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COUNTING DISCRETE, LEVEL-1, QUATERNIONIC AUTOMORPHIC REPRESENTATIONS ON G_2

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Abstract Quaternionic automorphic representations are one attempt to generalize to other groups the special place holomorphic modular forms have among automorphic representations of GL_2. Here, we use 'hyperendoscopy' techniques to develop a general trace formula and understand them on an arbitrary group. Then we specialize this general formula to study quaternionic automorphic representations on the exceptional group G_2, eventually getting an analog of the Eichler–Selberg trace formula for classical modular forms. We finally use this together with some techniques of Chenevier, Renard and Taïbi to compute dimensions of spaces of level-1 quaternionic representations. On the way, we prove a Jacquet–Langlands-style result describing them in terms of classical modular forms and automorphic representations on the compact-at-infinity form G_2^c.

The main technical difficulty is that the quaternionic discrete series that quaternionic automorphic representations are defined in terms of do not satisfy a condition of being 'regular'. A real representation theory argument shows that regularity miraculously does not matter for specifically the case of quaternionic discrete series.

We hope that the techniques and shortcuts highlighted in this project are of interest in other computations about discrete-at-infinity automorphic representations on arbitrary reductive groups instead of just classical ones.

Contents

Key words and phrases: Trace Formula; G2; Endoscopy; Dimension Formulas; Automorphic Representations

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1. Introduction

1.1. Context

This work first develops an 'explicit' trace formula (1) to study so-called quaternionic automorphic representations in general and then specializes it to describe level-1, discrete, quaternionic automorphic representations on G_2. Let $\mathcal{Q}_1(k)$ be the set of such representations of weight k counted with multiplicity. For each $k > 2$, we give a formula, (11), for $|\mathcal{Q}_1(k)|$ in terms of counts of automorphic representations on the compact-at-infinity inner form G_2^c that were calculated by Chenevier and Renard in [CR15] (In case the use of counts on G_2^c is bothersome, Section 9.4 also gives a relatively short closed-form formula, though this is less conceptually enlightening). We also give a Jacquet–Langlands-style result (Corollary 8.2.1) describing all elements of $\mathcal{Q}_1(k)$ in terms of certain automorphic representations on G_2^c and certain pairs of classical modular forms.

Quaternionic automorphic representations are one way to generalize to other groups the special place holomorphic modular forms have among automorphic representations of GL_2. Just like holomorphic modular forms, they are characterized by their infinite component being in a particular nice class of discrete series representations: the quaternionic discrete series of [GW96]. Just like modular forms, they also have many unexpected applications and connections to other areas of mathematics. For example, they have a nice theory of Fourier expansions with interesting arithmetic content – this was described for G_2 in [GGS02] and generalized to all exceptional groups in [Pol20]. They also somehow appear in certain string theory computations involving black holes – see the end of chapter 15 in [FGKP18] for example. Quaternionic forms have been studied a lot by Pollack: See [Pol21] for an introductory article on them and [Pol18] for good exposition specifically on G_2-quaternionic forms.

We study discrete, quaternionic representations in general using the trace formula: Arthur's invariant trace formula, as in [Art89], lets us analyze automorphic representations with infinite component contained in a fixed discrete series L-packet. However, quaternionic discrete series appear in L-packets with nonquaternionic members and therefore cannot be isolated within the packet without further techniques. A previous work, [Dal22], uses the stabilization of the trace formula to abstractly isolate members of the L-packet and prove general asymptotic bounds. Here, we demonstrate that the same techniques suffice for computing more explicit information.

As one technical point of interest, there is a particular miracle about quaternionic discrete series that crucially underpins the result. A priori, quaternionic discrete series are not regular, implying that there may not be a compact test function at infinity whose trace picks out exactly a quaternionic discrete series without also picking up some unwanted contributions from nontempered representations. This would preclude the use of easy trace formula arguments. However, it turns out that specifically quaternionic discrete series don't get entangled in this way, even though other members of their L-packet do. The proof is a computation in real representation theory.

Applying these generalities to G_2 lets us develop an 'Eicher–Selberg'-style trace formula for quaternionic discrete series there. The counts of level-1 forms and Jacquet–Langlands-style results come from computations with it. The level-1 computation in particular also relies heavily on powerful shortcuts developed in [CR15] and [Taï17] to get exact counts of level-1 automorphic representations on classical groups.

This work can be be compared to the much more difficult aforementioned papers of Chenevier, Renard and Taïbi counting level-1 representations with arbitrary discrete series at infinity on classical groups. Those avoided needing our real representation theory miracle by using the extremely powerful endoscopic classification of [Art13] which, in essence, gave a finer decomposition of the trace formula than pure stabilization. In particular, it allowed the isolation of summands in the trace formula that did not contain any contributions from automorphic representations with nontempered local components through a far more complicated inductive procedure. Unfortunately, there is no endoscopic classification currently available for G_2.

The methods here should be able to also compute averages of Hecke eigenvalues. We also hope that the computation highlights enough general methods and shortcuts to be helpful for people interested in doing explicit computations with discrete-at-infinity automorphic representations whenever an endoscopic classification might either be unavailable or be too complicated to use. See the end of Section 3.3 for some comments on this. In particular, a very similar method, albeit with more complicated computation at infinity, should be able to 'quickly' count the quaternionic forms on type D_4 studied by Martin Weissman in [Wei06].

1.2. Summary

We start with the case of general groups with quaternionic discrete series at infinity. Section 2 discusses quaternionic discrete series and their properties, culminating in Proposition 2.3.3 showing that they satisfy a property of being 'trace-distinguishable'. This is used in Theorem 3.1.1 to show that the spectral side of Arthur's invariant trace formula can be made equal to the trace of any desired finite-place test function against the space of all quaternionic representations of a fixed weight.

The second part of Section 3 uses 'hyperendoscopy' from [Fer07] to simplify the geometric side and develops an expression (1) for this trace that is explicit up to computing endoscopic transfers and orbital integrals. We conclude with some remarks on how to do the necessary computations and outline how they simplify in the special case of level-1 on G_2. This special case of G_2 then takes up the rest of the paper.

Specifically, after some setup work in Section 4, we work out what formula (1) reduces to in the unramified case for G_2 in Section 5 using a computation of the endoscopy of G_2. Instead of using formula (1) directly, we compare its application to G_2 to its application to the compact real form G_2^c to construct a formula for $I_{\mathrm{spec}}^{G_2}$ involving just $I_{\mathrm{spec}}^{G_2^c}$- and I_{spec}^{H}-terms. Here, H is the endoscopic group $\mathrm{SL}_2 \times \mathrm{SL}_2/\pm 1$ of G_2.

Section 6 then tells us which exact $I_{\mathrm{spec}}^{G_2^c}$- and I_{spec}^{H}-terms appear by computing endoscopic transfers at infinity. The difficult part of this computation is pinning down various signs coming from transfer factors. We present a shortcut to make it more

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manageable. The final result should be thought of as an 'Eichler–Selberg' trace formula for quaternionic automorphic representations on G_2. As a last piece of the puzzle, Section 7 uses results about level-1 forms from [CR15] to reduce counts of forms on H to counts of classical modular forms.

Section 8 uses all these formulas to characterize representations in $\mathcal{Q}_k(1)$ with $k > 2$ in terms of automorphic representations on G_2^c and certain pairs of classical modular forms. We substitute in values for the $I_{\text{spec}}^{G_2^c}$-terms from [CR15] and present a final table of dimensions: table 1, in Section 9. Finally, building off an impressive undergraduate thesis [Sul13] of Sullivan, we give a relatively simple closed-form formula for the G_2^c-term and present the resulting closed-form formula for $|\mathcal{Q}_k(1)|$ in Section 9.4.

1.3. Notation

Here is a list of notation used throughout:

The group G_2

- G_2 is the standard exceptional Chevalley group defined over $\mathbb{Z}$.
- G_2^c is the unique inner form of G_2 over $\mathbb{Q}$. Recall that $(G_2^c)_\mathbb{R}$ is the compact real form.
- $\alpha_i, \epsilon_i, \lambda_i$ are particular roots and coroots of G_2 defined in Section 4.1.
- s_λ is the simple reflection associated to root or coroot λ.
- ρ is half the sum of the positive roots of G_2.
- V_λ is the finite-dimensional representation of G_2 of highest weight λ.
- K is a choice of maximal compact subgroup $\mathrm{SU}(2) \times \mathrm{SU}(2)/\pm 1$ of $G_2(\mathbb{R})$.
- K^∞ is the product of maximal compact subgroups $G_2(\mathbb{Z}_p)$ over all p.
- $\Omega = \Omega_\mathbb{C}$ is the Weyl group of G_2.
- $\Omega_\mathbb{R}$ is the Weyl group of K as a subset of Ω.
- H will often refer to the specific endoscopic group $\mathrm{SL}_2 \times \mathrm{SL}_2/\pm 1$ of G_2.
- $\mathcal{Q}_k(1)$ is the set of discrete, quaternionic automorphic representations of G_2 of weight k and level 1 (see Section 4.2).
- π_k is the weight k quaternionic discrete series of G_2 (see Section 4.2).

General groups

- $G_\infty = \mathrm{Res}_\mathbb{Q}^F G(\mathbb{R})$ for G a reductive group over number field F. Since most groups here are over $\mathbb{Q}$, G_∞ is usually $G(\mathbb{R})$.
- G^S, G_S are more generally the standard upper- and lower-index notation for $G(\mathbb{A}^S), G(\mathbb{A}_S)$ – leaving out the places in S or only including the places in S, respectively.
- $\Omega(G)$ is the absolute Weyl group of G.
- $\Omega_\mathbb{R}(G)$ is the subset of the Weyl group of G_∞ with respect to an elliptic maximal torus (if one exists) generated by elements of G_∞.
- K_G is a maximal compact subgroup of G_∞.
- K_G^∞ for unramified G is the product of chosen hyperspecial subgroups at all finite places.
- ρ_G is half the sum of the positive roots of G.

- $[G(F)], [G(F)]_{\mathrm{ss}}, [G(F)]_{\mathrm{st}}, [G(F)]_{\mathrm{ell}}$ are the (semisimple, stable, elliptic) conjugacy classes of $G(F)$.

Real test functions

- φ_π for π a discrete series representation of G_∞ is the pseudocoefficient defined in the corollary to Proposition 4 in [CD90].
- $\Pi_{\mathrm{disc}}(\lambda)$ is the discrete series L-packet corresponding to dominant weight λ.
- η_λ is the Euler–Poincaré function from [CD90] for $\Pi_{\mathrm{disc}}(\lambda)$. We normalize it to be the average of the pseudocoefficients for $\pi \in \Pi_{\mathrm{disc}}(\lambda)$ instead of their sum.

Trace formula

- $\mathcal{AR}(G)$, $\mathcal{AR}_{\mathrm{disc}}(G)$ is the set of (discrete) automorphic representations on G.
- $\mathcal{AR}_{\mathrm{ur}}(G)$ for G unramified is the space of unramified automorphic representations of G.
- $m_{\mathrm{disc}}(\pi), m_{\mathrm{cusp}}(\pi)$ are the multiplicities of automorphic representation π in the discrete (cuspidal) subspace.
- $I^G_{\mathrm{spec}}, I^G_{\mathrm{geom}}, I^G_{\mathrm{disc}}$ are the distributions from Arthur's invariant trace formula on G.
- $S^G = S^G_{\mathrm{geom}}$ is the stable distribution defined in Theorem 3.2.1.

Miscellaneous

- $\mathbf{1}_S$ is the indicator function of set S.
- $\mathbf{1}_G$ is the trivial representation on group G.
- $\mathcal{S}_k(1)$ is the set of normalized, classical, cuspidal eigenforms on GL_2 of level 1 and weight k.

2. Quaternionic discrete series

2.1. Discrete series

2.1.1. Parametrization. For this section, let G be a reductive group over $\mathbb{R}$ and K a maximal compact of $G(\mathbb{R})$. Assume G has elliptic torus T so that $G(\mathbb{R})$ has discrete series. Without loss of generality, $T \subseteq K$. Recall the notation from [Dal22, §2.2.1] to discuss discrete series. In particular, recall the two parametrizations of discrete series on $G(\mathbb{R})$:

$$\pi^G_{\lambda,\omega} = \pi^G_{\omega(\lambda+\rho_G)}$$

for λ a dominant (but possibly irregular) weight of T and ω a Weyl-element that takes a chosen Ω_G dominant chamber into a chosen Ω_K-dominant one. Note that $\pi^G_{\lambda,\omega}$ has infinitesimal character $\lambda+\rho_G$. Recall that $\omega(\lambda+\rho_G)$ is called the *Harish–Chandra parameter* of this discrete series.

2.1.2. Their pseudocoefficients. Recall from the corollary on page 213 in [CD90] the notion of pseudocoefficients φ_π for discrete series π. They are defined by their traces against standard modules ρ:

$$\mathrm{tr}_\rho(\varphi_\pi) = \begin{cases} 1 & \rho = \pi \\ 0 & \rho \text{ standard}, \sigma \neq \pi \end{cases}.$$

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Here, a standard module is a parabolic induction of a discrete series or limit of discrete series.

Note. By the Langlands classification, every irreducible representation has a *character formula* writing it as a linear combination of standard modules in the Grothendieck group. By linearity of trace, if σ is an irreducible representation, then $\mathrm{tr}_\sigma(\varphi_\pi)$ is the coefficient of π in its character formula.

Recall also the Euler–Poincaré functions η_λ that we normalize to be the average of pseudocoefficients over an L-packet of infinitesimal character $\lambda+\rho_G$. For a quick summary of relevant properties of these functions in the notation used here, see [Dal22, §2.2.2].

2.2. Trace distinguishability

A priori, the trace against φ_π may be nonzero for certain nontempered representations in addition to just π. This could make φ_k unusable as a test function to pick out just automorphic representations π with $\pi_\infty = \pi$. We analyze when this happens.

Definition. Call discrete series π on group $G(\mathbb{R})$ *trace-distinguishable* if for all unitary representations σ of $G(\mathbb{R})$

$$\mathrm{tr}_\sigma(\varphi_k) = \begin{cases} 1 & \sigma = \pi_k \\ 0 & \text{else} \end{cases}.$$

To motivate this definition, the Paley–Weiner theorem of [CD90] shows that φ_π is the only compactly supported function that could have the property of isolating π in the unitary dual in this way – there are none if π isn't trace-distinguishable.

Proposition 2.2.1. *Let discrete series π on $G(\mathbb{R})$ have Harish–Chandra parameter ξ. Define*

$$S_\xi = \{\alpha \in \Phi_G : \langle \xi, \alpha^\vee \rangle = 1\},$$

where Φ_G is the set of roots of (G,T) for T elliptic. Then π is trace-distinguishable if and only if π contains no noncompact roots.

Proof. The following proof was described to me by David Vogan. Choose simple roots so that ξ is dominant. By the same argument of Vogan described in [Dal22, lem. 6.3.1], $\mathrm{tr}_\sigma(\varphi_\pi) = 0$ unless $\mathrm{tr}_\sigma(\eta_{\xi-\rho_G}) \neq 0$ for $\eta_{\xi-\rho_G}$ the Euler–Poincaré function at infinitesimal character ξ. If σ is unitary, this is only possible if σ has nonzero $(\mathfrak{g},K)$-cohomology with respect to the irreducible finite-dimensional representation of infinitesimal character ξ.

By the main classification result of [VZ84], the only representations that do so are the discrete-series packet $\Pi_\lambda(\xi - \rho_G)$ and certain cohomological inductions $A_\mathfrak{q}(\lambda)$ for θ-stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$ and λ a character of the Levi algebra $\mathfrak{l}$ associated to $\mathfrak{q}$ (see, for example, [AJ87, §2.1] for a definition of $A_\mathfrak{q}(\lambda)$). It therefore suffices to show that none of these except π itself have π appearing in their character formulas.

The only nontrivial case to check is that of nondiscrete-series $A_\mathfrak{q}(\lambda)$. Theorem 8.2 in [AJ87] provides its character formula and shows that the discrete series that appear are exactly those with Harish–Chandra parameters of the form $\lambda + \omega\rho_{\mathfrak{l}}$, where ω ranges over the Weyl group of $\mathfrak{l}$. For each ω, pick a set of simple roots of $\mathfrak{l}$ so that $\omega\rho_l$ is dominant. Then for simple root α of $\mathfrak{l}$,

$$\langle \lambda + \omega\rho_{\mathfrak{l}}, \alpha\rangle = \langle \omega\rho_{\mathfrak{l}}, \alpha\rangle = 1.$$

In particular, if π appears in the character formula for $A_{\mathfrak{q}}(\lambda)$, then there is a choice of simple roots of $\mathfrak{l}$ that are in S_ξ.

Finally, since λ is regular, for any root α of G, $|\langle \lambda, \alpha^\vee\rangle| \geq 1$. Therefore, S_ξ needs to be a subset our simple roots chosen to make λG-dominant. Let $\mathfrak{l}_\xi$ be the associated Levi subalgebra. If π appears in the character formula for $A_{\mathfrak{q}}(\lambda)$, the above gives that $\mathfrak{l} \subseteq \mathfrak{l}_\xi$. Therefore, if S_ξ has no noncompact roots, then $\mathfrak{l}$ is compact, so our condition on $\mathfrak{l}$ implies that $A_{\mathfrak{q}}(\lambda)$ is discrete series (see, for example, the bottom of [AJ87, pg. 272]) and therefore equal to π. In total, π cannot appear in other character formulas completing one direction.

In the other direction, if S_ξ has a noncompact root, then this can be used to construct a rank-1 Levi subalgebra $\mathfrak{l}$ that isn't compact. Pick corresponding $\mathfrak{q}$, and choose chamber for $\mathfrak{l}$ so that λ is $\mathfrak{l}$-dominant. Then π will appear in the character formula of $A_{\mathfrak{q}}(\lambda - \rho_{\mathfrak{l}})$ which isn't discrete series. □

2.3. Quaternionic discrete series

Quaternionic discrete series are a special class of discrete series picked out in [GW96]. We recall some needed definitions and properties:

Definition. Call $G(\mathbb{R})$ *quaternionic* if K is isogenous to a group of the form $\mathrm{SU}_2(\mathbb{R}) \times L$ (that has the same rank as G).

Definition. If $G(\mathbb{R})$ is quaternionic, call discrete series π *quaternionic* if its minimal K type lifts to a representation of the form $V \boxtimes \mathbf{1}_L$ on $\mathrm{SU}_2(\mathbb{R}) \times L$. Let the *weight* of π be $(\dim V - 1)/2$.

By looking at extended root diagrams:

Lemma 2.3.1. *Group $G(\mathbb{R})$ is quaternionic if and only if there is a choice of simple roots of $(G(\mathbb{R}), T)$ such that there is a is unique noncompact simple root that is also the unique simple root not perpendicular to the highest root.*

Then, by Blattner's formula for minimal K-types:

Lemma 2.3.2. *Let G have quaternionic discrete series with simple roots chosen as in the previous lemma. Then all quaternionic discrete series have Harish–Chandra parameter of the form $n\beta' + \rho_G$ for $n \in \mathbb{Z}_{\geq 0}$ and β' the highest root.*

Miraculously, almost all quaternionic discrete series are trace distinguishable:

Proposition 2.3.3. *Let π be a quaternionic discrete series of $G(\mathbb{R})$ with infinitesimal character not equal to ρ_G. Then π is trace-distinguishable.*

Proof. If $\lambda = n\beta' + \rho_G$ as in Lemma 2.3.2, then S_λ from Proposition 2.2.1 is a subset of the simple roots chosen in Lemma 2.3.2. Since β' is not perpendicular to the unique noncompact simple root and $n > 1$, S_λ can only contain compact roots. □

3. Trace formula

Let G be a reductive group over number field F such that G_∞ is quaternionic.

Definition. A *quaternionic automorphic representation* on G is an automorphic representation π such that π_∞ is quaternionic.

In this section, we construct an 'explicit' trace formula for studying almost all quaternionic automorphic representations.

3.1. Spectral side

The previous discussion on quaternionic discrete series shows:

Theorem 3.1.1. *Let G_∞ have quaternionic discrete series, and let π_0 be a quaternionic discrete series of G_∞ with infinitesimal character not equal to ρ_G. Then*

$$\begin{aligned} I_{\mathrm{spec}}(\varphi_{\pi_0} \otimes f^\infty) &= \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G_2)} m_{\mathrm{disc}}(\pi)\delta_{\pi_\infty = \pi_0} \operatorname{tr}_{\pi^\infty}(f^\infty) \\ &= \sum_{\pi \in \mathcal{AR}_{\mathrm{cusp}}(G_2)} m_{\mathrm{cusp}}(\pi)\delta_{\pi_\infty = \pi_0} \operatorname{tr}_{\pi^\infty}(f^\infty). \end{aligned}$$

Proof. The statement for discrete representations is the same argument as [Dal22, prop. 6.3.3] after we know Proposition 2.3.3 that these quaternionic discrete series are trace-distinguishable. Since $\pi_\infty = \pi_0$ is necessarily discrete series for the nonzero terms, the main result of [Wal84] shows that $m_{\mathrm{cusp}}(\pi) = m_{\mathrm{disc}}(\pi)$. □

Note. Of course, this theorem holds more generally for π_0 an arbitrary trace-distinguishable discrete series.

3.2. Geometric side/the hyperendoscopy formula

3.2.1. Notation. We will need to recall some extra notation related to general reductive group H over F to understand the geometric side

- Ω_H^c is the Weyl group generated by compact roots at infinity.
- $d(H_\infty)$ is the size of the discrete series L-packets of H_∞. Alternatively, $d(H_\infty) = |\Omega_H|/|\Omega_H^c|$.
- $k(H_\infty)$ is the size of the group $\mathfrak{K} = \ker(H^1(\mathbb{R}, T_{\mathrm{ell}}) \to H^1(\mathbb{R}, G_\infty))$ that appears in the theory of endoscopy for G_∞.
- $q(H_\infty) = \dim(H_\infty/K_\infty Z_{H_\infty})$ where K_∞ is a maximal compact subgroup of H_∞.
- H_∞^* is the quasisplit inner form of H_∞.
- $\bar{H}_\infty$ is the compact form. If H_∞ has an elliptic maximal torus, this is inner.
- $e(H_\infty)$ is the Kottwitz sign $(-1)^{q(H_\infty^*) - q(H_\infty)}$.

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- $[H:M]=[H:M]_F=\dim(A_M/A_G)$, where $A_\star$ is the maximal F-split torus in the center of $\star$. We call this the index of M in H.
- $\tau(H)$ is the Tamagawa number of H.
- Mot_H is the Gross motive for H.
- $L(\mathrm{Mot}_H)$ is the value of the corresponding L-function at 0 (or residue of the pole).
- $\iota^H(\gamma)=\iota^H_F(\gamma)$ for $\gamma\in H(F)$ is the number of connected components of H_γ that have an F-point.

3.2.2. Preliminaries. Let π_0 be a quaternionic discrete series on G_∞. We will use the hyperendoscopy formula of [Fer07] to compute $I_{\mathrm{geom}}(\varphi_{\pi_0}\otimes f^\infty)$. We need to apply the general case of [Dal22, Thm. 4.2.3] since G might have endoscopy without simply connected derived subgroup. We use notation from [Dal22, §3,4] to discuss endoscopy and hyperendoscopy. See [KS99] for a full reference to the theory of endoscopy and [Lab11] for a course-notes-style introduction.

Let η be the Euler–Poincaré function for the L-packet $\Pi_{\mathrm{disc}}(\lambda)$ that conatins π_0. Let $\mathcal{HE}_{\mathrm{ell}}(G)$ be the set of nontrivial hyperendoscopic paths for G. Then, in the notation of [Dal22, §4],

$$I^{G_2}_{\mathrm{geom}}(\varphi_{\pi_0}\otimes f^\infty)=I^{G_2}_{\mathrm{geom}}(\eta_k\otimes f^\infty)+\sum_{H\in\mathcal{HE}_{\mathrm{ell}}(G_2)}\iota(G,\mathcal{H})I^{\tilde{\mathcal{H}}}_{\mathrm{geom}}(((\eta-\varphi_{\pi_0})\otimes f^\infty)^{\tilde{\mathcal{H}}}),$$

where the $\tilde{\mathcal{H}}$ are choices of z-pair paths when they are needed.

3.2.3. Telescoping. Next, an unpublished result of Kottwitz summarized in [Mor10, §5.4] and proved by other methods in [Pen19] stabilizes $I_{\mathrm{geom}}(\varphi\otimes f^\infty)$ when φ satisfies a technical property of being stable-cuspidal (as EP-functions are but pseudocoefficients are not):

Theorem 3.2.1. *Let φ be stable cuspidal (e.g., an EP-function, but not a pseudocoefficient) on G_∞ and f^∞ a test function on $G(\mathbb{A}^\infty)$. Then*

$$I^G_{\mathrm{geom}}(\varphi\otimes f^\infty)=\sum_{H\in\mathcal{E}_{\mathrm{ell}}(G)}\iota(G,H)S^{\tilde H}_{\mathrm{geom}}((\varphi\otimes f^\infty)^{\tilde H}),$$

where $\mathcal{E}_{\mathrm{ell}}(G)$ is the set of elliptic endoscopic groups for G and the $\tilde H$ are z-extensions if necessary. The transfers $(\varphi\otimes f^\infty)^{\tilde H}$ depend on choices of measures for G and H.

The S_{geom} terms are defined by their values on Euler–Poincaré functions:

$$S^H_{\mathrm{geom}}(\eta_\lambda\otimes f^\infty)=\sum_{M\in\mathcal{L}^{\mathrm{cusp}}(H)}(-1)^{[H:M]}\frac{|\Omega_{M,F}|}{|\Omega_{H,F}|}\tau(M)$$
$$\times\sum_{\gamma\in[M(\mathbb{Q})]_{\mathrm{st,ell}^\infty}}|\iota^M(\gamma)|^{-1}\frac{e(\bar M_{\gamma,\infty})}{\mathrm{vol}(\bar M_{\gamma,\infty}/A_{\bar M_{\gamma,\infty}})}\frac{k(M_\infty)}{k(H_\infty)}\Phi^H_M(\gamma,\lambda)SO^\infty_\gamma((f^\infty)_M),$$

choosing Tamagawa globally measure on all centralizers. The volume on $\bar M_{\gamma,\infty}$ is transferred from that on $M_{\gamma,\infty}$ in the standard way for inner forms so that the entire term doesn't depend on a choice of measure at infinity.

There's an alternating sign in the hyperendoscopy formula: If $\mathcal{H}$ is a hyperendoscopic path, then $-\iota(G,\mathcal{H})\iota(\mathcal{H},H) = \iota(G,(\mathcal{H},H))$ for H any endoscopic group of $\mathcal{H}$. Here, $(\mathcal{H},H)$ represents the concatenation and $\mathcal{H}$ is overloaded to also refer to the last group in $\mathcal{H}$.

In particular, substituting in the stabilization telescopes the hyperendoscopy formula.

3.3. Final formula and usage notes

3.3.1. Formula. Telescoping together with Theorem 3.1.1 produces the final formula for quaternionic discrete series π_0 of G_∞ that has infinitesimal character not equal to ρ_G:

$$\sum_{\pi \in \mathcal{AR}_{\text{disc}}(G)} m_{\text{disc}}(\pi)\delta_{\pi_\infty = \pi_0} \operatorname{tr}_{\pi^\infty}(f^\infty)$$
$$= S^G_{\text{geom}}(\eta \otimes f^\infty) + \sum_{\substack{H \in \mathcal{E}_{\text{ell}}(G) \\ H \neq G}} \iota(G,H) S^{\tilde{H}}_{\text{geom}}((\varphi_{\pi_0} \otimes f^\infty)^{\tilde{H}}). \tag{1}$$

The right side can be evaluated with Theorem 3.2.1.

We recall

$$\iota(G,H) = |\Lambda(H,\mathcal{H},s,\eta)|^{-1} \frac{\tau(G)}{\tau(H)},$$

where $\Lambda(H,\mathcal{H},s,\eta)$ is the image in $\operatorname{Out}(\widehat{H})$ of the automorphisms of the endoscopic quadruple.

While getting a formula in terms of the distributions S_{geom} on smaller endoscopic groups comes immediately from stabilization, the above telescoping argument seems to be necessary to get explicit formulas for the S_{geom} when using a test factor at infinity that is just cuspidal instead of a stable cuspidal.

3.3.2. Usage. There are two possible methods to compute terms here. If we were interested in working with more general groups or at more general level, something like method 1 would have been necessary. However, our application case of level-1 representations on G_2 allows us to use the much easier method 2. Method 2 in fact does not even need an explicit expansion for S_{geom}.

Method 1:

We can try calculate the S_{geom} terms directly from their formula in Theorem 3.2.1. We will need to choose Euler–Poincaré measure at $\bar{M}_\gamma$ times canonical measure for the orbital integrals (canonical measure is the same for all inner forms). This adds an extra factor of

$$d(\bar{M}_{\gamma,\infty}) \frac{L(\operatorname{Mot}_{M_\gamma})}{e(\bar{M}_{\gamma,\infty}) 2^{\operatorname{rank}(M_{\gamma,\infty})}}$$

by [ST16, lem. 6.2]. Since $d(H_\infty) = 1$ and $\operatorname{vol}_{EP}(H_\infty / A_{H_\infty}) = 1$ for H compact, this expands the terms in Equation (1) as

$$S^H_{\text{geom}}(\eta_\lambda \otimes f^\infty) = \sum_{M \in \mathcal{L}^{\text{cusp}}(H)} \left((-1)^{[H:M]} \frac{|\Omega_{M,F}|}{|\Omega_{H,F}|} \right) \left(\tau(M) \frac{k(M_\infty)}{k(H_\infty)} \right)$$
$$\times \sum_{\gamma \in [M(\mathbb{Q})]_{\text{st, ell}^\infty}} 2^{-\operatorname{rank}(M_{\gamma,\infty})} \Phi^H_M(\gamma, \lambda) \left(L(\text{Mot}_{M_\gamma}) |\iota^M(\gamma)|^{-1} SO^\infty_\gamma((f^\infty)_M) \right),$$

where the stable orbital integrals are now computed using canonical measure on centralizers.

The hardest terms here are the stable orbital integrals, the L-values and the characters Φ. The constant terms $(f^\infty)_M$ are explicit integrals.

The L-values may be computed as products of values of Artin L-functions by explicitly describing the motives from [Gro97]. The terms Φ can be reduced to linear combinations of traces of γ against finite-dimensional representations of G by the algorithm on [Art89, pg. 273]. These can be computed by the Weyl character formula and its extension to irregular elements stated in, for example, [CR15, prop. 2.3].

The stable orbital integrals unfortunately cause far more difficulty. For specific groups, including our eventual application case of G_2, they are computed and listed in tables in [GP05, pg. 159]. First, they are interpreted as orbital integrals on compact-at-infinity form G^c by endoscopic transfer. The spectral side of the trace formula on G^c is then possible to compute, allowing the orbital integrals to be solved for once the coefficients in terms of L-values are known. Alternatively, [CT20] uses another trick, inputting vanishing results for small weight automorphic representations to solve for unstable orbital integrals in the resulting system of linear equations.

They can also be computed directly from unstable orbital integrals: [Kot77] and [Taï17] use Bruhat–Tits theory to do this for GL_3 and some classical groups respectively. Either way, all currently known methods are not fully general and extremely complicated.

Method 2:

Fortunately, there is a much simpler way to compute for our desired application of level-1 representations on G_2. Recalling that $I^{G_2^c}$ is known from [CR15], we can compare the expansions (1) for G_2 and G_2^c. The term for S^{G_2} a appears in the expansion for $I^{G_2^c}$ and can therefore be solved for and substituted in the expansion for I^{G_2}. In total, we get a formula

$$I^{G_2} = I^{G_2^c} + \text{corrections},$$

where the corrections are in terms of S^H for smaller endoscopic H.

In the next section, we will see that there aren't actually that many H appearing. Finally, Section 7 will show that the terms for these H are easily computed through another trick in the case of level-1. Method 2 also gives in Section 8 a Jacquet–Langlands-style result comparing quaternionic representations on G_2 to representations on G_2^c.

Note. We comment on possible extensions of method 2. The comparison to a compact form would work for any group with a form that is compact at infinity and unramified at all finite places. These appear in types, $G_2, B_3, D_4, B_4, F_4, B_7, D_8, B_8$ and E_8 as enumerated in [Gro96].

Being able to easily count the endoscopic terms spectrally is more rare and requires some kind of recursive expansion down to only terms of Lie type A_1^n. This in particular works out for type D_4, so level-1 forms on type D_4 should be countable analogously to method 2.

In another direction, plugging in other unramified test functions could compute counts weighted by Hecke eigenvalues. These would be in terms of the same weighted counts on G_2^c and certain other weighted counts of classical modular forms that are determined by combinatorial formulas for unramified transfers as explained in [Dal22, §5.4].

4. G_2 computation setup

From now on, we specialize to $G = G_2$ and discuss how to apply the previous theory to count $|\mathcal{Q}_k(1)|$.

4.1. Root system of G_2

4.1.1. Roots. We use notation from [LS93] to specify the root system of G_2. Let K be the maximal compact $\mathrm{SU}(2)\times\mathrm{SU}(2)/\pm 1$ of $G_2(\mathbb{R})$, and choose a maximal torus $T(\mathbb{R})$ that is inside K. Make a choice of simple roots of $G_2(\mathbb{R})$ that are noncompact; in this case determining a unique dominant chamber with respect to both G_2 and K. Let β be the highest root of G_2 with respect to the choice of simple roots and note that it is long.

We now give explicit coordinates. As a mnemonic convention, roots indexed 1 will be short and roots indexed 2 will be long. Figure 1 displays all the roots and shades our choices of dominant Weyl chambers for both G_2 and K. The compact roots at infinity are the four along the ϵ_i-coordinate axes.

If the roots of the short and long SU_2 are $2\epsilon_1$ and $2\epsilon_2$ respectively, then the simple roots of G_2 are

$$\text{(short) } \alpha_1 = -\epsilon_1 + \epsilon_2, \qquad \text{(long) } \alpha_2 = 3\epsilon_1 - \epsilon_2.$$

The other positive roots are

$$\begin{aligned} \text{(short) } 2\epsilon_1 &= \alpha_1 + \alpha_2, & \epsilon_1 + \epsilon_2 &= 2\alpha_1 + \alpha_2, \\ \text{(long) } 2\epsilon_2 &= 3\alpha_1 + \alpha_2, & 3\epsilon_1 + \epsilon_2 &= 3\alpha_2 + 2\alpha_2. \end{aligned}$$

The fundamental weights are

$$\lambda_1 = 2\alpha_1 + \alpha_2, \qquad \lambda_2 = 3\alpha_1 + 2\alpha_2.$$

Of course $\beta = \lambda_2$.

The Weyl group is generated by simple reflections

$$s_{\alpha_1}\begin{pmatrix} 2\epsilon_1 \\ 2\epsilon_2 \end{pmatrix} = \begin{pmatrix} \epsilon_1 + \epsilon_2 \\ 3\epsilon_1 - \epsilon_2 \end{pmatrix}, \qquad s_{\alpha_2}\begin{pmatrix} 2\epsilon_1 \\ 2\epsilon_2 \end{pmatrix} = \begin{pmatrix} -\epsilon_1 + \epsilon_2 \\ 3\epsilon_1 + \epsilon_2 \end{pmatrix}.$$

Finally,

$$\begin{aligned} \rho_K &= \epsilon_1 + \epsilon_2 = 2\alpha_1 + \alpha_2, \\ \rho_G &= 4\epsilon_1 + 2\epsilon_2 = 5\alpha_1 + 3\alpha_2. \end{aligned}$$

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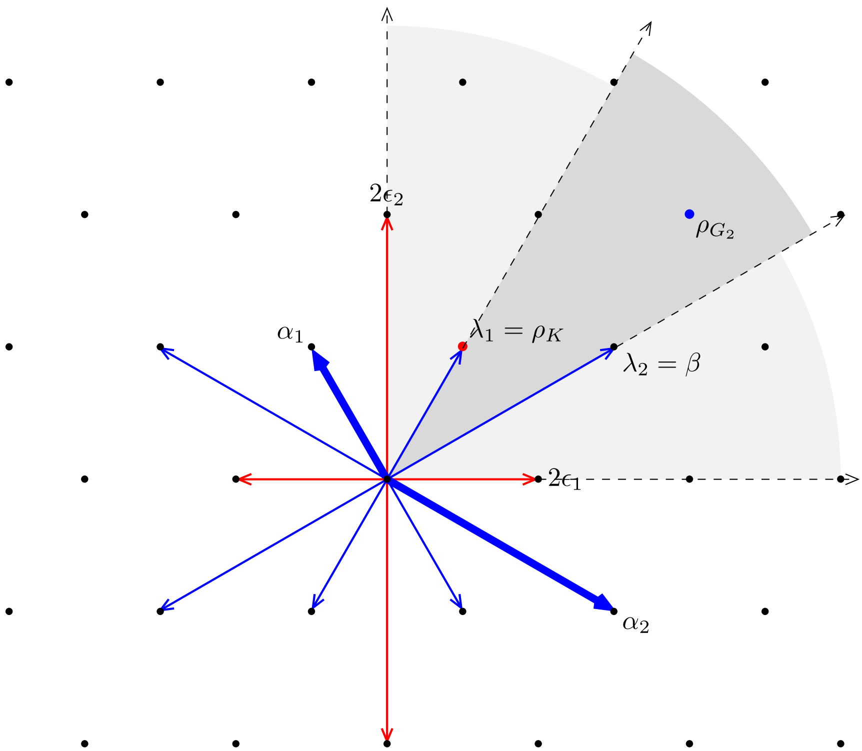

Figure 1. Character lattice, roots and choices of dominant chamber for G_2.

4.1.2. Coroots. Coroots will follow the opposite mnemonic: Coroots indexed 1 are long, and coroots indexed 2 are short.

Since G_2 has trivial center, $X^*(T)$ is the root lattice, which is exactly

$$X^*(T) = \{a\epsilon_1 + b\epsilon_2 : a,b \in \mathbb{Z}, a+b \in 2\mathbb{Z}\}.$$

Let (δ_1,δ_2) be the dual basis to $(2\epsilon_1,2\epsilon_2)$: that is, $(\delta_i,\epsilon_j) = 1/2\mathbf{1}_{i=j}$. Then

$$X_*(T) = \{a\delta_1 + b\delta_2 : a,b \in \mathbb{Z}, a+b \in 2\mathbb{Z}\}.$$

Since ϵ_1 and ϵ_2 are perpendicular,

$$(2\epsilon_1)^\vee = 2\delta_1,$$
$$(2\epsilon_2)^\vee = 2\delta_2.$$

More generally, the Weyl action gives

$$(\alpha_1^\vee, 2\epsilon_1) = -1, \qquad (\alpha_1^\vee, 2\epsilon_2) = 3,$$
$$(\alpha_2^\vee, 2\epsilon_1) = 1, \qquad (\alpha_2^\vee, 2\epsilon_2) = -1,$$

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so we get simple coroots

$$\alpha_1^\vee = -\delta_1 + 3\delta_2,$$
$$\alpha_2^\vee = \delta_1 - \delta_2.$$

This reproduces that the coroot lattice is $X_*(T)$, implying that G_2 is simply connected. For completeness,

$$\lambda_1^\vee = \delta_1 + 3\delta_2,$$
$$\lambda_2^\vee = \delta_1 + \delta_2.$$

4.2. Quaternionic discrete series for G_2

The quaternionic discrete series on G_2 of weight k for $k \geq 2$ lies in the L-packet

$$\Pi_{\text{disc}}^{G_2}((k-2)\beta).$$

The members of this L-packet have Harish–Chandra parameters:

$$(k-2)\beta + \rho_G, \qquad s_{\alpha_1}((k-2)\beta + \rho_G), \qquad s_{\alpha_2}((k-2)\beta + \rho_G).$$

As in [GGS02], the quaternionic member is the one with minimal K-type $\lambda_B = 2k\epsilon_2$. We know that the discrete series $\pi(\omega, \lambda)$ has minimal K-type

$$\lambda_B = \omega(\lambda + 2\rho_G) - 2\rho_K$$

by the Blattner formula [Kna01, Thm. 9.20]. Therefore, the weight-k quaternionic discrete series π_k is specifically $\pi(s_{\alpha_2}, (k-2)\beta)$ – computing, s_{α_2} fixes ρ_K so

$$s_{\alpha_2}(\lambda + 2\rho_G) - 2\rho_K = s_{\alpha_2}(\lambda + 2\rho_G - 2\rho_K) = s_{\alpha_2}(\lambda + 2\beta) = s_{\alpha_2}(k\beta) = 2k\epsilon_2.$$

This is the discrete series with Harish–Chandra parameter

$$\lambda_{k,H} := s_{\alpha_2}((k-2)\beta + \rho_G).$$

Call it π_k and its pseudocoefficient φ_k.

Theorem 3.1.1 then gives that for $k > 2$,

$$|\mathcal{Q}_k(1)| = I_{\text{spec}}(\varphi_k \otimes \mathbf{1}_K) \tag{2}$$

if we choose measures so that $\operatorname{vol} G_2(\widehat{\mathbb{Z}}) = 1$. Note again that this heavily depends on the miracle of Proposition 2.3.3 and a similar result does *not* hold either for pseudocofficients for the other members of $\Pi_{\text{disc}}((k-2)\beta)$ or for the Euler–Poincaré function.

Note. Theorem 3.1.1 for just the case of G_2 can be produced much more easily by the computation in [Mun20] of the A-packets of infinitesimal character $(k-2)\beta + \rho_G$ for $k > 2$. Mundy found that π_k appears in all of them. Therefore, trace-distinguishability follows immediately from [AJ87, Lemma 8.8] that a given discrete series appears in the character formula of exactly one element of such an A-packet.

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5. Groups contributing and related constants

5.1. Elliptic endoscopy of G_2

The elliptic endoscopic groups of G_2 are G_2, PGL_3 and SO_4. This is stated in a thesis [Alt13] but not fully explained, so we fill in some details here for reader convenience. We use notational conventions for endoscopy as in [Dal22, §3]. We compute the possible endoscopic pairs (s,ρ).

Since G_2 has trivial center, the cohomology condition on s is always satisfied so we don't bother checking it. Trivial center further gives that the isomorphism class of the pair only depends on s through its centralizer. Next, we note a result that we thank a referee for pointing out:

Lemma 5.1.1. *Let G be a split, adjoint and simply connected group over number field F. Then all its endoscopic groups are split.*

Proof. Since G is simply connected, there is an L-embedding ${}^LH \hookrightarrow {}^LG$. Since G is split, there is a projection ${}^LG \twoheadrightarrow \widehat{G}$. By inspecting the reconstruction of ${}^LH$ from (s,ρ), the image of ${}^LH$ in $\widehat{G}$ is connected if and only if ρ is trivial.

However, since $\widehat{G}$ also has trivial center, s is fixed by ρ which implies that the image of ${}^LH$ in $\widehat{G}$ is the centralizer of s. It is therefore necessarily connected since s is semisimple and $\widehat{G}$ is simply connected. □

In particular, ρ is always trivial and we can always find a valid s for any possible centralizer $\widehat{H}$. The possible elliptic $\widehat{H}$ with trivial ρ are G_2,SL_3 and $\mathrm{SL}_2 \times \mathrm{SL}_2/\{\pm 1\}$ corresponding to split endoscopic groups G_2,PGL_3, and $\mathrm{SL}_2 \times \mathrm{SL}_2/\{\pm 1\}$. In each of these cases, $\Lambda = 1$.

If a group contributes to the stabilization applied to our test function, then by the fundamental lemma, it needs to be unramified away from infinity. By formulas for transfers of pseudocoefficients (see [Dal22, lem. 5.6.1]), it also needs to have an elliptic maximal torus at infinity. The only groups contributing are therefore the G_2 and the $\mathrm{SL}_2 \times \mathrm{SL}_2/\{\pm 1\}$.

5.2. Endoscopic constants and normalizations

5.2.1. The ι. Let $H = \mathrm{SL}_2 \times \mathrm{SL}_2/\pm 1$, and let G_2^c be the unique nonsplit inner form of G_2 over $\mathbb{Q}$ which is compact at infinity. Then,

$$\iota(G_2^c,H) = \iota(G_2,H) = |\Lambda(H,\mathcal{H},s,\eta)|^{-1}\frac{\tau(G)}{\tau(H)} = 1\cdot\frac{1}{2},$$
$$\iota(G_2^c,G_2) = 1,$$

by Kottwitz's formula for Tamagawa numbers (note that $\ker^1(\mathbb{Q},Z_H) = \ker^1(\mathbb{Q},\{\pm 1\}) = 1$).

5.2.2. The transfer factors. We also need to fix transfer factors at all places for both G_2 and G_2^c to compute transfers. The computations in [Taï17] demonstrate how to do so explicitly. First, they can be chosen consistently by fixing a global Whittaker

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datum. The corresponding local Whittaker datum determines the local transfer factors on G_2 as in [KS99]. Since G_2 is defined over $\mathbb{Z}$, we can choose global data that are unramified/admissible at all finite places with respect to the $G_2(\mathbb{Z}_p)$ so we can use the fundamental lemma at all finite places as in [Hal93, §7]. By [Kal18, §4.4], the Whittaker datum on G_2 also gives compatible local transfer factors on G_2^c (note that G_2 is simply connected for applying Theorem 4.4.1). These allow us to use the fundamental lemma at finite places since transfer factors there stay the same as our choices for G_2.

We need to know two things about the Archimedean transfer factors. First, Whittaker normalization lets us use the formulas for discrete transfer from [She10] on both G_2 and G_2^c. Note here that G_2^c is in particular a pure inner form since G_2 is adjoint. This formula is stated in a slightly easier to use form in [Lab11, §IV.3] for our case of $\rho_G - \rho_H \in X^*(T)$ (the $\mathrm{inv}(\pi(1),\pi(w))$ of Shelstad is the $\kappa\cdot\epsilon$ term of Labesse).

Second, we need to know which element of $\Pi_{\mathrm{disc}}((k-2)\beta)$ our Archimedean Whittaker datum makes Whittaker-generic. This will have to be $\pi_{(k-2)\beta,1}$ since our choice of dominant Weyl chamber has all simple roots noncompact and is the only possible such choice up to Ω_K (see the discussion before Lemma 4.2.1 in [Taï17]. In fact, there is only one possible conjugacy class of Whittaker datum at infinity by considerations explained there).

5.2.3. The stabilizations. We fix canonical measure at finite places so that the fundamental lemma directly gives $\mathbf{1}^H_{K^\infty_{G_2}} = \mathbf{1}_{K^\infty_H}$. Recall that EP-functions and pseudo-coefficients are defined depending on measure so we don't need to fix measure at infinity.

Then, Equation (1) gives

$$I^{G_2}(\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)}\otimes\mathbf{1}_{K^\infty_{G_2}}) = S^{G_2}(\eta^{G_2}_{(k-2)\beta}\otimes\mathbf{1}_{K^\infty_{G_2}}) + \frac{1}{2}S^H((\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)})^H\otimes\mathbf{1}_{K^\infty_H}). \quad (3)$$

A simple case of the discrete transfer formula in [Lab11, §IV.3] computes that $(\eta^{G_2^c}_{(k-2)\beta})^{G_2} = \eta^{G_2}_{(k-2)\beta}$ (note that $\Omega_\mathbb{R}(G_2^c)\backslash\Omega_\mathbb{C}(G_2^c)$ is trivial so κ is too), so

$$I^{G_2^c}(\eta^{G_2^c}_{(k-2)\beta}\otimes\mathbf{1}_{K^\infty_{G_2^c}}) = S^{G_2}(\eta^{G_2}_{(k-2)\beta}\otimes\mathbf{1}_{K^\infty_{G_2}}) + \frac{1}{2}S^H((\eta^{G_2^c}_{(k-2)\beta})^H\otimes\mathbf{1}_{K^\infty_H}).$$

Since type $A_1\times A_1$ has no nontrivial centralizer of full semisimple rank, all elliptic endoscopy of $\mathrm{SL}_2\times\mathrm{SL}_2/\pm1$ is nonsplit. Therefore, it is ramified at some prime, so the transfers of $\mathbf{1}_{K^\infty_H}$ vanish, implying that $S^H = I^H$ on our test functions. Substituting one stabilization into another finally gives

$$I^{G_2}(\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)}\otimes\mathbf{1}_{K^\infty_{G_2}}) = I^{G_2^c}(\eta^{G_2^c}_{(k-2)\beta}\otimes\mathbf{1}_{K^\infty_{G_2^c}}) - \frac{1}{2}I^H((\eta^{G_2^c}_{(k-2)\beta})^H\otimes\mathbf{1}_{K^\infty_H}) + \frac{1}{2}I^H((\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)})^H\otimes\mathbf{1}_{K^\infty_H}) \quad (4)$$

under canonical measure at finite places.

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This is our realization of method 2. There are three steps remaining to get counts:

1. Compute the transfers of EP-functions to H.
2. Write the resulting $I^H(\eta_\lambda \otimes \mathbf{1}_{K_H})$ terms in terms of counts of level-1, classical modular forms.
3. Look up values for the G_2^c-term from [CR15].

6. Real endoscopic transfers

Let H again be the one endoscopic group we care about: $\mathrm{SL}_2 \times \mathrm{SL}_2/\{\pm 1\}$. We want to compute $(\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)})^H$ and $(\eta^{G_2^c}_{(k-2)\beta})^H$. By the choices of transfer factors in Section 5.2.2, we may do so by the formulas in [Lab11, §IV.3].

As a choice for computation that doesn't affect the final result, we realize the roots of H as $2\epsilon_1$ and $2\epsilon_2$. Orient $X^*(T)$ by setting the first quadrant in ϵ_1 and ϵ_2 to be H-dominant. The Weyl elements $\Omega(G,H)$ that send the G-dominant chamber to an H-dominant one are $\{1, s_{\alpha_1}, s_{\alpha_2}\}$.

6.1. Root combinatorics

Since $\rho_G - \rho_H \in X^*(T)$, [Lab11, §IV.3] gives the transfer of the pseudocoefficient of the quaternionic discrete series to H:

$$(\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)})^H = \kappa^H(s_{\alpha_2}^{-1})\eta^H_{(k-2)\beta+\rho_G-\rho_H} - \kappa^H(s_{\alpha_1}s_{\alpha_2}^{-1})\eta^H_{s_{\alpha_1}((k-2)\beta+\rho_G)-\rho_H} - \eta^H_{s_{\alpha_2}((k-2)\beta+\rho_G)-\rho_H} \tag{5}$$

for some signs $\kappa^H(\cdot)$.

We compute that $\rho_H = \epsilon_1 + \epsilon_2$. Then

$$(k-2)\beta + \rho_G - \rho_H = (k-2)(3\epsilon_1 + \epsilon_2) + (3\epsilon_1 + \epsilon) = 3(k-1)\epsilon_1 + (k-1)\epsilon_2.$$

In addition,

$$s_{\alpha_1}\rho_G = 5\epsilon_1 + \epsilon_2, \qquad s_{\alpha_1}\beta = \beta,$$
$$s_{\alpha_2}\rho_G = \epsilon_1 + 3\epsilon_2, \qquad s_{\alpha_2}\beta = 2\epsilon_2,$$

so

$$s_{\alpha_1}((k-2)\beta + \rho_G) - \rho_H = (k-2)(3\epsilon_1 + \epsilon_2) + (4\epsilon_1) = (3k-2)\epsilon_1 + (k-2)\epsilon_2$$

and

$$s_{\alpha_2}((k-2)\beta + \rho_G) - \rho_H = (k-2)(2\epsilon_2) + (2\epsilon_2) = 2(k-1)\epsilon_2.$$

6.2. Endoscopic characters

6.2.1. Setup. It remains to compute the κ terms in Equation (5). These signs depend in a very complicated way on the realization of H and the exact transfer factors chosen. We will therefore use an indirect trick to compute them more easily.

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Let ψ_H be a (discrete in our case) L-parameter for $H(\mathbb{R})$ and ψ_G the composition with ${}^LH \hookrightarrow {}^LG_2$. Then we have an identity of traces over L-packets:

$$S\Theta_{\psi_H}(f^H) = \sum_{\pi \in \Pi_{\psi_G}} \langle s_H, \pi \rangle \Theta_\pi(f),$$

where f^H is a transfer of f, Θ_π is the Harish–Chandra character, $S\Theta_{\psi_H}$ is the stable character corresponding to the L-packet, Π_{ψ_G} is the L-packet corresponding to the L-parameter, and $\langle s_H, \pi \rangle$ is shorthand for a sign depending on π and the choice of Whittaker data. This sign comes from pairing an element s_H of the centralizer of ψ_G determined by H with a character associated to π through the Whittaker datum. See [Kal16, §1] for an exposition of how this works in general.

If π on G_2 is discrete series, Labesse's formula tells us that we can choose

$$(\varphi_\pi^{G_2})^H = \sum_\lambda \epsilon(\lambda, \pi) \eta_\lambda^H$$

for some signs ϵ that depend on the transfer factor and some set of weights λ that only depends on the infinitesimal character of π.

Let ψ_λ be the L-parameter corresponding to weight-μ discrete series on H. Plugging this formula for $\varphi_\pi^{G_2}$ into the trace identity for ψ_λ gives that

$$\epsilon(\lambda, \pi) = \sum_\mu \epsilon(\mu, \pi) S\Theta_{\psi_\lambda}(\eta_\mu^H) = \sum_{\pi' \in \Pi_{\psi_G}} \langle s_H, \pi' \rangle \Theta_{\pi'}(\varphi_\pi^{G_2}) = \begin{cases} \langle s_H, \pi \rangle & \pi \in \Pi_{\psi_G} \\ 0 & \text{else} \end{cases},$$

where ψ_G is the pushforward of ψ_λ. The last equality is the definition of pseudocoefficient since all π in a packet for an L-parameter should be tempered. This computation shows that ψ_λ is required to push forward to the parameter for π and that $\epsilon(\lambda, \pi) = \langle s_H, \pi \rangle$.

6.2.2. The trick. Now, we are ready to compute the signs. Instead of doing the hard work of figuring out how the transfer factor directly affects the signs in Labesse's formulation, we will use the key fact that $\epsilon(\lambda, \pi) = \langle s_H, \pi \rangle = 1$ whenever π is the Whittaker-generic member of its L-packet. Therefore, in Labesse's formula for the *generic* member $\pi_{1,(k-2)\beta}$,

$$(\varphi_{\pi_{G_2}(1,(k-2)\beta)})^H = \eta^H_{(k-2)\beta+\rho_G-\rho_H} + \kappa^H(s_{\alpha_1})\operatorname{sgn}(s_{\alpha_1})\eta^H_{s_{\alpha_1}((k-2)\beta+\rho_G)-\rho_H}$$
$$+ \kappa^H(s_{\alpha_2})\operatorname{sgn}(s_{\alpha_2})\eta^H_{s_{\alpha_2}((k-2)\beta+\rho_G)-\rho_H},$$

all the coefficients need to be 1. The allows to solve

$$\kappa^H(s_{\alpha_1}) = \kappa^H(s_{\alpha_2}) = -1$$

for our choice of transfer factors. Right-$\Omega_\mathbb{R}$-invariance of Labesse's κ then also gives that

$$\kappa^H(s_{\alpha_1}s_{\alpha_2}) = -1.$$

6.3. Final formulas for transfers

Therefore, our final transfer is

$$(\varphi_{\pi_{G_2}(s_{\alpha_2},(k-2)\beta)})^H = -\eta^H_{3(k-1)\epsilon_1+(k-1)\epsilon_2} + \eta^H_{(3k-2)\epsilon_1+(k-2)\epsilon_2} - \eta^H_{2(k-1)\epsilon_2}. \quad (6)$$

Transfers from G_2^c are easier. Here, $\Omega_{\mathbb{R}}(G_2^c)\backslash\Omega_{\mathbb{C}}(G_2^c)$ is trivial so the average value of κ is 1. Averaging Labesse's formula as in [Dal22, cor. 5.1.5] therefore gives

$$(\eta^{G_2^c}_{(k-2)\beta})^H = \eta^H_{3(k-1)\epsilon_1+(k-1)\epsilon_2} - \eta^H_{(3k-2)\epsilon_1+(k-2)\epsilon_2} - \eta^H_{2(k-1)\epsilon_2}. \quad (7)$$

7. The $H = \mathrm{SL}_2 \times \mathrm{SL}_2/\pm 1$ term

Here, we compute the terms $I^H(\eta_\lambda \otimes \mathbf{1}_{K_H})$ for Euler–Poincaré functions η_λ. Any $\lambda = a\epsilon_1 + b\epsilon_2$ is a weight of H if $a+b$ is even. Note first that

$$I^H(\eta^H_\lambda \otimes \mathbf{1}_{K_H}) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(H)} \mathrm{tr}_{\pi_\infty}(\eta^H_\lambda)\,\mathrm{tr}_{\pi^\infty}(\mathbf{1}_{K_H}) = \sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(H) \\ \pi \text{ unram.}}} \mathrm{tr}_{\pi_\infty}(\eta^H_\lambda),$$

by Arthur's simple trace formula and using our choice of canonical measure at finite places.

To move forward, we need to understand automorphic reps on H by relating them to other groups. Consider the sequence

$$1 \to \pm 1 \to \mathrm{SL}_2 \times \mathrm{SL}_2 \to H \to 1.$$

It induces on local or global F

$$1 \to \pm 1 \to \mathrm{SL}_2 \times \mathrm{SL}_2(F) \to H(F) \to F^\times/(F^\times)^2 \to 1,$$

using that $H^1(F, \pm 1) = F^\times/(F^\times)^2$ and $H^1(F, \mathrm{SL}_2) = 1$ for the F we care about (the $\mathbb{R}$ case of the second equality comes from the determinant exact sequence on GL_2). The image of $\mathrm{SL}_2 \times \mathrm{SL}_2(F)$ is the connected component $H(F)^0$.

Note. As pointed out by a referee, we may also think of H as $(\mathrm{GL}_2 \times \mathrm{GL}_2)^{\det}/Z$, where the superscript det signifies that the two coordinates have the same determinant and the Z is the center of GL_2 embedded diagonally. Then, since $\mathbb{G}_m$ has trivial cohomology, $H(F) = (\mathrm{GL}_2(F) \times \mathrm{GL}_2(F))^{\det}/F^\times$. This suggests an alternate way to perform the ensuing computations that may be conceptually clearer.

7.1. Cohomological representations of $H(\mathbb{R})$

Next, we recall that the infinite trace measures an Euler characteristic against $(\mathfrak{h}, K_{H,\infty})$-cohomology:

$$\mathrm{tr}_{\pi_\infty}(\eta^H_\lambda) = \chi(H^*(\mathfrak{h}, K_{H,\infty}, \pi_\infty \otimes V_\lambda)),$$

where $\mathfrak{h}$ is the Lie algebra of H_∞ and V_λ is the finite-dimensional representation of weight λ of H^0_∞ pulled back to H_∞. Using the definition from [BW00, §5.1],

$$H^*(\mathfrak{h}, K_{H,\infty}, \pi_\infty \otimes V_\lambda) = H^*(\mathfrak{h}, K^0_{H,\infty}, \pi_\infty \otimes V_\lambda)^{K_{H,\infty}/K^0_{H,\infty}},$$

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it suffices to consider the π_∞ whose restrictions to H^0_∞ contain a component that is cohomological when pulled back to $[\mathrm{SL}_2\times\mathrm{SL}_2](\mathbb{R})$. By Frobenius reciprocity and semisimplicity of inductions, these are exactly the irreducible constituents of $\mathrm{Ind}_{H^0_\infty}^{H_\infty}\pi'$ for π' cohomological of H^0_∞.

Next, H^0_∞ is index 2 in H_∞. Pick $h\in H_\infty - H^0_\infty$, and let $\pi'^{(h)}$ be the representation $\gamma\mapsto\pi'(h^{-1}\gamma h)$. Define character

$$\chi: H_\infty \mapsto H_\infty/H^0_\infty \simeq \{\pm 1\}.$$

As noted in the proof of Lemma 2.5 in [LL79], there are two cases for H^0_∞-representations π':

1. $\pi'\neq\pi'^{(h)}$: Then $\mathrm{Ind}_{H^0_\infty}^{H_\infty}\pi'$ is irreducible and $\mathrm{Res}_{H^0_\infty}^{H_\infty}\mathrm{Ind}_{H^0_\infty}^{H_\infty}\pi' = \pi'\oplus\pi'^{(h)}$.
2. $\pi'=\pi'^{(h)}$: Then $\mathrm{Ind}_{H^0_\infty}^{H_\infty}\pi' = V\oplus(V\otimes\chi)$ for some irreducible V. Also, $\mathrm{Res}_{H^0_\infty}^{H_\infty}\mathrm{Ind}_{H^0_\infty}^{H_\infty}\pi'$ $=\pi'\oplus\pi'$

Recalling a standard result, the cohomological representations of $\mathrm{SL}_2(\mathbb{R})$ with respect to λ are:

- A discrete series L-packet $\{\pi_{\lambda,1},\,\pi_{\lambda,s}\}$ (where $\Omega_{\mathrm{SL}_2}=\{1,s\}$),
- The trivial representation $\mathbf{1}_{\mathrm{SL}_2}$ if $\lambda=0$.

By the Künneth rule, cohomological representations of $\mathrm{SL}_2\times\mathrm{SL}_2(\mathbb{R})$ are exactly products of those on $\mathrm{SL}_2(\mathbb{R})$. Those of H^0_∞ are exactly those of $\mathrm{SL}_2\times\mathrm{SL}_2(\mathbb{R})$ that are trivial on ±1 – in other words, with $\lambda=a\epsilon_1+b\epsilon_2$ and $a+b$ even.

Consider such λ. There are three cases of inductions to consider to compute the cohomological representations of H_∞. Note that conjugation by $h\in H_\infty-H^0_\infty$ swaps the two members of a discrete-series L-packet of an embedded SL_2 factor and fixes the trivial representation.

- $a,b\neq 0$: We look at the inductions of products of discrete series. This is case (1) so the 4 products pair up in sums that are 2 members of an L-packet. These are of course $\pi^H_{\lambda,1}$ and $\pi^H_{\lambda,s}$, where s is a length-1 element of Ω_H:

$$\pi^H_{\lambda,1}|_{H^0_\infty} = (\pi_{a\epsilon_1,1}\boxtimes\pi_{b\epsilon_2,1})\oplus(\pi_{a\epsilon_1,s}\boxtimes\pi_{b\epsilon_2,s}),$$
$$\pi^H_{\lambda,s}|_{H^0_\infty} = (\pi_{a\epsilon_1,1}\boxtimes\pi_{b\epsilon_2,s})\oplus(\pi_{a\epsilon_1,s}\boxtimes\pi_{b\epsilon_2,1}).$$

- Without loss of generality, $a=0, b\neq 0$: We also need to consider inductions of $\mathbf{1}\boxtimes\pi_{b\epsilon_2,\star}$. This is case (1), and both induce to a single irreducible σ^H_λ:

$$\sigma^H_\lambda|_{H^0_\infty} = (\mathbf{1}\boxtimes\pi_{b\epsilon_2,1})\oplus(\mathbf{1}\boxtimes\pi_{b\epsilon_2,s}).$$

- $a=b=0$: In addition to both the above, we need to consider the induction of $\mathbf{1}_{\mathrm{SL}_2}\boxtimes\mathbf{1}_{\mathrm{SL}_2}$. This is case (2). This trivial representation induces to $\mathbf{1}_{H_\infty}\oplus\chi$ on H_∞. Both factors are cohomological.

Grothendieck group relations stay true restricted to H^0_∞, so we can compute traces against η_λ. Recall that in $\mathrm{SL}_2(\mathbb{R})$

$$\mathbf{1} = I - \pi_{0,1} - \pi_{0,s},$$

where I is some parabolically induced representation with trivial trace against $\eta_0^{\mathrm{SL}_2}$.

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First, by our normalization

$$\mathrm{tr}_{\pi^H_{\lambda,1}}(\eta^H_\lambda) = \mathrm{tr}_{\pi^H_{\lambda,s}}(\eta^H_\lambda) = 1/2.$$

Next, working in H^0_∞ and for $\lambda = b\epsilon_2$,

$$\mathbf{1} \boxtimes \pi_{\lambda,\star} = (I - \pi_{0,1} - \pi_{0,s}) \boxtimes \pi_{\lambda,\star} = I \boxtimes \pi_{\lambda,\star} - \pi_{0,1} \boxtimes \pi_{\lambda,\star} - \pi_{0,s} \boxtimes \pi_{\lambda,\star},$$

so

$$\sigma^H_\lambda = \mathbf{1} \boxtimes \pi_{\lambda,1} + \mathbf{1} \boxtimes \pi_{\lambda,s} = I \boxtimes (\pi_{\lambda,1} + \pi_{\lambda,s}) - \pi^H_{0+\lambda,1} - \pi^H_{0+\lambda,s},$$

implying

$$\mathrm{tr}_{\sigma^H_\lambda}(\eta^H_\lambda) = -1.$$

Finally,

$$\begin{aligned}\mathbf{1} \boxtimes \mathbf{1} &= (I - \pi_{0,1} - \pi_{0,s}) \boxtimes (I - \pi_{0,1} - \pi_{0,s}) \\ &= I \boxtimes I - I \boxtimes (\pi_{0,1} + \pi_{0,s}) - (\pi_{0,1} + \pi_{0,s}) \boxtimes I + \pi^H_{0+0,1} + \pi^H_{0+0,s},\end{aligned}$$

so

$$\mathrm{tr}_{\mathbf{1}}(\eta^H_\lambda) = 1.$$

Since η_λ is supported on H^0_∞, we similarly have

$$\mathrm{tr}_\chi(\eta^H_\lambda) = 1.$$

In total, our H-term becomes a count

$$I^H(\eta^H_\lambda \otimes \mathbf{1}_{K_H}) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc,ur}}(H)} m^H_{\mathrm{disc}}(\pi) w^H(\pi_\infty), \tag{8}$$

where w^H is a weight

$$w^H(\pi_\infty) = \begin{cases} 0 & \pi_\infty \text{ not cohomological of weight } \lambda \\ 1/2 & \pi_\infty \text{ one of the } \pi^H_{\lambda,*} \\ -1 & \pi_\infty = \sigma^H_\lambda \\ 1 & \pi_\infty \text{ trivial or } \chi \text{ and } \lambda = 0 \end{cases}.$$

Call the cohomological cases type I, II and III in order.

7.2. Reduction to modular form counts

We now recall two results from [CR15]. Consider central isogeny $G \to G'$ of semisimple algebraic groups over $\mathbb{Z}$. If $\pi' = \pi'_\infty \otimes \pi'^\infty$ is an unramified, discrete automorphic representation of G', let $R(\pi')$ be the set of unitary, admissible representations $\pi = \pi_\infty \otimes \pi^\infty$ of $G(\mathbb{A})$ that satisfy:

- π^∞ is unramified with set of Satake parameters $c^\infty(\pi^\infty)$ induced from that of π'^∞ through $T^G_{G'} : \widehat{G}' \to \widehat{G}$.
- π_∞ is a constituent of the restriction of π'_∞ through $G(\mathbb{R}) \to G'(\mathbb{R})$.

Note that the size of $R(\pi')$ is the number of constituents of the restriction $\pi'_\infty|_{G(\mathbb{R})}$.

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Theorem 7.2.1 [CR15, prop. 4.7]. *Let π be an automorphic representation of G. Then*

$$m^G_{\mathrm{disc}}(\pi) = \sum_{\pi' : \pi \in R(\pi')} m^{G'}_{\mathrm{disc}}(\pi')[\pi_\infty, \pi'_\infty],$$

where $[\pi_\infty, \pi'_\infty]$ is the multiplicity of π_∞ in $\pi'_\infty|_{G(\mathbb{R})}$.

We will apply this with $G = H$ and $G' = \mathrm{PGL}_2 \times \mathrm{PGL}_2$. Make similar definitions of type I, II and III for representations of $[\mathrm{PGL}_2 \times \mathrm{PGL}_2](\mathbb{R})$. Type I on $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ restricts to the sum over a discrete L-packet on H_∞. Type II and III on $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ have irreducible restrictions. These restrictions partition the cohomological representations of H except for χ so

$$m^H_{\mathrm{disc}}(\pi_\infty \otimes \pi^\infty) = \sum_{c^\infty(\pi'^\infty) \in (T^H_{G'})^{-1}(c^\infty(\pi^\infty))} m^{G'}_{\mathrm{disc}}(\pi'_\infty \oplus \pi'^\infty)$$

when $\pi_\infty \subseteq \pi'_\infty|_{H_\infty}$ and the multiplicity is 0 when $\pi_\infty = \chi$. Now, we sum over the constituents of $\pi'_\infty|_{H_\infty}$ and the possible values of $c^\infty(\pi^\infty)$, noting that $T^H_{G'}$ is surjective. This gives

Corollary 7.2.2.

$$\sum_{\pi \in \mathcal{AR}_{\mathrm{disc,ur}}(H)} m^H_{\mathrm{disc}}(\pi) w^H(\pi_\infty) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc,ur}}(G')} m^{G'}_{\mathrm{disc}}(\pi) w^{G'}(\pi_\infty),$$

where $w^{G'}$ is the weight

$$w^{\mathrm{PGL}_2 \times \mathrm{PGL}_2}(\pi_\infty) = \begin{cases} 1 & \pi_\infty \text{ type I} \\ -1 & \pi_\infty \text{ type II} \\ 1 & \pi_\infty \text{ type III} \end{cases}$$

that only differs from w^H by multiplying the type I case by two.

Let $\mathcal{S}_k(1)$ be the set of normalized, level-1, weight-k cuspidal (new)eigenforms. If $\lambda = a\epsilon_1 + b\epsilon_2$, then type I representations on $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ correspond to pairs in $S_{a+2}(1) \times S_{b+2}(1)$. Type II is a single form times the trivial representation, and type III is only the trivial representation.

7.3. Final formula for S^H

Therefore, if

$$S_k = |\mathcal{S}_k(1)|,$$

we get:

$$I^H(\eta^H_{a\epsilon_1 + b\epsilon_2} \otimes \mathbf{1}_{K_H}) = (S_{a+2} - \mathbf{1}_{a=0})(S_{b+2} - \mathbf{1}_{b=0}), \tag{9}$$

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using canonical measure at finite places. By a classical formula ([DS05, Thm. 3.5.2], for example),

$$S_{a+2}=\begin{cases}0 & a+2=2 \text{ or } a+2 \text{ odd}\\ \lfloor \frac{a+2}{12}\rfloor -1 & a+2\equiv 2 \pmod{12}\\ \lfloor \frac{a+2}{12}\rfloor & \text{else}\end{cases}.$$

8. A Jacquet–Langlands-style result

8.1. First form

Generalizing Equation (4) slightly and substituting in Equations (6) and (7) gives

$$I^{G_2}(\varphi_{\pi_k}\otimes f^\infty)=I^{G_2^c}(\eta^{G_2^c}_{(k-2)\beta}\otimes f^\infty)-I^H(\eta^H_{(3k-3)\epsilon_1+(k-1)\epsilon_2}\otimes (f^\infty)^H)\\ +I^H(\eta^H_{(3k-2)\epsilon_1+(k-2)\epsilon_2}\otimes (f^\infty)^H) \tag{10}$$

for any unramified function f^∞ (we use here that $(G_2^c)^\infty=(G_2)^\infty$). This will let us describe the set $\mathcal{Q}_k(1)$ for $k>2$ in terms of certain representations of G_2^c and H.

Choose $\pi=\pi_k\otimes\pi^\infty\in\mathcal{Q}_k(1)$. Since π^∞ is unramified, it can be described by a sequence of Satake parameters: For each prime p, a semisimple conjugacy class $c_p(\pi^\infty)\in[\widehat{G_2}]_{\mathrm{ss}}$ (note that G_2 is split so we don't need to worry about the full Langlands dual and see [ST16, §3.2] for full background).

The endoscopic datum for H also gives an embedding $\widehat{H}\hookrightarrow\widehat{G_2}$ (noting again that everything is split) whose image contains a chosen maximal torus and therefore induces a map

$$T^{G_2}_H:[\widehat{H}]_{\mathrm{ss}}\twoheadrightarrow[\widehat{G_2}]_{\mathrm{ss}}.$$

The fibers of this map are Ω_{G_2}-orbits of conjugacy classes in H and have size 3 at G_2-regular elements.

Proposition 8.1.1. *Let $k>2$ and π^∞ an unramified representation of $(G_2)^\infty$. Then*

$$m^{G_2}_{\mathrm{disc}}(\pi_k\otimes\pi^\infty)=m^{G_2^c}_{\mathrm{disc}}(V_{(k-2)\beta}\otimes\pi^\infty)-\frac{1}{2}|S^H(\pi^\infty,(3k-3)\epsilon_1+(k-1)\epsilon_2)|\\ +\frac{1}{2}|S^H(\pi^\infty,(3k-2)\epsilon_1+(k-2)\epsilon_2)|.$$

Recall here that V_λ is the finite-dimensional representation of G_2^c with highest weight λ. Also, $S^H(\pi^\infty,\lambda)$ is the multiset of $\pi_\infty\otimes\pi_1^\infty\in\mathcal{AR}_{\mathrm{disc}}(H)$ with multiplicity such that

- $\pi_\infty\in\Pi^H_{\mathrm{disc}}(\lambda)$,
- *For all p, $c_p(\pi_1^\infty)\in(T^{G_2}_H)^{-1}(c_p(\pi^\infty))$.*

Proof. This is a standard Jacquet–Langlands-style argument. Through the Satake isomorphism, each f_p can be thought of as a function $[\widehat{G_2}]_{\mathrm{ss}}\to\mathbb{C}$ through $f_p(c_p(\pi))=\mathrm{tr}_{\pi_p}(f_p)$. It is in fact a Weyl-invariant regular function on a maximal torus in $\widehat{G_2}$. The full version of the fundamental lemma (see the introduction to [Hal95],

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for example) shows that

$$f_p^H(c_p) = f_p(T_H^{G_2}(c_p))$$

for all $c_p \in \widehat{H}$.

There are only finitely many sequences $c_p(\pi_1^\infty)$ and $T_H^{G_2}(c_p(\pi_1^\infty))$ for π_1^∞ the unramified finite component of an automorphic representation either:

- of G_2 with infinite part π_k,
- of G_2^c with infinite part $V_{(k-2)\beta}$,
- or of H with infinite part in $\Pi_{\mathrm{disc}}((3k-3)\epsilon_1 + (k-1)\epsilon_2)$ or $\Pi_{\mathrm{disc}}((3k-2)\epsilon_1 + (k-2)\epsilon_2)$.

Therefore, we can choose an f^∞ that is 0 on all of these sequences $c_p(\pi_1^\infty)$ except 1 on exactly the sequence $c_p(\pi^\infty)$ (this reduces to finding Weyl-invariant polynomials on $(\mathbb{C}^\times)^2$ that take specified values on certain Weyl orbits). The result follows from plugging this f^∞ into Equation (10), noting that the w^H from Equation (8) is always 1/2 in the relevant cases. □

8.2. In terms of modular forms

We can use the argument from Section 7.2 to reduce the H-multiplicity terms to PGL_2-multiplicity ones.

First, we have a map on conjugacy classes

$$T^H_{\mathrm{PGL}_2\times\mathrm{PGL}_2} : [\widehat{\mathrm{PGL}_2\times\mathrm{PGL}_2}]_{\mathrm{ss}} \twoheadrightarrow [\widehat{H}]_{\mathrm{ss}}.$$

Since the first group is $\mathrm{SL}_2 \times \mathrm{SL}_2(\mathbb{C})$, the fibers of this map are of the form $\{c, -c\}$ for some $c \in [\mathrm{SL}_2 \times \mathrm{SL}_2(\mathbb{C})]_{\mathrm{ss}}$. Composing then gives map

$$T^{G_2}_{\mathrm{PGL}_2\times\mathrm{PGL}_2} : [\widehat{\mathrm{PGL}_2\times\mathrm{PGL}_2}]_{\mathrm{ss}} \twoheadrightarrow [\widehat{G_2}]_{\mathrm{ss}}.$$

This allows us to define $S^{\mathrm{PGL}_2\times\mathrm{PGL}_2}(\pi^\infty,\lambda)$ analogous to $S^H(\pi^\infty,\lambda)$ for all $\lambda = a\epsilon_1 + b\epsilon_2$ with both a and b even. For indexing purposes, set it to be empty when a and b aren't even.

Formula (9) also gives us that $S^H(\pi^\infty, a\epsilon_1 + b\epsilon_2) = \emptyset$ when a and b aren't both even. Recall from §7.2 that the restriction of discrete series $\pi_\lambda^{\mathrm{PGL}_2\times\mathrm{PGL}_2}$ to $H(\mathbb{R})$ has as components the two members of the L-packet $\Pi^H_{\mathrm{disc}}(\lambda)$. Therefore, a similar analysis using Theorem 7.2.1 shows that

$$|S^H(\pi^\infty,\lambda)| = 2|S^{\mathrm{PGL}_2\times\mathrm{PGL}_2}(\pi^\infty,\lambda)|.$$

Finally, PGL_2 is a quotient of GL_2 by a central torus with trivial Galois cohomology, so automorphic representations on PGL_2 are just those on GL_2 with all components having trivial central character. Recalling injection

$$\iota : [\mathrm{SL}_2 \times \mathrm{SL}_2(\mathbb{C})]_{\mathrm{ss}} \hookrightarrow [\mathrm{GL}_2 \times \mathrm{GL}_2(\mathbb{C})]_{\mathrm{ss}},$$

this gives:

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Corollary 8.2.1. *Let $k>2$ and π^∞ an unramified representation of $(G_2)^\infty$. Then*

$$m^{G_2}_{\mathrm{disc}}(\pi_k\otimes\pi^\infty)=m^{G_2^c}_{\mathrm{disc}}(V_{(k-2)\beta}\otimes\pi^\infty)-|S^{\mathrm{GL}_2\times\mathrm{GL}_2}(\pi^\infty,(3k-3)\epsilon_1+(k-1)\epsilon_2)|$$
$$+|S^{\mathrm{GL}_2\times\mathrm{GL}_2}(\pi^\infty,(3k-2)\epsilon_1+(k-2)\epsilon_2)|.$$

Recall here that V_λ is the finite-dimensional representation of G_2^c with highest weight λ. Also, $S^{\mathrm{GL}_2\times\mathrm{GL}_2}(\pi^\infty,\lambda)$ is the set of $\pi_\infty\otimes\pi_1^\infty\in\mathcal{AR}_{\mathrm{disc}}(\mathrm{GL}_2\times\mathrm{GL}_2)$ such that

- *π_∞ is the discrete series $\pi_\lambda^{\mathrm{GL}_2\times\mathrm{GL}_2}$,*
- *For all p, $c_p(\pi_1^\infty)=\iota(c'_p)$ for some $c'_p\in(T^{G_2}_{\mathrm{PGL}_2\times\mathrm{PGL}_2})^{-1}(c_p(\pi^\infty))$. Here ι is the map $[\mathrm{SL}_2\times\mathrm{SL}_2(\mathbb{C})]_{\mathrm{ss}}\hookrightarrow[\mathrm{GL}_2\times\mathrm{GL}_2(\mathbb{C})]_{\mathrm{ss}}$.*

Of course, since all infinite factors in sight are discrete series, we may again replace the m_{disc} by m_{cusp} using [Wal84].

Note of course that $S^{\mathrm{GL}_2\times\mathrm{GL}_2}(\pi^\infty,a\epsilon_1+b\epsilon_2)=\emptyset$ unless both a and b are even. Therefore, we can interpret this as, for $k>2$:

- If k is even: $\mathcal{Q}_k(1)$ is the corresponding set of representations transferred from G_2^c *in addition to* representations transferred from pairs of cuspidal eigenforms in $\mathcal{S}_{3k}(1)\times\mathcal{S}_k(1)$.
- If k is odd: $\mathcal{Q}_k(1)$ is the corresponding set of representations transferred from G_2^c *except for* representations that are also transferred from pairs of cuspidal eigenforms in $\mathcal{S}_{3k-1}(1)\times\mathcal{S}_{k+1}(1)$.

Results for level >1 would be a lot more complicated since formula (4) would have many further hyperendoscopic terms and the comparison to $\mathrm{GL}_2\times\mathrm{GL}_2$ would not work as nicely.

9. Counts of forms

9.1. Formula in terms of $I^{G_2^c}$

To get counts instead of a list, combining formulas (2),(10) and (9) gives that

$$|\mathcal{Q}_k(1)|=I^{G_2^c}(\eta_\lambda^{G_2^c}\otimes\mathbf{1}_{K^\infty_{G_2^c}})-(S_{3k-1}-\mathbf{1}_{3k-3=0})(S_{k+1}-\mathbf{1}_{k-1=0})$$
$$+(S_{3k}-\mathbf{1}_{3k-2=0})(S_k-\mathbf{1}_{k-2=0}),\tag{11}$$

where S_k as before represents the count of classical modular forms of weight k.

Substituting in the formulas for S_k, for $k>2$:

$$|\mathcal{Q}_k(1)|=I^{G_2^c}(\eta_\lambda\otimes\mathbf{1}_{K^\infty_{G_2^c}})$$
$$+\begin{cases}\lfloor\frac{k}{4}\rfloor\left(\lfloor\frac{k}{12}\rfloor-1\right) & k\equiv 2\pmod{12}\\ \lfloor\frac{k}{4}\rfloor\lfloor\frac{k}{12}\rfloor & k\equiv 0,4,6,8,10\pmod{12}\\ -\left(\lfloor\frac{3k-1}{12}\rfloor-1\right)\left(\lfloor\frac{k+1}{12}\rfloor-1\right) & k\equiv 1\pmod{12}\\ -\left(\lfloor\frac{3k-1}{12}\rfloor-1\right)\lfloor\frac{k+1}{12}\rfloor & k\equiv 5,9\pmod{12}\\ -\lfloor\frac{3k-1}{12}\rfloor\lfloor\frac{k+1}{12}\rfloor & k\equiv 3,7,11\pmod{12}\end{cases}.$$

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9.2. Computing $I^{G_2^c}$

The group $G_2^c(\mathbb{R})$ is compact, so the $I^{G_2^c}$ term takes a very simple form: $L^2(G_2^c(\mathbb{Q})\backslash G_2^c(\mathbb{A}))$ decomposes as a direct sum of automorphic representations and the EP-functions η_λ are just scaled matrix coefficients of the finite-dimensional representations V_λ with highest weight λ on $G_2^c(\mathbb{R})$. Therefore,

$$I^{G_2^c}(\eta_\lambda \otimes \mathbf{1}_{K^\infty_{G_2^c}}) = \sum_{\pi \in \mathcal{AR}(G_2^c)} \mathbf{1}_{\pi_\infty = V_\lambda} \operatorname{tr}_{\pi^\infty}(\mathbf{1}_{K^\infty_{G_2^c}}),$$

which is just counting the number of unramified automorphic reps of G_2^c that have infinite component V_λ.

For reader convenience, we now explain in detail an argument well known to experts. Since unramified representations have one-dimensional spaces of K^∞-fixed vectors, taking $K^\infty_{G_2^c}$ invariants sends each such π to a linearly independent copy of V_λ that together span the V_λ-isotypic component of

$$L^2(G_2^c(\mathbb{Q})\backslash G_2^c(\mathbb{A})/K^\infty_{G_2^c}) = L^2(G_2^c(\mathbb{Z})\backslash G_2^c(\mathbb{R})) \subseteq L^2(G_2^c(\mathbb{R})).$$

By Peter–Weyl, $L^2(G_2^c(\mathbb{R}))$ has V_λ-isotypic component $V_\lambda^{\oplus \dim V_\lambda}$. In fact, this component for both the left and right actions is the same subspace. Therefore, the number of copies of $V_\lambda \subseteq L^2(G_2^c(\mathbb{Z})\backslash G_2^c(\mathbb{R}))$ is $\dim\left(V_\lambda^{G_2^c(\mathbb{Z})}\right)$ by a dimension count.

Summarizing:

$$I^{G_2^c}(\eta_\lambda \otimes \mathbf{1}_{K^\infty_{G_2^c}}) = \dim\left(V_\lambda^{G_2^c(\mathbb{Z})}\right). \tag{12}$$

A PARI/GP 2.5.0 program in the online appendix to [CR15] computes this for all λ by pairing the trace character of $V_\lambda|_{G_2(\mathbb{Z})}$ with the trivial character.

An explicit paper formula for this computation is more-or-less written out in an honors thesis of Steven Sullivan [Sul13]. Sullivan writes out the traces of all 16 conjugacy classes in $G_2^c(\mathbb{Z})$ against $V_{(k-2)\beta}$ as polynomials of k with coefficients that are sums of kth powers of 7th, 8th and 12th roots of unity. This gets a polynomial expression for the trace character pairing and therefore $I^{G_2^c}(\eta_\lambda \otimes \mathbf{1}_{K^\infty_{G_2^c}})$ in cases (mod 168). Simplifications in Mathematica give a reasonable closed-form version in Section 9.4.

It is important to note here that getting the explicit descriptions and sizes of the conjugacy classes in $G_2^c(\mathbb{Z})$ was nontrivial and required some trickery in both Sullivan's and Chenevier-Taïbi's computations. This step would be an obstacle to any generalizations.

9.3. Table of counts

Table 1 gives values of $|\mathcal{Q}_k(1)|$ for $k = 3$ to 52 produced by formula (11) and [CR15]'s table for formula (12). The lowest-weight example is bolded, although this work does not rule out the existence of an example with weight 2 or weight 1 (as defined by [Pol20, §1.1]).

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Table 1. Counts of discrete, quaternionic automorphic representations of level 1 on G_2.

k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$
3	0	13	5	23	76	33	478	43	1792
4	0	14	13	24	126	34	610	44	2112
5	0	15	8	25	121	35	637	45	2250
6	**1**	16	23	26	175	36	807	46	2619
7	0	17	17	27	173	37	849	47	2790
8	2	18	37	28	248	38	1037	48	3233
9	1	19	30	29	250	39	1097	49	3447
10	4	20	56	30	341	40	1332	50	3938
11	1	21	50	31	349	41	1412	51	4201
12	9	22	83	32	460	42	1686	52	4780

9.4. Explicit formula

For the reader's amusement, we build off the work of [Sul13] to present a closed-form formula for $|\mathcal{Q}_k(1)|$ that fits in a few lines:

$$
\begin{aligned}
&|\mathcal{Q}_{n+2}(1)| = \\
&\frac{1}{12096}\frac{1}{120}(n+1)(3n+4)(n+2)(3n+5)(2n+3) + \frac{1}{216}\frac{1}{6}(n+1)(n+2)(2n+3) \\
&+ \frac{5}{192}\frac{1}{8}\begin{cases} (n+2)(3n+4) & n=0 \pmod 2 \\ -(n+1)(3n+5) & n=1 \pmod 2 \end{cases} + \frac{1}{18}\begin{cases} \frac{2n}{3}+1 & n=0 \pmod 3 \\ -\lfloor \frac{n}{3} \rfloor - 1 & n=1,2 \pmod 3 \end{cases} \\
&+ \frac{1}{32}\begin{cases} \frac{3n}{2}+10 & n=0 \pmod 4 \\ 6\lfloor \frac{n}{4} \rfloor - 4 & n=1 \pmod 4 \\ -2\lfloor \frac{n}{4} \rfloor - 2 & n=2,3 \pmod 4 \end{cases} + \frac{1}{24}\begin{cases} 3\lfloor \frac{n}{6} \rfloor + 5 & n=0,1 \pmod 6 \\ 3\lfloor \frac{n}{6} \rfloor - 2 & n=2,3 \pmod 6 \\ 3\lfloor \frac{n}{6} \rfloor + 3 & n=4,5 \pmod 6 \end{cases} \\
&+ \frac{1}{7}\begin{cases} 1 & n=0 \pmod 7 \\ -1 & n=4 \pmod 7 \\ 0 & n=1,2,3,5,6 \pmod 7 \end{cases} + \frac{1}{4}\begin{cases} 1 & n=0 \pmod 8 \\ -1 & n=5 \pmod 8 \\ 0 & n=1,2,3,4,6,7 \pmod 8 \end{cases} \\
&+ \begin{cases} \lfloor \frac{n+2}{4} \rfloor \left(\lfloor \frac{n+2}{12} \rfloor - 1\right) & n=0 \pmod{12} \\ \lfloor \frac{n+2}{4} \rfloor \lfloor \frac{n+2}{12} \rfloor & n=2,4,6,8,10 \pmod{12} \\ -\left(\lfloor \frac{3n+5}{12} \rfloor - 1\right)\left(\lfloor \frac{n+3}{12} \rfloor - 1\right) & n=11 \pmod{12} \\ -\left(\lfloor \frac{3n+5}{12} \rfloor - 1\right)\lfloor \frac{n+3}{12} \rfloor & n=3,7 \pmod{12} \\ -\lfloor \frac{3n+5}{12} \rfloor \lfloor \frac{n+3}{12} \rfloor & n=1,5,9 \pmod{12} \end{cases}.
\end{aligned}
$$

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References

[AJ87] J. Adams and J. F. Johnson, 'Endoscopic groups and packets of nontempered representations', *Compositio Math.* **64**(3) (1987), 271–309. MR 918414

[Alt13] S. Altschul, *Endoscopy for Nilpotent Orbits of g_2* (University of Michigan Deep Blue Repositories, 2013).

[Art89] J. Arthur, 'The L^2-Lefschetz numbers of Hecke operators', *Invent. Math.* **97**(2) (1989), 257–290. MR 1001841

[Art13] J. Arthur, *The Endoscopic Classification of Representations*, *American Mathematical Society Colloquium Publications*, vol. 61 (American Mathematical Society, Providence, RI, 2013). MR 3135650

[BW00] A. Borel and N. Wallach, *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, second edn., Mathematical Surveys and Monographs, vol. 67, (American Mathematical Society, Providence, RI, 2000). MR 1721403

[CD90] L. Clozel and P. Delorme, 'Le théorème de Paley-Wiener invariant pour les groupes de Lie réductifs. II', *Ann. Sci. École Norm. Sup. (4)* **23**(2) (1990), 193–228. MR 1046496

[CR15] G. Chenevier and D. Renard, 'Level one algebraic cusp forms of classical groups of small rank', *Mem. Amer. Math. Soc.* **237**(1121) (2015), v+122. MR 3399888

[CT20] G. Chenevier and O. Taïbi, 'Discrete series multiplicities for classical groups over Z and level 1 algebraic cusp forms', *Publ. Math. Inst. Hautes Études Sci.* **131** (2020), 261–323. MR 4106796

[Dal22] R. Dalal, 'Sato–Tate equidistribution for families of automorphic representations through the stable trace formula', *Algebra Number Theory* **16**(1) (2022), 59–137. MR 4384564

[DS05] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics, vol. 228 (Springer-Verlag, New York, 2005). MR 2112196

[Fer07] Axel Ferrari, 'Théorème de l'indice et formule des traces', *Manuscripta Math.* **124**(3) (2007), 363–390. MR 2350551

[FGKP18] P. Fleig, Henrik P. A. Gustafsson, A. Kleinschmidt and D. Persson, *Eisenstein Series and Automorphic Representations*, Cambridge Studies in Advanced Mathematics, vol. 176 (Cambridge University Press, Cambridge, 2018). MR 3793195

[GGS02] W. T. Gan, B. Gross and G. Savin, 'Fourier coefficients of modular forms on G_2', *Duke Math. J.* **115**(1) (2002), 105–169. MR 1932327

[GP05] B. H. Gross and D. Pollack, 'On the Euler characteristic of the discrete spectrum', *J. Number Theory* **110**(1) (2005), 136–163. MR 2114678

[Gro96] B. H. Gross, 'Groups over Z', *Invent. Math.* **124**(1–3) (1996), 263–279. MR 1369418

[Gro97] B. H. Gross, 'On the motive of a reductive group', *Invent. Math.* **130**(2) (1997), 287–313. MR 1474159

[GW96] B. H. Gross and Nolan R. Wallach, 'On quaternionic discrete series representations, and their continuations', *J. Reine Angew. Math.* **481** (1996), 73–123. MR 1421947

[Hal93] T. C. Hales, 'A simple definition of transfer factors for unramified groups', in *Representation Theory of Groups and Algebras*, Contemp. Math., vol. 145 (Amer. Math. Soc., Providence, RI, 1993), 109–134. MR 1216184

[Hal95] T. C. Hales, 'On the fundamental lemma for standard endoscopy: Reduction to unit elements', *Canad. J. Math.* **47**(5) (1995), 974–994. MR 1350645

[Kal16] T. Kaletha, *The Local Langlands Conjectures for Non-Quasi-Split Groups*, Families of Automorphic Forms and the Trace Formula, Simons Symp. (Springer, Cham, 2016), 217–257. MR 3675168

[Kal18] T. Kaletha, 'Global rigid inner forms and multiplicities of discrete automorphic representations', *Invent. Math.* **213**(1) (2018), 271–369. MR 3815567

[Kna01] A. W. Knapp, *Representation Theory of Semisimple Groups*, Princeton Landmarks in Mathematics (Princeton University Press, Princeton, NJ, 2001). Reprint of the 1986 original. MR 1880691

[Kot77] R. E. Kottwitz, *Orbital Integral on GL(3)* (ProQuest LLC, Ann Arbor, MI, 1977), Thesis (Ph.D.)–Harvard University. MR 2940737

[KS99] R. E. Kottwitz and D. Shelstad, 'Foundations of twisted endoscopy', *Astérisque* (**255**) (1999), vi+190. MR 1687096

[Lab11] J.-P. Labesse, 'Introduction to endoscopy: Snowbird lectures, revised version, May 2010 [revision of mr2454335], On the stabilization of the trace formula', in *Stab. Trace Formula Shimura Var. Arith. Appl.*, vol. 1 (Int. Press, Somerville, MA, 2011), 49–91. MR 2856367

[LL79] J.-P. Labesse and R. P. Langlands, 'L -indistinguishability for SL(2)', *Canadian J. Math.* **31**(4) (1979), 726–785. MR 540902

[LS93] J.-S. Li and J. Schwermer, 'Constructions of automorphic forms and related cohomology classes for arithmetic subgroups of G_2', *Compositio Math.* **87**(1) (1993), 45–78. MR 1219452

[Mor10] S. Morel, *On the Cohomology of Certain Noncompact Shimura Varieties*, Annals of Mathematics Studies, vol. 173 (Princeton University Press, Princeton, NJ, 2010). With an appendix by Robert Kottwitz. MR 2567740

[Mun20] S. Mundy, 'Multiplicity of eisenstein series in cohomology and applications to gsp_4 *and* g_2', 2020.

[Pen19] Z. Peng, 'Multiplicity formula and stable trace formula', *Amer. J. Math.* **141**(4) (2019), 1037–1085.

[Pol18] A. Pollack, 'Modular forms on g_2 and their standard l-function', 2018.

[Pol20] A. Pollack, 'The Fourier expansion of modular forms on quaternionic exceptional groups', *Duke Math. J.* **169**(7) (2020), 1209–1280. MR 4094735

[Pol21] A. Pollack, 'Exceptional groups and their modular forms', *Notices Amer. Math. Soc.* **68**(2) (2021), 194–203. MR 4202334

[She10] D. Shelstad, 'A note on real endoscopic transfer and pseudo-coefficients', 2010.

[ST16] S. W. Shin and N. Templier, 'Sato–Tate theorem for families and low-lying zeros of automorphic L-functions', *Invent. Math.* **203**(1) (2016), 1–177, Appendix A by Robert Kottwitz, and Appendix B by Raf Cluckers, Julia Gordon and Immanuel Halupczok. MR 3437869

[Sul13] S. M. Sullivan, 'A trace formula for G_2' (University of Utah J. Willard Marriott Digital Libraries, May 2013).

[Taï17] O. Taïbi, 'Dimensions of spaces of level one automorphic forms for split classical groups using the trace formula', *Ann. Sci. Éc. Norm. Supér. (4)* **50**(2) (2017), 269–344. MR 3621432

[VZ84] D. A. Vogan, Jr. and Gregg J. Zuckerman, 'Unitary representations with nonzero cohomology', *Compositio Math.* **53**(1) (1984), 51–90. MR 762307

[Wal84] N. R. Wallach, 'On the constant term of a square integrable automorphic form', in *Operator Algebras and Group Representations, Vol. II (Neptun, 1980)*, Monogr. Stud. Math., vol. 18 (Pitman, Boston, MA, 1984), 227–237. MR 733320

[Wei06] M. H. Weissman, 'D_4 modular forms', *Amer. J. Math.* **128**(4) (2006), 849–898. MR 2251588