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FORCED OSCILLATIONS OF SOLUTIONS OF PARABOLIC EQUATIONS

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Parabolic equations with forcing terms are studied and sufficient conditions are given that all solutions of boundary value problems are oscillatory in a cylindrical domain.

Recently there has been much interest in studying the oscillatory behaviour of solutions of parabolic equations with functional arguments. We refer the reader to Bykov and Kultaev [1], Kreith and Ladas [2] and the author [3]. However, forced oscillations have not been discussed.

In this paper we are concerned with the forced oscillation of solutions of the parabolic equation

$$(1)\qquad u_t - a(t)\Delta u + c(x,t,u(x,t),u(x,\sigma(t))) = f(x,t),\ (x,t) \in \Omega \times \mathbb{R}_+,$$

where Δ is the Laplacian in Euclidean n-space $\mathbb{R}^n$, $\mathbb{R}_+ = [0,\infty)$ and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. It is assumed that

(A_1) $a(t)$ is a nonnegative continuous function in $\mathbb{R}_+$ and $f(x,t)$ is a continuous function in $\overline{\Omega} \times \mathbb{R}_+$;

(A_2) $c(x,t,\xi,\eta) \geqq 0$ for $(x,t) \in \Omega \times \mathbb{R}_+$, $\xi \geqq 0$, $\eta \geqq 0$, and

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$$c(x,t,\xi,\eta) \leq 0 \quad \text{for} \quad (x,t) \in \Omega \times \mathbb{R}_+ , \ \xi \leq 0 , \ \eta \leq 0 ;$$

(A_3) $\sigma(t)$ is a continuous function in $\mathbb{R}_+$ such that $\lim_{t\to\infty} \sigma(t) = \infty$.

Our objective is to present conditions which imply that every (classical) solution u of (1) satisfying a certain boundary condition is oscillatory in $\Omega \times \mathbb{R}_+$ in the sense that u has a zero in $\Omega \times [t,\infty)$ for any $t > 0$. We consider three kinds of boundary conditions:

(B_1) $u = \phi$ on $\partial\Omega \times \mathbb{R}_+$,

(B_2) $\dfrac{\partial u}{\partial \nu} = \psi$ on $\partial\Omega \times \mathbb{R}_+$,

(B_3) $\dfrac{\partial u}{\partial \nu} + \mu u = 0$ on $\partial\Omega \times \mathbb{R}_+$,

where ϕ, ψ, μ are continuous functions on $\partial\Omega \times \mathbb{R}_+$, ν denotes the unit exterior normal vector to $\partial\Omega$ and $\mu \geq 0$ on $\partial\Omega \times \mathbb{R}_+$.

It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\Delta w + \lambda w = 0 \quad \text{in} \quad \Omega$$
$$w = 0 \quad \text{on} \quad \partial\Omega$$

is positive and the corresponding eigenfunction Φ is positive in Ω.

THEOREM 1. *Assume that* (A_1)-(A_3) *hold. Every solution* u *of the problem* (1), (B_1) *is oscillatory in* $\Omega \times \mathbb{R}_+$ *if*

$$\liminf_{s\to\infty} \int_{\tilde{s}}^{s} \exp(\lambda_1 A(t)) \left(-a(t) \int_{\partial\Omega} \phi \frac{\partial \Phi}{\partial \nu} d\omega + \int_{\Omega} f(x,t)\Phi \, dx \right) dt = -\infty ,$$

$$\limsup_{s\to\infty} \int_{\tilde{s}}^{s} \exp(\lambda_1 A(t)) \left(-a(t) \int_{\partial\Omega} \phi \frac{\partial \Phi}{\partial \nu} d\omega + \int_{\Omega} f(x,t)\Phi \, dx \right) dt = \infty$$

for all large $\tilde{s}$, *where* $A(t) = \int_0^t a(\tau) d\tau$.

Proof. Suppose to the contrary that there is a solution u of the problem (1), (B_1) which has no zero in $\Omega \times [t_0,\infty)$ for some $t_0 > 0$. Let $u > 0$ in $\Omega \times [t_0,\infty)$. Since $\lim_{t\to\infty} \sigma(t) = \infty$, there is a number t_1 such that $t_1 > t_0$ and $\sigma(t) \geq t_0$ $(t \geq t_1)$. Hence $u(x,\sigma(t)) > 0$ in

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$\Omega \times [t_1,\infty)$. From assumption (A_2) we see that $c(x,t,u(x,t),u(x,\sigma(t))) \geqq 0$ in $\Omega \times [t_1,\infty)$, and therefore

$$u_t - a(t)\Delta u \leqq f(x,t) \quad \text{in} \quad \Omega \times [t_1,\infty) . \tag{2}$$

Multiplying (2) by Φ and integrating over Ω , we obtain

$$\frac{d}{dt}\int_\Omega u\Phi\, dx - a(t)\int_\Omega (\Delta u)\Phi\, dx \leqq \int_\Omega f(x,t)\Phi\, dx , \quad t \geqq t_1 . \tag{3}$$

It follows from Green's formula that

$$\begin{aligned}\int_\Omega (\Delta u)\Phi\, dx &= \int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\Phi - u\frac{\partial\Phi}{\partial\nu}\right)d\omega + \int_\Omega u\Delta\Phi\, dx \\ &= -\int_{\partial\Omega}\phi\,\frac{\partial\Phi}{\partial\nu}d\omega - \lambda_1\int_\Omega u\Phi\, dx .\end{aligned} \tag{4}$$

Combining (3) with (4) yields

$$\frac{d}{dt}\int_\Omega u\Phi\, dx + \lambda_1 a(t)\int_\Omega u\Phi\, dx \leqq -a(t)\int_{\partial\Omega}\phi\,\frac{\partial\Phi}{\partial\nu}d\omega + \int_\Omega f(x,t)\Phi\, dx ,$$

which is equivalent to

$$(\exp(\lambda_1 A(t))U(t))' \leqq \exp(\lambda_1 A(t))\left(-a(t)\int_{\partial\Omega}\phi\frac{\partial\Phi}{\partial\nu}d\omega + \int_\Omega f(x,t)\Phi\, dx\right) , \tag{5}$$

where $A(t) = \int_0^t a(\tau)d\tau$ and $U(t) = \int_\Omega u\Phi\, dx$. Integrating (5) over $[t_1,s]$, we obtain

$$\begin{aligned}&\exp(\lambda_1 A(s))U(s) - \exp(\lambda_1 A(t_1))U(t_1) \\ &\leqq \int_{t_1}^{s} \exp(\lambda_1 A(t))\left(-a(t)\int_{\partial\Omega}\phi\frac{\partial\Phi}{\partial\nu}d\omega + \int_\Omega f(x,t)\Phi\, dx\right)dt .\end{aligned} \tag{6}$$

The hypothesis implies that the right hand side of (6) is not bounded from below, and hence $\exp(\lambda_1 A(s))U(s)$ cannot be eventually positive. This contradicts the positivity of $\exp(\lambda_1 A(s))U(s)$ $(s \in [t_1,\infty))$. If $u < 0$ in $\Omega \times [t_0,\infty)$, $v \equiv - u$ satisfies

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$$\frac{d}{dt}\int_\Omega v\Phi\, dx + \lambda_1 a(t)\int_\Omega v\Phi\, dx \leqq - a(t)\int_{\partial\Omega}(-\phi)\frac{\partial\Phi}{\partial\nu}d\omega + \int_\Omega (-f(x,t))\Phi\, dx .$$

Proceeding as in the case where $u > 0$, we are led to a contradiction. The proof is complete.

A special case of the problem (1), (B_1) is the following:

(7) $$u_t - \Delta u + c(x,t,u(x,t),u(x,\sigma(t))) = f(x,t), \quad (x,t) \in \Omega\times\mathbb{R}_+ ,$$

(8) $$u = 0 \quad \text{on} \quad \partial\Omega\times\mathbb{R}_+ .$$

COROLLARY. *Assume that* (A_1)-(A_3) *hold. Every solution* u *of the problem (7), (8) is oscillatory in* $\Omega\times\mathbb{R}_+$ *if*

$$\liminf_{s\to\infty}\int_{\tilde{s}}^{s} exp(\lambda_1 t)\left(\int_\Omega f(x,t)\Phi\, dx\right)dt = -\infty,$$

$$\limsup_{s\to\infty}\int_{\tilde{s}}^{s} exp(\lambda_1 t)\left(\int_\Omega f(x,t)\Phi\, dx\right)dt = \infty$$

for all large $\tilde{s}$.

Proof. Since $A(t) = t$ and $\phi \equiv 0$, the conclusion follows from Theorem 1.

THEOREM 2. *Assume that* (A_1)-(A_3) *hold. Every solution* u *of the problem (1),* (B_2) *is oscillatory in* $\Omega\times\mathbb{R}_+$ *if*

(9) $$\liminf_{s\to\infty}\int_{\tilde{s}}^{s}\left(a(t)\int_{\partial\Omega}\psi\, d\omega + \int_\Omega f(x,t)dx\right)dt = -\infty,$$

(10) $$\limsup_{s\to\infty}\int_{\tilde{s}}^{s}\left(a(t)\int_{\partial\Omega}\psi\, d\omega + \int_\Omega f(x,t)dx\right)dt = \infty$$

for all large $\tilde{s}$.

Proof. Suppose that the problem (1),(B_2) has a solution u which has no zero in $\Omega\times[t_0,\infty)$ for some $t_0 > 0$. We may suppose that $u > 0$ in $\Omega\times[t_0,\infty)$. As in the proof of Theorem 1, we see that the inequality (2) holds. Integration of (2) over Ω gives

$$\frac{d}{dt}\int_\Omega u\, dx \leqq a(t)\int_{\partial\Omega}\psi\, d\omega + \int_\Omega f(x,t)dx , \quad t \geqq t_1 .$$

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Arguing as in the proof of Theorem 1, we are led to a contradiction. The proof is complete.

THEOREM 3. *Assume that* (A_1)-(A_3) *hold. Every solution* u *of the problem* (1),(B_3) *is oscillatory in* $\Omega \times \mathbb{R}_+$ *if*

$$\liminf_{s\to\infty} \int_{\tilde{s}}^{s}\left(\int_{\Omega} f(x,t)dx\right)dt = -\infty , \tag{11}$$

$$\limsup_{s\to\infty} \int_{\tilde{s}}^{s}\left(\int_{\Omega} f(x,t)dx\right)dt = \infty \tag{12}$$

for all large $\tilde{s}$.

Proof. Let u be a solution of (1),(B_3), which has no zero in $\Omega \times [t_0,\infty)$ for some $t_0 > 0$. We may suppose that $u > 0$ in $\Omega \times [t_0,\infty)$. Integrating (2) over Ω and taking into account (B_3), we obtain

$$\frac{d}{dt}\int_{\Omega} u\, dx \leqq a(t)\int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\omega + \int_{\Omega} f(x,t)dx$$

$$= - a(t)\int_{\partial\Omega} \mu u\, d\omega + \int_{\Omega} f(x,t)dx \leqq \int_{\Omega} f(x,t)dx,\ t \geqq t_1 .$$

The same argument as in the proof of Theorem 1 leads us to a contradiction.

EXAMPLE 1. We consider the problem

$$u_t - u_{xx} + e^{\pi/2}u(x,t-\pi/2) = 2(\cos x)e^t \sin t,\ (x,t) \in (0,\pi/2) \times \mathbb{R}_+ , \tag{13}$$

$$- u_x(0,t) = 0 ,\ u_x(\pi/2,t) = - e^t \sin t,\quad t \in \mathbb{R}_+ . \tag{14}$$

Here $n = 1$, $a(t) \equiv 1$, $\Omega = (0,\pi/2)$, $f(x,t) = 2(\cos x)e^t \sin t$ and $\int_{\partial\Omega} \psi\, d\omega = - e^t \sin t$. We easily see that

$$\int_{\tilde{s}}^{s}\left(\int_{\partial\Omega} \psi\, d\omega + \int_{\Omega} f(x,t)dx\right)dt$$

$$= \int_{\tilde{s}}^{s} e^t \sin t\, dt$$

$$= 2^{-1/2}e^s \sin (s-\pi/4) + 2^{-1}e^{\tilde{s}}(\cos \tilde{s} - \sin \tilde{s}) .$$

Hence, we find that conditions (9) and (10) are satisfied. It follows from Theorem 2 that every solution u of (13), (14) is oscillatory in $(0,\pi/2) \times \mathbb{R}_+$. One such solution is $u = (\cos x)e^t \sin t$.

EXAMPLE 2. We consider the problem

(15) $$u_t - u_{xx} + e^{\pi/2}u(x,t-\pi/2) = (2\cos x + 1)e^t \cos t, \ (x,t) \in (0,\pi) \times \mathbb{R}_+,$$

(16) $$-u_x(0,t) = u_x(\pi,t) = 0, \quad t \in \mathbb{R}_+.$$

Here $n = 1$, $a(t) \equiv 1$, $\Omega = (0,\pi)$ and $f(x,t) = (2\cos x + 1)e^t \cos t$. Since

$$\int_{\tilde{s}}^{s}\left(\int_{\Omega} f(x,t)dx\right)dt$$

$$= \int_{\tilde{s}}^{s} \pi e^t \cos t \, dt$$

$$= 2^{-1/2}\pi e^s \sin(s + \pi/4) - (\pi/2)e^{\tilde{s}}(\cos \tilde{s} + \sin \tilde{s}),$$

conditions (11) and (12) are satisfied. Theorem 3 implies that every solution u of (15), (16) is oscillatory in $(0,\pi) \times \mathbb{R}_+$. In fact, there is an oscillatory solution $u = (\cos x + 1)e^t \cos t$ of the problem (15), (16).

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