whittaker [**(4)**, p. 106, (3), (4)] had been used by Whittaker and Wilson (1939) to prove (5) with $\{n'\}$ of unit density for a more restricted S, thus partly confirming a conjecture of Wilson (fully confirmed below). It was Macintyre who perceived that the connection

https://doi.org/10.1017/S0013091500013043 Published online by Cambridge University Press

between (4), (5) and " flat region " theorems was fundamental. Briefly, if $G(z)$ is the function approximately interpreting the c_n, so that $c_n = G(n)+c_n^*$, where $\overline{\lim}_{n\to\infty} | c_n^* |^{1/n}<1$, and $G(z)$ is an integral function of at most order 1, minimum type, then along any radius, by (2), (3),

$$\log | G(z) | > -\gamma | z |, \quad | G'(z)/G(z) | < \varepsilon \quad (| z | = r) \tag{6}$$

for a set of r of linear density 1. From (6), new proofs of Pólya's results for (4), as well as analogous results for (5), easily follow.

In the same paper (1941) by Macintyre and Wilson the virtually isolated singularity is defined; for this the authors obtain (4), (5) with $\{n'\}$ of density 1. Their method contained an error, difficult to discern, but the result is correct and was brilliantly proved by Macintyre in 1958 using Bourion's theory of overconvergence.

In 1940 Macintyre and Wilson had supposed that there is a set $\{S\}$ of singularities on $| z | = 1$, and considered the density of the sequence $\{n''\}$ of n for which $| c_n |<e^{-\gamma n}$, i.e. $| c_n |^{1/n}<e^{-\gamma}$, so that $\{S\}$ contributes nothing to such coefficients, which are therefore called *small*, because their effects cancel out. Briefly, it was found that the density (or lower density) of $\{n''\}$ is zero, unless the singularities are of the same kind and placed at the vertices of a regular polygon. In that case, if the polygon is k-sided with an identical singularity at each vertex, the density is $(k-1)/k$. F. W. Ponting, a colleague of Macintyre at Aberdeen, has dealt in detail with similar cases giving a density $(k-2)/(k-1)$, etc.

Methods used in the paper on Laplace transforms mentioned earlier were applied in at least two further ways in joint papers with Wilson. In the first (1947) it is shown that the orders and types of $F(z)$, $z^{-1}f(z^{-1})$ and $G(z)$ are all related, and the directions of strongest growth of $F(z)$ as $z\to\infty$ are the same to within reflection as the corresponding directions of $\psi(z)$ as $z\to 1$, where

$$\psi(z) = \sum_{n=0}^{\infty} c_n z^n = G[1/(1-z)]$$

with $G(z)$ an integral function, $c_n = F(n)$, and $f(z)$ is the Laplace transform of $F(z)$. This result was considerably sharpened by Wilson in 1961. The second application (1954) was in the determination of the asymptotic form of the c_n arising from singularities like $\exp P[1/(1-z)]$, where $P(z)$ is a polynomial, and from critical points derived from this singularity. In the introduction Macintyre sketched an elegant theory which provided the basis required.

It was in the field of Tauberian theorems that I began my research in 1946, with Macintyre as my supervisor. The initial problem was to obtain the asymptotic relation $n(r)\sim Kr^\rho$ $(0<K<\infty, 0<\rho<1, r\to+\infty)$ for the number $n(r)$ of zeros in $|z|<r$ of the integral function $f(z) = \prod_{n=1}^{\infty} (1+z/a_n), 0<a_n \leqq a_{n+1}$ $(n = 1, 2, 3, \ldots)$ given that

$$\log f(z) \sim K\pi z^\rho \operatorname{cosec} \pi\rho \tag{7}$$

for $z = +x\to\infty$. Lengthy real variable proofs had been given by Valiron

(1914) and Titchmarsh (1926). A simple function theory proof, valid under less restrictive hypotheses, was evolved under Macintyre's guiding hand. See Chapter 4 of (2) for details of some of the theorems mentioned here. Montel's limit theorem gives (7) for $|\arg z|<\pi-\delta$ $(\delta>0)$, from the imaginary parts of which, with δ small, we have

$$\int_0^\infty \frac{\sin\delta\,.\,n(ru)\,.\,du}{1+u^2-2u\cos\delta} \sim K\pi r^\rho \sin\rho(\pi-\delta)\operatorname{cosec}\pi\rho,$$

which gives the required result by an elementary argument, since the integral is significant only when $u \doteqdot \cos\delta$. Trivial modifications are required to get the result for any such canonical product of genus p, where $p<\rho<p+1$, so that a complete proof of Valiron's theorem is thus obtained.

Relaxations which we found possible in the hypotheses are of two kinds. In the first place the path P to infinity may be taken, as Montel's theorem allows, to be *any* continuous path having no point in common with the negative real axis, and P may even be replaced by a sequence $\{z_n\}$ such that $|z_n|\uparrow\infty$ and $|z_{n+1}/z_n|\to 1$ as $n\to\infty$. We had in fact proved (1954) a refined version of Montel's theorem, and also of Vitali's convergence theorem, from a lemma due to J. M. Whittaker [(4), p. 57].

The second kind of relaxation is the replacement of (7) by the real parts of the relation for $\arg z = \alpha$ with $\alpha \neq \pm\pi$ constant, with certain side conditions [(2), p. 59]; and even for nonintegral $\rho<3/2$ with $\alpha=\pi$, thus giving a new proof and an extension of a result due to Titchmarsh (1926).

We also showed (1949) that Valiron's theorem can be regarded as the limiting case of an oscillation theorem; that, in fact, from

$$0<l<\overline{\lim_{x\to\infty}}\; x^{-\rho}\log f(x) \leqq L<\infty$$

follows

$$0<\phi(l, L) \leqq \overline{\lim_{n\to\infty}}\; n^{-1/\rho}a_n \leqq \Phi(l, L)<\infty,$$

where Φ/ϕ is arbitrarily near unity when L/l is sufficiently near unity.

As far as I can discover, only one paper (1952) was prepared jointly by Macintyre and his wife Sheila Scott Macintyre. Sheila, herself a classical analyst, had numerous research publications to her credit, especially on the Whittaker constant [defined in (2), p. 173] and on interpolation series of various kinds for integral functions. The object of the joint paper is to investigate conditions under which the Abel series

$$\sum_{n=0}^{\infty} z(z-n)^{n-1}F^{(n)}(n)/n!$$

(i) converges;

(ii) is asymptotically equivalent to $F(z)$ in $\operatorname{Re} z>0$, under the assumption that $F(z)$ is regular in $|\arg z| \leqq 3\pi/4$ where it satisfies $|F(re^{i\theta})|<Kr^{-\gamma}e^{rb(\theta)}$ with

$K>0$, γ real and $b(\theta)$ the supporting function of a certain convex set. The proof of (i) is immediate, but that of (ii) depends on the use of the Laplace transform $f(z)$ of $F(z)$—in the expression

$$F(z) = (2\pi i)^{-1} \int_\Gamma e^{z\zeta} f(\zeta) d\zeta \tag{8}$$

with a suitable contour Γ, $e^{z\zeta}$ is expanded in powers of ζe^{ζ} and (ii) follows on integration term by term.

In 1954 Macintyre alone obtained interesting results, applicable to the Lidstone and Whittaker two-point series as well as the Poritsky and Gontcharoff n-point series, by expanding $f(\zeta)$ in (8) and integrating term by term, with $F(z)$ now an *integral* function of exponential type, and then gave an explicit solution to a problem of Pólya by modifying his method to apply when $F(z)$ is of *any* finite order.

On several occasions Macintyre took up the topic of gap power series. He considered (1952) whether the existence of asymptotic paths of an integral function

$$F(z) = \sum_{n=0}^{\infty} c_n z^{\lambda_n} \quad (0 \leqq \lambda_n \uparrow \infty)$$

can or cannot be excluded by a knowledge of the sequence $\{\lambda_n\}$ alone. With Paul Erdös (1954) he amplified and sharpened Pólya's [*loc. cit.*] deductions that

$$\overline{\lim_{r\to\infty}}\, m(r)/M(r) = \overline{\lim_{r\to\infty}}\, \mu(r)/M(r) = 1$$

from information about $\{\lambda_n\}$; here $m(r)$, $\mu(r)$ denote min $|f(z)|$, max $|c_n z^{\lambda_n}|$ respectively on $|z| = r$. In 1959 he gave new results on overconvergence for gap series.

In this brief account I have selected what I consider to be the highlights of Macintyre's mathematical output. Others would probably have chosen differently, but I think that all will agree what a great debt Mathematics owes, and will owe, to the fertility of his mind.

In conclusion I offer my sincere thanks to all those who have helped me to prepare this essay, in particular to the Registrars of Swansea University College and of Sheffield University and to Mr W. S. Angus, formerly Secretary of Aberdeen University, for biographical details; to H. S. A. Potter, G. E. H. Reuter, J. M. Whittaker and especially H. Shankar and R. Wilson for assistance in various ways concerning the mathematical content.

REFERENCES

(1) L. Bieberbach, *Analytische Fortsetzung* (Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Heft, Springer-Verlag, 1955).

(2) R. P. Boas, *Entire Functions* (Academic Press, 1954).

(3) M. L. Cartwright, *Integral Functions* (Cambridge Tracts in Mathematics and Mathematical Physics No. 44, Cambridge 1956).

(4) J. M. Whittaker, *Interpolatory Function Theory* (Cambridge Tracts in Mathematics and Mathematical Physics No. 33, Cambridge, 1935; reprinted Stechert-Hafner, New York, 1964).

N. A. Bowen